The Invertible Matrix Theorem

Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following twelve statements are equivalent:

(a) $\mathbf{A}$ is an invertible matrix.

(b) $\mathbf{A}$ is row equivalent to the identity matrix.

(c) $\mathbf{A}$ has n pivot positions

(d) The equation $\mathbf{Ax} = \mathbf{0}$ has no nontrivial solutions.

(e) The columns of $\mathbf{A}$ are linearly independent.

(f) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ is one-to-one.

(g) The equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.

(h) The columns of $\mathbf{A}$ span $\mathbb{R}^n$.

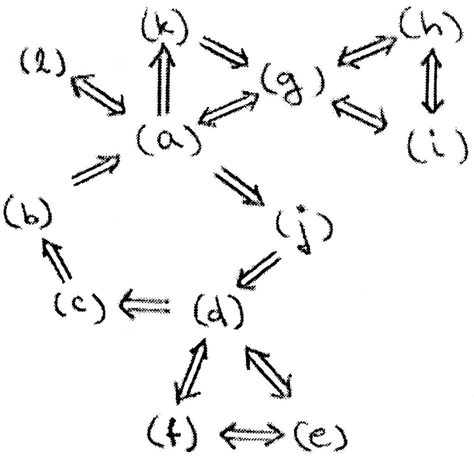
(i) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.

(k) There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}$.

(l) $\mathbf{A}^T$ is an invertible matrix.

Comments on the proof:



Computer graphics

Homogeneous coordinates are useful when we want to do computer graphics with matrices.

Definition. A point (x, y) in $\mathbb{R}^2$ can be represented by the point $(x, y, 1)$ in $\mathbb{R}^3$. The coordinates $(x, y, 1)$ are called the homogeneous coordinates of the point (x, y).

Homogeneous coordinates are useful because translation in $\mathbb{R}^2$ can be represented by a linear transformation in $\mathbb{R}^3$.

Fact 1. A translation by (h, k) in $\mathbb{R}^2$ can be obtained by matrix multiplication of homogeneous coordinates. That is,

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}.$$

Fact 2. Any linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ can be represented as a transformation of homogeneous coordinates by matrix multiplication. In particular, if the transformation is represented by the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the corresponding matrix for homogeneous coordinates is

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example. What is the matrix that represents rotation by 45° in terms of homogeneous coordinates?

Example. The *Mathematica* notebook `ComputerGraphics.nb` on the course website illustrates how homogeneous coordinates can be used to produce animated images. We start with a flag represented by the 3×4 matrix

$$\text{flag} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 1.5 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Rotation by 45 degrees in homogeneous coordinates is produced by the matrix

$$\mathbf{A}_r = \begin{bmatrix} 0.707107 & -0.707107 & 0. \\ 0.707107 & 0.707107 & 0. \\ 0. & 0. & 1. \end{bmatrix}.$$

We rotate the flag by multiplying the flag matrix on the left by $\mathbf{A}_r$, and we get

$$\begin{bmatrix} 0. & -1.41421 & 0.353553 & -0.707107 \\ 0. & 1.41421 & 2.47487 & 0.707107 \\ 1. & 1. & 1. & 1. \end{bmatrix}.$$

We translate the flag one unit to the right and up one-half of a unit using the matrix

$$\mathbf{A}_t = \begin{bmatrix} 1. & 0. & 1. \\ 0. & 1. & 0.5 \\ 0. & 0. & 1. \end{bmatrix}.$$

We scale the translated flag using the matrix

$$\mathbf{A}_s = \begin{bmatrix} -0.5 & 0. & 0. \\ 0. & 0.5. & 0. \\ 0. & 0. & 1. \end{bmatrix}.$$

Waving the flag:

We can wave the flag using matrix multiplication. We start with the original flag rotated clockwise by 36 degrees

$$\text{rflag} = \begin{bmatrix} 0. & 1.17557 & 2.49971 & 0.587785 \\ 0. & 1.61803 & 0.037955 & 0.809017 \\ 1. & 1. & 1. & 1. \end{bmatrix}.$$

Then we rotate it counterclockwise in increments of 9 degrees by repeatedly multiplying on the left by the matrix

$$\textbf{rotationStep} = \begin{bmatrix} 0.987688 & -0.156434 & 0. \\ 0.156434 & 0.987688 & 0. \\ 0. & 0. & 1. \end{bmatrix}.$$

Rising spinning pinwheel

The 3×13 matrix pinwheel in the notebook describes a basic looking pinwheel.

We rotate the pinwheel using a rotation matrix **rotationStep** that rotates counterclockwise by 18 degrees.

We translate the pinwheel vertically using the translation matrix

$$\textbf{upStep} = \begin{bmatrix} 1. & 0. & 0. \\ 0. & 1. & 0.2 \\ 0. & 0. & 1. \end{bmatrix}.$$

When we try to rotate and spin the pinwheel at the same time using the product

$$\textbf{rotationStep} \circ \textbf{upStep},$$

we get an unexpected result. Why is that and how to we overcome this problem?