Projection Matrices

We discussed projection matrices briefly when we discussed orthogonal projection. In particular, we discussed the following theorem.

**Theorem.** Let \{u_1, \ldots, u_k\} be an orthonormal basis for a subspace W of \(\mathbb{R}^n\). Form the \(n \times k\) matrix

\[
U = \begin{bmatrix} u_1 & u_2 & \ldots & u_k \end{bmatrix}.
\]

Then \(\text{proj}_W v = UU^T v\).

The matrix \(UU^T\) is called the projection matrix for the subspace W. It does not depend on the choice of orthonormal basis.

What if we do not start with an orthonormal basis of W?

**Theorem.** Let \{a_1, \ldots, a_k\} be any basis for a subspace W of \(\mathbb{R}^n\). Form the \(n \times k\) matrix

\[
A = \begin{bmatrix} a_1 & a_2 & \ldots & a_k \end{bmatrix}.
\]

Then the projection matrix for W is \(A(A^T A)^{-1}A^T\).

To see why this formula is true, we need a lemma.

**Lemma.** Suppose A is an \(n \times k\) matrix whose columns are linearly independent. Then \(A^T A\) is invertible.

To see why this lemma is true, consider the transformation \(A : \mathbb{R}^k \rightarrow \mathbb{R}^n\) determined by A. Since the columns of A are linearly independent, this transformation is one-to-one. Moreover, the null space of \(A^T\) is orthogonal to the column space of A. Consequently, \(A^T\) is one-to-one on the column space of A, and as a result, \(A^T A : \mathbb{R}^k \rightarrow \mathbb{R}^k\) is one-to-one. By the Invertible Matrix Theorem, \(A^T A\) is invertible.

Now we can compute the projection matrix for the column space of A. (Note that \(W = \text{Col} A\).) Any element of the column space of the matrix A is a linear combination of the columns of A, that is,

\[x_1a_1 + x_2a_2 + \ldots + x_ka_k.\]

If we let

\[x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix},\]

1
then
\[ x_1a_1 + x_2a_2 + \ldots + x_ka_k = Ax. \]

Given \( v \) in \( \mathbb{R}^n \), we denote by \( x_p \) the \( x \) that corresponds to the projection of \( v \) onto \( W \). In other words, let
\[ \text{proj}_W v = Ax_p. \]

We find the projection matrix by calculating \( x_p \).

The projection of \( v \) onto \( W \) is characterized by the fact that
\[ v - \text{proj}_W v \]
is orthogonal to each vector \( w \) in \( W \), that is,
\[ w \cdot (v - \text{proj}_W v) = 0 \]
for all \( w \) in \( W \). Since \( w = Ax \) for some \( x \), we have
\[ Ax \cdot (v - Ax_p) = 0 \]
for all \( x \) in \( \mathbb{R}^k \). Writing this dot product in terms of matrices yields
\[ (Ax)^T(v - Ax_p) = 0, \]
which is equivalent to
\[ (x^TA^T)(v - Ax_p) = 0. \]
Converting back to dot products, we have
\[ x \cdot A^T(v - Ax_p) = 0. \]
In other words, the vector \( A^T(v - Ax_p) \) is orthogonal to every vector \( x \) in \( \mathbb{R}^k \). The only vector in \( \mathbb{R}^k \) with this property is the zero vector, so we may conclude that
\[ A^T(v - Ax_p) = 0. \]
We get
\[ A^Tv = A^TAx_p. \]

From the lemma, we know that \( A^TA \) is invertible, and we have
\[ (A^TA)^{-1}A^Tv = x_p. \]
Since \( Ax_p \) is the desired projection, we have
\[ A(A^TA)^{-1}A^Tv = \text{proj}_W v. \]
We conclude that the projection matrix for \( W \) is
\[ A(A^TA)^{-1}A^T. \]
Note that any projection matrix $P$ satisfies the two properties

1. $P^2 = P$, and

2. $P$ is symmetric.

It is also true that any matrix that satisfies these two properties is the projection matrix for some subspace of $\mathbb{R}^n$ (see Exercise 36 in Section 7.1 of Lay).