The characteristic polynomial of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$

is $\det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -4 & 0 & 3 \end{bmatrix} = -\lambda^3 + 3\lambda^2 - 4.$

Guessing $\lambda = 1$ is a root, and using $\frac{-\lambda^3+3\lambda^2-4}{\lambda+1} = -\lambda^2+4\lambda-4$

we see that the eigenvalues are $\lambda = -1$, $\lambda = 2$ (double)

The eigenvector for $\lambda = -1$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -u \\ -v \\ -w \end{pmatrix}$$

so $\left.\begin{array}{l} v = -u \\ w = -v \end{array}\right\} \Rightarrow w = u$

$-4u + 3w = -w$ so $-4u + 3u = -u.$ ✓

and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector.

Hence we guess $\vec{u}(x) = k e^{-x} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a soln of the system or

$u(x) = k e^{-x}$ is a soln of the equation (Check it, it works).

For $\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u \\ 2v \\ 2w \end{pmatrix}$$

or $\left.\begin{array}{l} v = 2u \\ w = 2v \end{array}\right\}$ $w = 4u$, $u = \frac{1}{4}w$

$-4u + 3w = 2w.$

So $-4u + 12u = 8u$ ✓ and $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector