From Schaum's Calculus 3rd Ed. Ayres + Mendelson

$$f'(x) = 3x^2 + 6x - 8 \quad \text{and} \quad f'(a) = 3a^2 + 6a - 8$$
$$f''(x) = 6x + 6 \quad \text{and} \quad f''(a) = 6a + 6$$
$$f'''(x) = 6 \quad \text{and} \quad f'''(a) = 6$$

All derivatives of higher order exist and are identically zero.

22. Investigate the successive derivatives of $f(x) = x^{4/3}$ at $x = 0$.

$$f'(x) = \frac{4}{3}x^{1/3} \quad \text{and} \quad f'(0) = 0$$

$$f''(x) = \frac{4}{9x^{2/3}} \quad \text{and} \quad f''(0) \text{ does not exist}$$

Thus the first derivative, but no derivative of higher order, exists at $x = 0$.

23. Given $f(x) = \dfrac{2}{1-x} = 2(1-x)^{-1}$, find $f^{(n)}(x)$.

$$f'(x) = 2(-1)(1-x)^{-2}(-1) = 2(1-x)^{-2} = 2(1!)(1-x)^{-2}$$
$$f''(x) = 2(1!)(-2)(1-x)^{-3}(-1) = 2(2!)(1-x)^{-3}$$
$$f'''(x) = 2(2!)(-3)(1-x)^{-4}(-1) = 2(3!)(1-x)^{-4}$$

which suggest $f^{(n)}(x) = 2(n!)(1-x)^{-(n+1)}$. This result may be established by mathematical induction by showing that if $f^{(k)}(x) = 2(k!)(1-x)^{-(k+1)}$, then

$$f^{(k+1)}(x) = -2(k!)(k+1)(1-x)^{-(k+2)}(-1) = 2[(k+1)!](1-x)^{-(k+2)}$$

Supplementary Problems

24. Establish formula 10 for $m = -1/n$, n a positive integer, by using formula 9 to compute $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right)$. (For the case $m = p/q$, p and q integers, see Problem 4 of Chapter 11.)

In Problems 25 to 43, find the derivative.

25. $y = x^5 + 5x^4 - 10x^2 + 6$ *Ans.* $dy/dx = 5x(x^3 + 4x^2 - 4)$

26. $y = 3x^{1/2} - x^{3/2} + 2x^{-1/2}$ *Ans.* $dy/dx = \dfrac{3}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} - 1/x^{3/2}$

27. $y = \dfrac{1}{2x^2} + \dfrac{4}{\sqrt{x}} = \dfrac{1}{2}x^{-2} + 4x^{-1/2}$ *Ans.* $\dfrac{dy}{dx} = -\dfrac{1}{x^3} - \dfrac{2}{x^{3/2}}$

28. $y = \sqrt{2x} + 2\sqrt{x}$ *Ans.* $y' = (1 + \sqrt{2})/\sqrt{2x}$

29. $f(t) = \dfrac{2}{\sqrt{t}} + \dfrac{6}{\sqrt[3]{t}}$ *Ans.* $f'(t) = -\dfrac{t^{1/2} + 2t^{2/3}}{t^2}$

30. $y = (1 - 5x)^6$ *Ans.* $y' = -30(1 - 5x)^5$

31. $f(x) = (3x - x^3 + 1)^4$ *Ans.* $f'(x) = 12(1 - x^2)(3x - x^3 + 1)^3$

32. $y = (3 + 4x - x^2)^{1/2}$ *Ans.* $y' = (2 - x)/y$

33. $\theta = \dfrac{3r + 2}{2r + 3}$ *Ans.* $\dfrac{d\theta}{dr} = \dfrac{5}{(2r + 3)^2}$

34. $y = \left(\dfrac{x}{1+x}\right)^5$ *Ans.* $y' = \dfrac{5x^4}{(1+x)^6}$

35. $y = 2x^2\sqrt{2-x}$ *Ans.* $y' = \dfrac{x(8-5x)}{\sqrt{2-x}}$

36. $f(x) = x\sqrt{3-2x^2}$ *Ans.* $f'(x) = \dfrac{3-4x^2}{\sqrt{3-2x^2}}$

37. $y = (x-1)\sqrt{x^2-2x+2}$ *Ans.* $\dfrac{dy}{dx} = \dfrac{2x^2-4x+3}{\sqrt{x^2-2x+2}}$

38. $z = \dfrac{w}{\sqrt{1-4w^2}}$ *Ans.* $\dfrac{dz}{dw} = \dfrac{1}{(1-4w^2)^{3/2}}$

39. $y = \sqrt{1+\sqrt{x}}$ *Ans.* $y' = \dfrac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$

40. $f(x) = \sqrt{\dfrac{x-1}{x+1}}$ *Ans.* $f'(x) = \dfrac{1}{(x+1)\sqrt{x^2-1}}$

41. $y = (x^2+3)^4(2x^3-5)^3$ *Ans.* $y' = 2x(x^2+3)^3(2x^3-5)^2(17x^3+27x-20)$

42. $s = \dfrac{t^2+2}{3-t^2}$ *Ans.* $\dfrac{ds}{dt} = \dfrac{10t}{(3-t^2)^2}$

43. $y = \left(\dfrac{x^3-1}{2x^3+1}\right)^4$ *Ans.* $y' = \dfrac{36x^2(x^3-1)^3}{(2x^3+1)^5}$

44. For each of the following, compute dy/dx by two different methods and check that the results are the same: (*a*) $x = (1+2y)^3$, (*b*) $x = 1/(2+y)$.

In Problems 45 to 48, use the chain drule to find dy/dx.

45. $y = \dfrac{u-1}{u+1}$, $u = \sqrt{x}$ *Ans.* $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}$

46. $y = u^3+4$, $u = x^2+2x$ *Ans.* $dy/dx = 6x^2(x+2)^2(x+1)$

47. $y = \sqrt{1+u}$, $u = \sqrt{x}$ *Ans.* See Problem 39.

48. $y = \sqrt{u}$, $u = v(3-2v)$, $v = x^2$ $\left(\textit{Hint: } \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dx}.\right)$ *Ans.* See Problem 36.

In Problems 49 to 52, find the indicated derivative.

49. $y = 3x^4 - 2x^2 + x - 5$; y''' *Ans.* $y''' = 72x$

50. $y = 1/\sqrt{x}$; $y^{(\text{iv})}$ *Ans.* $y^{(\text{iv})} = \dfrac{105}{16x^{9/2}}$

51. $f(x) = \sqrt{2-3x^2}$; $f''(x)$ *Ans.* $f''(x) = -6/(2-3x^2)^{3/2}$

52. $y = x/\sqrt{x-1}$, y'' *Ans.* $y'' = \dfrac{4-x}{4(x-1)^{5/2}}$

In Problems 53 and 54, find the nth derivative.

Supplementary Problems

In Problems 13 to 29 and 32 to 40 evaluate the indefinite integral at left.

13. $\int x \cos x \, dx = x \sin x + \cos x + C$

14. $\int x \sec^2 3x \, dx = \frac{1}{3}x \tan 3x - \frac{1}{9} \ln|\sec 3x| + C$

15. $\int \arccos 2x \, dx = x \arccos 2x - \frac{1}{2}\sqrt{1-4x^2} + C$

16. $\int \arctan x \, dx = x \arctan x - \ln \sqrt{1+x^2} + C$

17. $\int x^2\sqrt{1-x} \, dx = -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C$

18. $\int \frac{xe^x \, dx}{(1+x)^2} = \frac{e^x}{1+x} + C$

19. $\int x \arctan x \, dx = \frac{1}{2}(x^2+1) \arctan x - \frac{1}{2}x + C$

20. $\int x^2 e^{-3x} \, dx = -\frac{1}{3}e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$

21. $\int \sin^3 x \, dx = -\frac{2}{3}\cos^3 x - \sin^2 x \cos x + C$

22. $\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

23. $\int \frac{x \, dx}{\sqrt{a+bx}} = \frac{2(bx-2a)\sqrt{a+bx}}{3b^2} + C$

24. $\int \frac{x^2 \, dx}{\sqrt{1+x}} = \frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x} + C$

25. $\int x \arcsin x^2 \, dx = \frac{1}{2}x^2 \arcsin x^2 + \frac{1}{2}\sqrt{1-x^4} + C$

26. $\int \sin x \sin 3x \, dx = \frac{1}{8}\sin 3x \cos x - \frac{3}{8}\sin x \cos 3x + C$

27. $\int \sin(\ln x) \, dx = \frac{1}{2}x(\sin \ln x - \cos \ln x) + C$

28. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2+b^2} + C$

29. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$

30. (*a*) Write $\int \frac{a^2 \, dx}{(a^2 \pm x^2)^m} = \int \frac{(a^2 \pm x^2) \mp x^2}{(a^2 \pm x^2)^m} dx = \int \frac{dx}{(a^2 \pm x^2)^{m-1}} \mp \int \frac{x^2 \, dx}{(a^2 \pm x^2)^m}$ and use the result of Problem 10(*a*) to obtain (*31.2*).

(*b*) Write $\int (a^2 \pm x^2)^m \, dx = a^2 \int (a^2 \pm x^2)^{m-1} \, dx \pm \int x^2(a^2 \pm x^2)^{m-1} \, dx$ and use the result of Problem 10(*b*) to obtain (*31.3*).

31. Derive reduction formulas (*31.4*) to (*31.11*).

32. $\int \frac{dx}{(1-x^2)^3} = \frac{x(5-3x^2)}{8(1-x^2)^2} + \frac{3}{16} \ln \left|\frac{1+x}{1-x}\right| + C$

Let $u = x^{1/6}$, so that $x = u^6$, $dx = 6u^5\,du$, $x^{1/2} = u^3$, and $x^{1/3} = u^2$. Then we obtain

$$\int \frac{6u^5\,du}{u^3+u^2} = 6\int \frac{u^3}{u+1}\,du = 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)du = 6\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right) + C$$
$$= 2x^{1/2} - 3x^{1/3} + x^{1/6} - \ln|x^{1/6}+1| + C$$

Supplementary Problems

In Problems 14 to 39, evaluate the integral at the left.

14. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx = 2\sqrt{x} - 2\arctan\sqrt{x} + C$

15. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\ln(1+\sqrt{x}) + C$

16. $\displaystyle\int \frac{dx}{3+\sqrt{x+2}} = 2\sqrt{x+2} - 6\ln(3+\sqrt{x+2}) + C$

17. $\displaystyle\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}\,dx = -x + \frac{4}{3}\left\{\sqrt{3x+2} - \ln(1+\sqrt{3x+2})\right\} + C$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-x+1}} = \ln|2\sqrt{x^2-x+1} + 2x - 1| + C$

19. $\displaystyle\int \frac{dx}{x\sqrt{x^2+x-1}} = 2\arctan(\sqrt{x^2+x-1} + x) + C$

20. $\displaystyle\int \frac{dx}{\sqrt{6+x-x^2}} = \arcsin\frac{2x-1}{5} + C$

21. $\displaystyle\int \frac{\sqrt{4x-x^2}}{x^3}\,dx = -\frac{(4x-x^2)^{3/2}}{6x^3} + C$

22. $\displaystyle\int \frac{dx}{(x+1)^{1/2}+(x+1)^{1/4}} = 2(x+1)^{1/2} - 4(x+1)^{1/4} + 4\ln(1+(x+1)^{1/4}) + C$

23. $\displaystyle\int \frac{dx}{2+\sin x} = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{1}{2}x+1}{\sqrt{3}} + C$

24. $\displaystyle\int \frac{dx}{1-2\sin x} = \frac{\sqrt{3}}{3}\ln\left|\frac{\tan\frac{1}{2}x - 2 - \sqrt{3}}{\tan\frac{1}{2}x - 2 + \sqrt{3}}\right| + C$

25. $\displaystyle\int \frac{dx}{3+5\sin x} = \frac{1}{4}\ln\left|\frac{3\tan\frac{1}{2}x+1}{\tan\frac{1}{2}x+3}\right| + C$

26. $\displaystyle\int \frac{dx}{\sin x - \cos x - 1} = \ln|\tan\tfrac{1}{2}x - 1| + C$

27. $\displaystyle\int \frac{dx}{5+3\sin x} = \frac{1}{2}\arctan\frac{5\tan\frac{1}{2}x+3}{4} + C$

28. $\displaystyle\int \frac{\sin x\,dx}{1+\sin^2 x} = \frac{\sqrt{2}}{4}\ln\left|\frac{\tan^2\frac{1}{2}x + 3 - 2\sqrt{2}}{\tan^2\frac{1}{2}x + 3 + 2\sqrt{2}}\right| + C$

29. $\displaystyle\int \frac{dx}{1+\sin x+\cos x} = \ln|1+\tan\tfrac{1}{2}x| + C$

30. $\displaystyle\int \frac{dx}{2-\cos x} = \frac{2}{\sqrt{3}}\arctan(\sqrt{3}\tan\tfrac{1}{2}x) + C$

31. $\displaystyle\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

32. $\displaystyle\int \frac{dx}{x\sqrt{3x^2+2x-1}} = -\arcsin\frac{1-x}{2x} + C$ (*Hint*: Let $x = 1/z$.)

33. $\displaystyle\int \frac{(e^x-2)e^x}{e^x+1}\,dx = e^x - 3\ln(e^x+1) + C$ (*Hint*: Let $e^x + 1 = z$.)