More matrix algebra

**Definition.** The product $C$ of $A$ and $B$ is the matrix

$$ C = \begin{bmatrix} AB_1 & AB_2 & \ldots & AB_p \end{bmatrix} $$

where $B_1, B_2, \ldots, B_p$ are the columns of the matrix $B$.

**Example.** Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $45^\circ$ and let $B : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation determined by the matrix

$$ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. $$

At the end of last class we determined that the matrix representation for the transformation $A \circ B$ is

$$ \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 2 & \frac{\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} & \sqrt{2} \end{bmatrix}. $$

**Row-column dot product definition:** The columns of $AB$ are linear combinations of the columns of $A$. In fact, consider the $j$th column of $AB$.

**Row-column rule:** $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$
Definition. The \( n \times n \) (square) matrix

\[
I_n = \begin{bmatrix}
1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & \ldots & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 1
\end{bmatrix}
\]

with 1’s down the diagonal and 0’s everywhere else is called the \( n \times n \) identity matrix.

Theorem 2. Let \( A \) be an \( m \times n \) matrix, and let \( B \) and \( C \) be matrices of appropriate sizes. Then

1. \( A(BC) = (AB)C \)
2. \( A(B + C) = AB + AC \)
3. \( (B + C)A = BA + CA \)
4. \( r(AB) = (rA)B = A(rB) \) for any scalar \( r \)
5. \( I_m A = A = AI_n \)

Three warnings.

1. \( AB \) does not always equal \( BA \).
2. \( AB = AC \) does not necessarily imply that \( B = C \).
3. \( AB = 0 \) does not necessarily imply that \( A = 0 \) or \( B = 0 \).

We will occasionally need to use the transpose of a matrix.

Definition. Given an \( m \times n \) matrix \( A \), its transpose \( A^T \) is the \( n \times m \) matrix such that

\[
(A^T)_{ij} = A_{ji}.
\]

Example. Consider

\[
M = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6\pi
\end{bmatrix}.
\]
Theorem 3. Let $A$ and $B$ be matrices whose sizes are appropriate for the following sums and products. Then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Matrix inverses

Invertible square matrices are more well behaved than arbitrary square matrices.

Definition. Let $A$ be a square matrix for which there exists a square matrix $B$ such that either

1. $AB = I$
2. $BA = I$.

Then we say that $A$ is invertible and that $B$ is the inverse of $A$.

Examples. Consider

$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. 
There is a simple formula for the inverse of a $2 \times 2$ matrix.

**Theorem 4.** Consider the $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

If $ad - bc \neq 0$, then $A$ is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ 

If $ad - bc = 0$, then $A$ is not invertible.

Here are some basic properties of inverses.

**Theorem 6.**

1. If $A$ is an invertible matrix, then $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

2. If $A$ and $B$ are $n \times n$ invertible matrices, then $AB$ is invertible. Moreover, $(AB)^{-1} = B^{-1}A^{-1}$.

3. If $A$ is an invertible matrix, then $A^T$ is invertible, and $(A^T)^{-1} = (A^{-1})^T$. 

4