Linear independence

**Definition.** A (linear) dependence relation among a set of vectors \{v_1, v_2, \ldots, v_k\} is an equation of the form

\[ r_1v_1 + r_2v_2 + \ldots + r_kv_k = 0, \]

where \( r_i \neq 0 \) for some vector \( v_i \in \{v_1, v_2, \ldots, v_k\} \).

**Example.**

\[
\begin{bmatrix}
2 & 0 & 2 \\
2 & 3 & 5 \\
\end{bmatrix}
- \begin{bmatrix}
3 & 1 & 3 \\
\end{bmatrix}
+ \begin{bmatrix}
4 & 5 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

**Definition.** If there exists a dependence relation

\[ r_1v_1 + r_2v_2 + \ldots + r_kv_k = 0 \]

among a set of vectors \{v_1, v_2, \ldots, v_k\}, then we say that the set is *linearly dependent*. A set is *linearly independent* if it is not linearly dependent.

Matrix characterization

A dependence relation \( r_1v_1 + r_2v_2 + \ldots + r_kv_k = 0 \) can be rewritten as the matrix equation

\[ Ar = 0 \]

where

\[
A = \begin{bmatrix}
v_1 & v_2 & \ldots & v_k \\
\end{bmatrix}
\quad \text{and} \quad
r = \begin{bmatrix}
r_1 \\
\vdots \\
r_k \\
\end{bmatrix}.
\]

Therefore, a dependence relation among the vectors \( v_1, \ldots, v_k \) is the same as a nontrivial solution to \( Ar = 0 \).

**Example.** Consider the vectors

\[
v_1 = \begin{bmatrix}
1 \\
-1 \\
0 \\
\end{bmatrix} \quad v_2 = \begin{bmatrix}
0 \\
1 \\
-1 \\
\end{bmatrix} \quad v_3 = \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}.
\]

We can determine that these three vectors are linearly independent by considering the matrix

\[
\begin{bmatrix}
1 & 0 & 1 \\
-1 & 1 & 1 \\
0 & -1 & 1 \\
\end{bmatrix}.
\]
Example. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent? Why?

1. \[ \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\} \]

2. \[ \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\} \]

3. \[ \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ \pi \end{bmatrix} \right\} \]

4. \[ \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

5. \[ \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\} \]
Theorem. A nonzero set \( \{v_1, \ldots, v_k\} \) of vectors is linearly dependent if and only if, for some index \( j \), the vector \( v_j \) is a linear combination of the vectors \( v_1, \ldots, v_{j-1} \).
Theorem. If \( \{v_1, \ldots, v_k\} \) is a linearly independent set of vectors in \( \mathbb{R}^n \), then \( k \leq n \).

Example. We know that the set of vectors
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}
\]
is linearly dependent. What are all possible dependence relations among this set of vectors?