

More on subspaces of vector spaces

Last class we learned about three subspaces associated to a matrix—the column space $\text{Col } \mathbf{A}$, the row space $\text{Row } \mathbf{A}$, and the null space $\text{Nul } \mathbf{A}$.

Here is how Lay (p. 232) contrasts $\text{Nul } \mathbf{A}$ and $\text{Col } \mathbf{A}$ for an $m \times n$ matrix \mathbf{A} :

$\text{Nul } \mathbf{A}$	$\text{Col } \mathbf{A}$
1. $\text{Nul } \mathbf{A}$ is a subspace of \mathbb{R}^n .	1. $\text{Col } \mathbf{A}$ is a subspace of \mathbb{R}^m .
2. $\text{Nul } \mathbf{A}$ is implicitly defined; that is, you are given only a condition ($\mathbf{Ax} = \mathbf{0}$) that vectors in $\text{Nul } \mathbf{A}$ must satisfy.	2. $\text{Col } \mathbf{A}$ is explicitly defined; that is, you are told how to build vectors in $\text{Col } \mathbf{A}$.
3. It takes time to find vectors in $\text{Nul } \mathbf{A}$. Row operations on $[\mathbf{A} \ \mathbf{0}]$ are required.	3. It is easy to find vectors in $\text{Col } \mathbf{A}$. The columns of \mathbf{A} are displayed; others are formed from them.
4. There is no obvious relation between $\text{Nul } \mathbf{A}$ and the entries in \mathbf{A} .	4. There is an obvious relation between $\text{Col } \mathbf{A}$ and the entries in \mathbf{A} , since each column of \mathbf{A} is in $\text{Col } \mathbf{A}$.
5. A typical vector \mathbf{v} in $\text{Nul } \mathbf{A}$ has the property that $\mathbf{Av} = \mathbf{0}$.	5. A typical vector \mathbf{v} in $\text{Col } \mathbf{A}$ has the property that the equation $\mathbf{Ax} = \mathbf{v}$ is consistent.
6. Given a specific vector \mathbf{v} , it is easy to tell if \mathbf{v} is in $\text{Nul } \mathbf{A}$. Just compute \mathbf{Av} .	6. Given a specific vector \mathbf{v} , it may take time to tell if \mathbf{v} is in $\text{Col } \mathbf{A}$. Row operations on $[\mathbf{A} \ \mathbf{v}]$ are required.
7. $\text{Nul } \mathbf{A} = \{\mathbf{0}\}$ if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.	7. $\text{Col } \mathbf{A} = \mathbb{R}^m$ if and only if the equation $\mathbf{Ax} = \mathbf{b}$ has a solution for every \mathbf{b} in \mathbb{R}^m .
8. $\text{Nul } \mathbf{A} = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one.	8. $\text{Col } \mathbf{A} = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ maps \mathbb{R}^n onto \mathbb{R}^m .

Item 8 in both lists suggest two subspaces that are intimately connected with any linear transformation from one vector space to another.

Definition. A transformation $L : V_1 \rightarrow V_2$ from a vector space V_1 to a vector space V_2 is linear if

1. $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ for all vectors \mathbf{v}_1 and \mathbf{v}_2 in V_1 , and
2. $L(r\mathbf{v}) = rL(\mathbf{v})$ for all \mathbf{v} in V_1 and all r in \mathbb{R} .

Example. Let V_1 be the vector space of all continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and let V_2 be the vector space of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$. The operation of differentiation is a linear transformation from V_1 to V_2 . That is, the transformation $D : V_1 \rightarrow V_2$ given by

$$D(f) = f'$$

is a linear transformation.

Associated to any linear transformation are two important subspaces.

Definition. The kernel of $L : V_1 \rightarrow V_2$ is the subset of V_1 given by

$$\{\mathbf{v}_1 \mid L(\mathbf{v}_1) = \mathbf{0}\}.$$

The range of L is the subset of V_2 given by

$$\{\mathbf{v}_2 \mid L(\mathbf{v}_1) = \mathbf{v}_2 \text{ for some } \mathbf{v}_1 \text{ in } V_1\}.$$

Fact. Both the kernel and the range of a linear transformation are subspaces. The kernel is a subspace of V_1 , and the range is a subspace of V_2 .

For a matrix transformation from \mathbb{R}^n to \mathbb{R}^m determined by the matrix \mathbf{A} , its range is $\text{Col } \mathbf{A}$, and its kernel is $\text{Nul } \mathbf{A}$.

Example. What are the kernel and range of the transformation $p : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ determined by the matrix

$$\frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} ?$$

(This example was first introduced on September 22.)

Example. What are the kernel and the range of the differentiation transformation D mentioned above?

If we are careful, we can also use integration to define a linear transformation.

Example. Given a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ in \mathbb{P}_n (the vector space of all polynomial functions of degree at most n), we can define

$$I(p) = \int_0^x p(t) dt = a_n \frac{x^{n+1}}{n+1} + a_{n-1} \frac{x^n}{n} + \dots + a_0 x.$$

The map $I : \mathbb{P}_n \rightarrow \mathbb{P}_{n+1}$ is a linear transformation. What are its kernel and range?

Bases for vector spaces and subspaces

Given a vector space or subspace V , we often find it convenient to express it as the span of a few vectors. A basis for V is a spanning set that contains as few vectors as possible.

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a *basis* for V if

1. it is linearly independent, and
2. it spans V .

Example. The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of \mathbb{R}^n .

Example. The two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis of \mathbb{R}^2 .

Example. The set $\{x^3, x^2, x, 1\}$ is a basis of \mathbb{P}_3 .

Example. The set $\{x^3, x^3 + x^2, x, 1\}$ is another basis of \mathbb{P}_3 .