Now we choose $\alpha, \beta, \gamma, \delta$ so that

$$\begin{cases} a\alpha + b\beta = a^2+b^2 \\ a\gamma + b\delta = 0 \end{cases} \text{ i.e. } \alpha = a, \beta = b, \gamma = b, \delta = -a.$$

So the equation in the new coordinates is

$$(a^2+b^2)\,\tilde{u}_{\tilde{x}} + c\tilde{u} = 0.$$

This says that

$$\tilde{u}_{\tilde{x}} = \frac{-c}{a^2+b^2}\tilde{u}$$

or there is exponential growth in the x direction

or

$$\tilde{u}(\tilde{x}, \tilde{y}) = K e^{\frac{-c}{a^2+b^2}\tilde{x}}$$ where K is a

constant in x, hence can depend on y or

$$\tilde{u}(\tilde{x}, \tilde{y}) = f(\tilde{y})\, e^{\frac{-c}{a^2+b^2}\tilde{x}}.$$

Converting back to original variables

$$u(x,y) = \tilde{u}(ax+by,\ bx-ay) = f(bx-ay)\, e^{\frac{-c}{a^2+b^2}(ax+by)},$$

and this is the solution to the original equation.

(Check $u_x = f'(bx-ay)\cdot b\, e^{\frac{-c}{a^2+b^2}(ax+by)} + f(bx-ay)\frac{-ac}{a^2+b^2} e^{\frac{-c}{a^2+b^2}(ax+by)}$

$$u_y = f'(bx-ay)(-a)\, e^{\frac{-c}{a^2+b^2}(ax+by)} + f(bx-ay)\frac{-bc}{a^2+b^2} e^{\frac{-c}{a^2+b^2}(ax+by)}$$

So

$a u_x + b u_y + cu = 0$ ✓.)