Calculus Schaum's Outline 2nd Ed F. Ayers

14. $\frac{1}{3}+\frac{2!}{3^2}+\frac{3!}{3^3}+\frac{4!}{3^4}+\cdots$ $\qquad s_n=\frac{n!}{3^n},\quad s_{n+1}=\frac{(n+1)!}{3^{n+1}},\quad \frac{s_{n+1}}{s_n}=\frac{n+1}{3}.$

Then $\lim_{n\to+\infty}\frac{s_{n+1}}{s_n}=\lim_{n\to+\infty}\frac{n+1}{3}=\infty$ and the series diverges.

15. $1+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\cdots$

$$s_n=\frac{n!}{1\cdot 3\cdot 5\ldots(2n-1)},\quad s_{n+1}=\frac{(n+1)!}{1\cdot 3\cdot 5\ldots(2n+1)},\quad \frac{s_{n+1}}{s_n}=\frac{n+1}{2n+1}.$$

Then $\lim_{n\to+\infty}\frac{n+1}{2n+1}=\frac{1}{2}$ and the series converges.

16. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$ $\qquad s_n=\frac{1}{n\cdot 2^n},\quad s_{n+1}=\frac{1}{(n+1)2^{n+1}},\quad \frac{s_{n+1}}{s_n}=\frac{n}{2(n+1)}.$

Then $\lim_{n\to+\infty}\frac{n}{2(n+1)}=\frac{1}{2}$ and the series converges.

17. $2+\frac{3}{2}\cdot\frac{1}{4}+\frac{4}{3}\cdot\frac{1}{4^2}+\frac{5}{4}\cdot\frac{1}{4^3}+\cdots$ $\qquad s_n=\frac{n+1}{n}\cdot\frac{1}{4^{n-1}},\quad s_{n+1}=\frac{n+2}{n+1}\cdot\frac{1}{4^n},\quad \frac{s_{n+1}}{s_n}=\frac{n(n+2)}{4(n+1)^2}.$

Then $\lim_{n\to+\infty}\frac{n(n+2)}{4(n+1)^2}=\frac{1}{4}$ and the series converges.

18. $1+\frac{2^2+1}{2^3+1}+\frac{3^2+1}{3^3+1}+\frac{4^2+1}{4^3+1}+\cdots$

$$s_n=\frac{n^2+1}{n^3+1},\quad s_{n+1}=\frac{(n+1)^2+1}{(n+1)^3+1},\quad \frac{s_{n+1}}{s_n}=\frac{(n+1)^2+1}{(n+1)^3+1}\cdot\frac{n^3+1}{n^2+1}.$$

Then $\lim_{n\to+\infty}\frac{s_{n+1}}{s_n}=1$ and the test fails. See Problem 11 above.

Supplementary Problems

19. Verify that the integral test may be applied and use the test to determine convergence or divergence.

(a) $\sum\frac{1}{n}$ (c) $\sum\frac{1}{n\ln n}$ (e) $\sum\frac{n}{n^2+1}$ (g) $\sum\frac{2n}{(n+1)(n+2)(n+3)}$

(b) $\sum\frac{50}{n(n+1)}$ (d) $\sum\frac{n}{(n+1)(n+2)}$ (f) $\sum\frac{n}{e^n}$ (h) $\sum\frac{1}{(2n+1)^2}$

Ans. (a), (c), (d), (e) divergent.

20. Determine convergence or divergence using the comparison test.

(a) $\sum\frac{1}{n^3-1}$ (e) $\sum\frac{n+2}{n(n+1)}$ (i) $\sum\frac{1}{3^n+1}$ (m) $\sum\frac{n}{3n^2-4}$

(b) $\sum\frac{n-2}{n^3}$ (f) $\sum\frac{1}{n^{n-1}}$ (j) $\sum\frac{\ln n}{\sqrt{n}}$ (n) $\sum\frac{1}{1+\ln n}$

(c) $\sum\frac{1}{\sqrt[3]{n}}$ (g) $\sum\frac{1}{3n+1}$ (k) $\sum\frac{1}{3^n-1}$ (o) $\sum\frac{n^4+5}{n^5}$

(d) $\sum\frac{1}{n^2+5}$ (h) $\sum\frac{\ln n}{n}$ (l) $\sum\frac{\ln n}{n^p}$ (p) $\sum\frac{n+1}{n\sqrt{3n-2}}$

Ans. (a), (b), (d), (f), (i), (k), (l) for $p>2$ convergent.

21. Determine convergence or divergence, using the ratio test.

(a) $\sum \frac{(n+1)(n+2)}{n!}$ (d) $\sum \frac{3^{2n-1}}{n^2+n}$ (g) $\sum \frac{n^3}{(\ln 2)^n}$ (j) $\sum \frac{n^n}{n!}$

(b) $\sum \frac{5^n}{n!}$ (e) $\sum \frac{(n+1)2^n}{n!}$ (h) $\sum \frac{n^3}{(\ln 3)^n}$ (k) $\sum \frac{2^n}{2n-1}$

(c) $\sum \frac{n}{2^{2n}}$ (f) $\sum n\left(\frac{3}{4}\right)^n$ (i) $\sum \frac{2^n}{n(n+2)}$ (l) $\sum \frac{n^3}{3^n}$

Ans. (a), (b), (c), (e), (f), (h), (l) convergent.

22. Determine convergence or divergence.

(a) $\frac{1}{4^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{13^2} + \cdots$

(b) $3 + \frac{3}{\sqrt[3]{2}} + \frac{3}{\sqrt[3]{3}} + \frac{3}{\sqrt[3]{4}} + \cdots$

(c) $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots$

(d) $\frac{1}{2} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5\cdot 6} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots$

(e) $3 + \frac{3}{4} + \frac{11}{27} + \frac{9}{32} + \cdots$

(f) $\frac{2}{3} + \frac{3}{2\cdot 3^2} + \frac{4}{3\cdot 3^3} + \frac{5}{4\cdot 3^4} + \cdots$

(g) $\frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \cdots$

(h) $\frac{2}{1\cdot 3} + \frac{3}{2\cdot 4} + \frac{4}{3\cdot 5} + \frac{5}{4\cdot 6} + \cdots$

(i) $\frac{1}{2} + \frac{2}{3^2} + \frac{3}{4^3} + \frac{4}{5^4} + \cdots$

(j) $1 + \frac{1}{2^2} + \frac{1}{3^{5/2}} + \frac{1}{4^3} + \cdots$

(k) $2 + \frac{3}{5} + \frac{4}{10} + \frac{5}{17} + \cdots$

(l) $\frac{2}{5} + \frac{2\cdot 4}{5\cdot 8} + \frac{2\cdot 4\cdot 6}{5\cdot 8\cdot 11} + \frac{2\cdot 4\cdot 6\cdot 8}{5\cdot 8\cdot 11\cdot 14} + \cdots$

Ans. (a), (d), (f), (g), (i), (j), (l) convergent.

23. Prove the comparison test for convergence. *Hint.* If $\Sigma c_n = C$, then $\{S_n\}$ is bounded.

24. Prove the comparison test for divergence. *Hint.* $\sum_1^n s_i \geqq \sum_1^n d_i > M$ for $n > m$.

25. Prove the Polynomial Test: If $P(n)$ and $Q(n)$ are polynomials of degree p and q respectively, the series $\sum \frac{P(n)}{Q(n)}$ converges if $q > p+1$ and diverges if $q \leqq p+1$. *Hint.* Compare with $1/n^{q-p}$.

26. Use the polynomial test to determine convergence or divergence.

(a) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$

(b) $\frac{1}{2} + \frac{1}{7} + \frac{1}{12} + \frac{1}{17} + \cdots$

(c) $\frac{3}{2} + \frac{5}{10} + \frac{7}{30} + \frac{9}{68} + \cdots$

(d) $\frac{3}{2} + \frac{5}{24} + \frac{7}{108} + \frac{9}{320} + \cdots$

(e) $\frac{1}{2^2-1} + \frac{2}{3^2-2} + \frac{3}{4^2-3} + \frac{4}{5^2-4} + \cdots$

(f) $\frac{1}{2^3-1^2} + \frac{1}{3^3-2^2} + \frac{1}{4^3-3^2} + \frac{1}{5^3-4^2} + \cdots$

(g) $\frac{2}{1\cdot 3} + \frac{3}{2\cdot 4} + \frac{4}{3\cdot 5} + \frac{5}{4\cdot 6} + \cdots$

Ans. (a), (c), (d), (f) convergent.

27. Prove the Root Test: A positive series Σs_n converges if $\lim_{n\to+\infty} \sqrt[n]{s_n} < 1$ and diverges if $\lim_{n\to+\infty} \sqrt[n]{s_n} > 1$. The test fails if $\lim_{n\to+\infty} \sqrt[n]{s_n} = 1$. *Hint.* If $\lim_{n\to+\infty} \sqrt[n]{s_n} < 1$, then $\sqrt[n]{s_n} < r < 1$, for $n > m$, and $s_n < r^n$.

28. Determine convergence or divergence, using the Root Test.

(a) $\sum \frac{1}{n^n}$, (b) $\sum \frac{1}{(\ln n)^n}$, (c) $\sum \frac{2^{n-1}}{n^n}$, (d) $\sum \left(\frac{n}{n^2+2}\right)^n$ *Ans.* All convergent.