More matrix algebra

Definition. The product $\mathbf{C}$ of $\mathbf{A}$ and $\mathbf{B}$ is the matrix

$$\mathbf{C} = \left[\begin{array}{c|c|c|c} \mathbf{AB}_1 & \mathbf{AB}_2 & \dots & \mathbf{AB}_p \end{array}\right]$$

where $\mathbf{B}_1$, $\mathbf{B}_2$, $\dots$, $\mathbf{B}_p$ are the columns of the matrix $\mathbf{B}$.

Example. Let $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 45° and let $\mathbf{B} : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation determined by the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

At the end of last class we determined that the matrix representation for the transformation $\mathbf{A} \circ \mathbf{B}$ is

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{3\sqrt{2}}{2} & \sqrt{2} \end{bmatrix}.$$

Row-column dot product definition: The columns of $\mathbf{AB}$ are linear combinations of the columns of $\mathbf{A}$. In fact, consider the jth column of $\mathbf{AB}$.

Row-column rule: $(\mathbf{AB})_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

Definition. The $n \times n$ (square) matrix

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & 1 \end{bmatrix}$$

with 1's down the diagonal and 0's everywhere else is called the $n \times n$ *identity matrix.*

Theorem 2. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ and $\mathbf{C}$ be matrices of appropriate sizes. Then

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
4. $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B} = \mathbf{A}(r\mathbf{B})$ for any scalar r
5. $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{AI}_n$

Three warnings.

1. $\mathbf{AB}$ does not always equal $\mathbf{BA}$.
2. $\mathbf{AB} = \mathbf{AC}$ does not necessarily imply that $\mathbf{B} = \mathbf{C}$.
3. $\mathbf{AB} = \mathbf{0}$ does not necessarily imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

We will occasionally need to use the transpose of a matrix.

Definition. Given an $m \times n$ matrix $\mathbf{A}$, its transpose $\mathbf{A}^T$ is the $n \times m$ matrix such that

$$(\mathbf{A}^T)_{ij} = \mathbf{A}_{ji}.$$

Example. Consider

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6\pi \end{bmatrix}.$$

Theorem 3. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices whose sizes are appropriate for the following sums and products. Then

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
3. $(r\mathbf{A})^T = r\mathbf{A}^T$
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Matrix inverses

Invertible square matrices are more well behaved than arbitrary square matrices.

Definition. Let $\mathbf{A}$ be a square matrix for which there exists a square matrix $\mathbf{B}$ such that either

1. $\mathbf{AB} = \mathbf{I}$ or
2. $\mathbf{BA} = \mathbf{I}$.

Then we say that $\mathbf{A}$ is *invertible* and that $\mathbf{B}$ is the *inverse* of $\mathbf{A}$.

Examples. Consider

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

There is a simple formula for the inverse of a 2×2 matrix.

Theorem 4. Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $\mathbf{A}$ is not invertible.

Here are some basic properties of inverses.

Theorem 6.

1. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
2. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then $\mathbf{AB}$ is invertible. Moreover, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
3. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^T$ is invertible, and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.