More on the matrix-vector product $Ax$

Let $A$ be an $m \times n$ matrix and $x$ be a vector in $\mathbb{R}^n$. We define the product $Ax$ as a linear combination of the vectors that come from the columns of $A$.

Let

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} = \begin{bmatrix}
    A_1 \\
    A_2 \\
    \vdots \\
    A_n
\end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_n
\end{bmatrix}.$$

Then the matrix-vector product $Ax$ is the linear combination

$$x_1 A_1 + x_2 A_2 + \ldots + x_n A_n.$$

Note that $Ax$ is a vector in $\mathbb{R}^m$.

Example.

$$\begin{bmatrix}
    3 & -8 \\
    -1 & 5 \\
    2 & 3
\end{bmatrix} \begin{bmatrix}
    -4 \\
    2
\end{bmatrix} = -4 \begin{bmatrix}
    3 \\
    -1 \\
    2
\end{bmatrix} + 2 \begin{bmatrix}
    -8 \\
    5 \\
    3
\end{bmatrix} = \begin{bmatrix}
    -28 \\
    14 \\
    -2
\end{bmatrix}$$

Remark. Given an $m \times n$ matrix $A$ and $x \in \mathbb{R}^n$, then the matrix equation

$$Ax = b$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix}
    A_1 & A_2 & \cdots & A_n & b
\end{bmatrix}.$$
Theorem. Let \( A \) be an \( m \times n \) matrix. Then the following three statements are equivalent:

1. For each \( b \) in \( \mathbb{R}^m \), the equation \( Ax = b \) has at least one solution.
2. The columns of \( A \) span \( \mathbb{R}^m \).
3. The matrix \( A \) has a pivot position in every row.

Warning: In this theorem, \( A \) is a coefficient matrix. The three statements are not equivalent if \( A \) is an augmented matrix.
Observation. Note that the kth entry in $Ax$ is

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n.$$ 

For example,

$$\begin{bmatrix}
* & * \\
5 & 6 \\
* & *
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
* \\
5x_1 + 6x_2 \\
*
\end{bmatrix}.$$ 

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n$$

is called the dot product of $[a_{k1} \ a_{k2} \ \ldots \ a_{kn}]$ and the vector $x$.

**Theorem.** Let $A$ be an $m \times n$ matrix. Then the matrix-vector product $Ax$ is “linear” in $x$. That is,

1. $A(u + v) = Au + Av$ for all $u$ and $v$ in $\mathbb{R}^n$, and
2. $A(cu) = cAu$ for all $u$ in $\mathbb{R}^n$ and all $c$ in $\mathbb{R}$.

Solution sets of systems of linear equations

**Definition.** Consider a linear system $Ax = b$. We say that it is *homogeneous* if $b = 0$ and *nonhomogeneous* otherwise.

The homogeneous case $Ax = 0$

**Observation.** Note that every homogeneous system is consistent. The solution $x = 0$ is called the trivial solution. All other solutions are said to be nontrivial.

**Theorem.** If $v_1$ and $v_2$ are two solutions to the homogeneous system $Ax = 0$, then any linear combination of $v_1$ and $v_2$ is also a solution.
Example. Let

\[ A = \begin{bmatrix} 1 & 6 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]

Express the solution set for \( Ax = 0 \) as a span. (Note that \( A \) is a coefficient matrix, not an augmented matrix.)