NAME Key

Justify all steps. Neatness definitely counts.

12) 1. Compute

$$\int x \ln\left(\sqrt{1+x^2}\right)\, dx$$

Try $u = x^2$ so $du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$\int x \ln(\sqrt{1+x^2})\,dx = \frac{1}{2}\int \ln(\sqrt{1+u})\,du$$

$$= \frac{1}{2}\int \frac{1}{2}\ln(1+u)\,du$$

Let $v = 1+u$

$dv = du$

$$= \frac{1}{4}\int \ln(v)\,dv$$

By parts

$\mathcal{U} = \ln(v)$ $\quad d\nu = dv$

$d\mathcal{U} = \frac{1}{v}dv$ $\quad \nu = v$

$$= \frac{1}{4}\left[v\ln(v) - \int v \cdot \frac{1}{v}\,dv\right]$$

$$= \frac{1}{4}\left[v\ln(v) - \int 1\,dv\right]$$

$$= \frac{1}{4}\left[v\ln(v) - v + c\right]$$

$$= \frac{1}{4}(1+x^2)\ln(1+x^2) - \frac{1}{4}(1+x^2) + c$$

(Check by differentiating!)

2. Compute

$$\int \sqrt{1-\sin(x)}\,dx = \int \sqrt{1-u} \cdot \frac{1}{\sqrt{1-u^2}}\,du$$

Let $u = \sin(x)$

$du = \cos(x)dx$

$du = \sqrt{1-\sin^2 x}\,dx$

$\frac{1}{\sqrt{1-u^2}}\,du = dx$

$$= \int \frac{\sqrt{1-u}}{\sqrt{1-u}\,\sqrt{1+u}}\,du$$

$$= \int \frac{1}{\sqrt{1+u}}\,du$$

$$= 2(1+u)^{1/2} + c$$

$$= 2\sqrt{1+\sin(x)} + c$$

3. Find the cubic Taylory polynomial centered at zero of the solutions of the differential equation

$$\frac{dy}{dx} = 2xy + e^x.$$

Given $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

So

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = 2x\left(a_0 + a_1 x + a_2 x^2 + \dots\right) + \left(1 + x + \frac{x^2}{2} + \dots\right)$$

So $a_1 = 1$

$2a_2 = 2a_0 + 1 \Rightarrow a_2 = a_0 + 1/2$

$3a_3 = 2a_1 + \frac{1}{2} \Rightarrow a_3 = \frac{2}{3}a_1 + \frac{1}{6} = \frac{2}{3}\cdot 1 + 1/6 = 5/6$

So $y(x) = a_0 + x + (a_0 + \frac{1}{2})x^2 + \frac{5}{6}x^3 + \dots$

10

4. Find the solution of

$$\frac{dy}{dx} = y(x+3), \quad y(0) = 2.$$

This equation is separable

$\frac{1}{y}\frac{dy}{dx} = x+3$

$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int (x+3)\,dx$

$\int \frac{1}{y}\,dy = \frac{x^2}{2} + 3x + c$

$\ln|y| = \frac{x^2}{2} + 3x + c$

So $|y| = ke^{x^2/2+3x}$ where $k = e^c$

$y(0) = 2 > 0$

So $y(x) = 2e^{x^2/2+3x}$

(no absolute values needed because the right hand side is +)

12

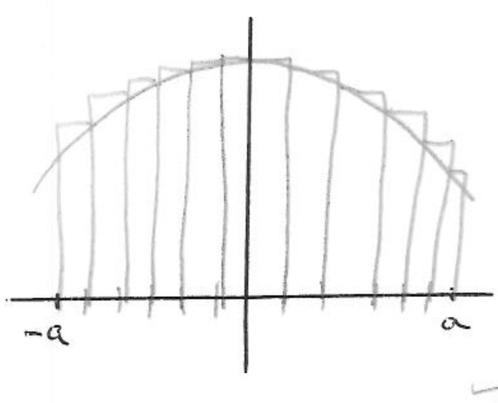
5. Suppose you are assigned to design a method to approximate value of $\int_{-a}^{a} f(x)\,dx$ for functions $f(x)$ that satisfy $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $x > 0$ and $f''(x) < 0$ for all x values.

Your method must give an asnwer which is LARGER than the actual integral and the user will provide the "step size" Δx.

(a) Sketch the graph of a "typical" function $f(x)$ satisfying the conditions above.

(b) Describe your method for approximating the integral on the figure from part (a) AND in one sentence.

(c) Give a "worst case" bound on the error for your method.



Use the right hand rule on $-a \le x \le 0$

and the left hand rule on $0 \le x \le a$.

The worst case error on $-a \le x < 0$ is $\frac{M_1}{2} \cdot a \cdot \Delta x$

The " " " $0 \le x < a$ is $\frac{M_1}{2} a \cdot \Delta x$

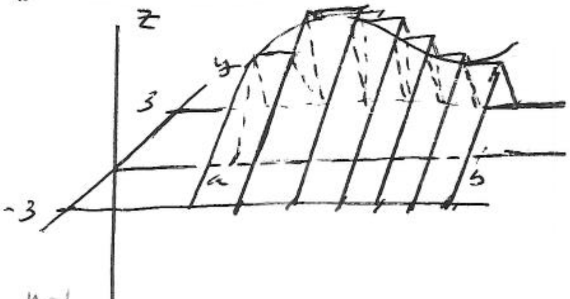
where $M_1 = \max |f'(x)|$ on $-a \le x \le a$

So the total worst case error is

$\underline{\underline{M_1 \cdot a \cdot \Delta x}}$

6. Given a curve in the xz–plane ($y = 0$) given by $z = f(x), a \leq x \leq b$ we make a solid from the curve out of boards of width Δx by placing one end of the board on the lines $y = \pm 3, z = 0$ and leaning the other end on the curve (see figure).

   (a) Give an expression for the volume of the object pictured (you will have to define a bit more notation to give your expression).



Let $x_0 = a$, $x_1 = a + \Delta x$, ...

The volume between x_i and x_{i+1} is

$\frac{1}{2} \cdot (6) \cdot f(x_i) \cdot \Delta x$

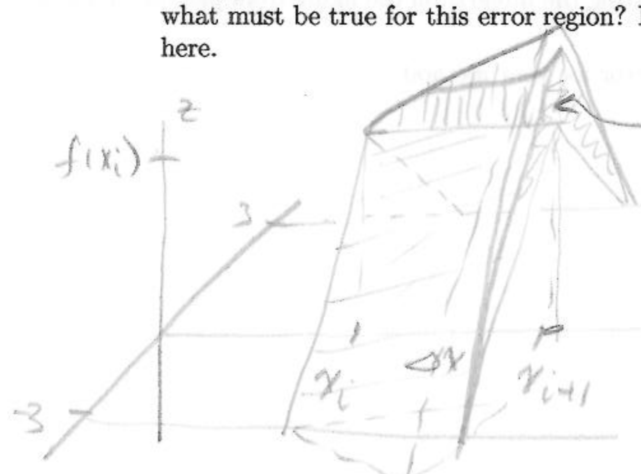
So the total approximate volume is $\sum_{i=0}^{n-1} 3f(x_i)\Delta x$

   (b) What is the formula for the volume solid formed as $\Delta x \to 0$?

As $\Delta x \to 0$ this tends to

$$\int_a^b 3f(x)\,dx$$

   (c) The expression in part (a) is an approximation of the volume of the solid with a smooth surface that is formed when by taking the limit as the board width Δx tends to zero. Draw (as best you can) the region between one piece of "slice" of the smooth surface and the approximation with board width Δx. In order for your limit in part (b) to be correct, what must be true for this error region? Explain in a sentence why that condition holds here.



The error is the volume of shaded region

This error needs to be very small -- so small that even $\frac{b-a}{\Delta x}$ of such errors is still small

In this case the region is trapped in a prism shaped region of volume $\leq 2 \cdot \frac{1}{2} M_1 \Delta x^2 \cdot \sqrt{3^2 + f(x_i)^2}$

The important thing is that this error is order Δx^2!

20.

7. (a) Find the Taylor series for the solution of the differential equation

$$\frac{d^2y}{dx^2} = y \quad y(0) = a_0, y'(0) = a_1.$$

(Hint: Do enough terms of the Taylor polynomial so that you can see the pattern–you do not need to justify the pattern, just state it).

Guess $y(x) = a_0 + a_1 x + \cdots$

$y'(x) = a_1 + 2a_2 x + \cdots$

$y'' = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3 a_4 x^2 + \cdots$

So $2a_2 + 3\cdot 2 a_3 x + 4\cdot 3\, a_4 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$

So
$2\cdot 1\, a_2 = a_0 \Rightarrow a_2 = \frac{a_0}{2\cdot 1}$

$3\cdot 2 a_3 = a_1 \Rightarrow a_3 = \frac{a_1}{3\cdot 2}$

$4\cdot 3\, a_4 = a_2 \Rightarrow a_4 = \frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}$

So $y(x) = a_0 + a_1 x + \frac{a_0}{2!} x^2 + \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$

So the even power coefficients are $\frac{a_0}{n!}$

the odd power coefficients are $\frac{a_1}{n!}$

(b) What are the solutions with $a_0 = 1, a_1 = 1$ and $a_0 = -1, a_1 = 1$?

If $a_1 = a_0 = 1$ then $y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

If $a_0 = -1, a_1 = 1$ then $y(x) = -1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots = -e^{-x}$