The Casting-Out Procedure

Given a vector subspace $S$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can obtain a basis $B$ for $S$ by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
   
   (a) (cast out) $B_i = B_{i-1}$ if $\mathbf{v}_i$ is in $\text{Span} B_{i-1}$, or
   
   (b) (keep) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if $\mathbf{v}_i$ is not in $\text{Span} B_{i-1}$.

Then the final result $B_k$ is a basis for $S$.

To prove this theorem, we must show that the casting-out procedure produces a linearly independent set that still spans $S$.

Linear independence: Let $B_i$ be the first step in the procedure for which $B_i$ is linearly dependent. Then $\mathbf{v}_i$ is an element of $B_i$, but it is also a linear combination of vectors in $B_{i-1}$. This situation contradicts part 2 of the procedure. Consequently, the sets $B_i$ are linearly independent for $i = 1, \ldots, k$.

Spanning: We must show that $\text{Span} B_k = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. To do so, we prove that

$$\text{Span} B_i = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$$

for $i = 1, \ldots, k$ by induction on $i$.

Certainly $\text{Span} B_1 = \text{Span}\{\mathbf{v}_1\}$, so we assume that $\text{Span} B_{i-1} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$ and show that $\text{Span} B_i = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$. If $B_i = B_{i-1}$, then $\mathbf{v}_i$ is a linear combination of the vectors in $B_{i-1}$, and therefore,

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$$

$$= \text{Span} B_{i-1}$$

$$= \text{Span} B_i.$$ 

If $B_i \neq B_{i-1}$, then every vector $\mathbf{v}$ in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ can be written as

$$\mathbf{v} = \mathbf{w} + r_i \mathbf{v}_i,$$

where $\mathbf{w}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$. By the inductive hypothesis, $\mathbf{w}$ is in $\text{Span} B_{i-1}$, and therefore, $\mathbf{v}$ is in $\text{Span} B_i$. 