MA 713

Properties of the Winding Number

The winding number of a contour γ about a point a is

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz.$$

On p. 116, Ahlfors proves the following properties of the winding number:

1. If γ is contained in a circle, then $n(\gamma, a) = 0$ for all points a outside the circle.
2. The contour γ determines regions in $\overline{\mathbb{C}}$, i.e., the components of $\overline{\mathbb{C}} - \gamma$. Considered as a function of a, the winding number $n(\gamma, a)$ is constant on each of these regions. Moreover, it vanishes on the component that contains the point at infinity.
3. Suppose a contour γ contains a point z_1 with $\operatorname{Im} z_1 > 0$ and a point z_2 with $\operatorname{Im} z_2 < 0$. Moreover, suppose that γ can be split into two contours γ_1 and γ_2 such that

    (a) γ_1 goes from z_1 to z_2 while missing $\mathbb{R}^+$, and

    (b) γ_2 goes from z_2 to z_1 while missing $\mathbb{R}^-$.

    Then $n(\gamma, 0) = 1$.