Selected Solutions 13 Nov.

6.1 # 29. Area of the "lune"

Place center of axes at center of the small circle. So the formula of the small circle is

$$x^2+y^2=r^2$$

and the top of the small circle is

$$y=\sqrt{r^2-x^2}.$$

The center of the large circle is at $(0,-\sqrt{R^2-r^2})$

So the equation of the large circle is

$$x^2+(y+\sqrt{R^2-r^2})^2=R^2$$

or

$$x^2+y^2+2y\sqrt{R^2-r^2}+(R^2-r^2)=R^2$$

or

$$y^2+2y\sqrt{R^2-r^2}+(x^2-r^2)=0.$$

Solving for y, we have

$$y=\frac{-2\sqrt{R^2-r^2}\pm\sqrt{4(R^2-r^2)-4(x^2-r^2)}}{2}$$

or $y=-\sqrt{R^2-r^2}\pm\sqrt{R^2-x^2}.$

The bottom of the lune is the top of the large circle, so

$$y=\sqrt{R^2-x^2}-\sqrt{R^2-r^2}.$$

Hence, the region is defined by $-r\le x\le r$

$$\sqrt{R^2-x^2}-\sqrt{R^2-r^2}\le y\le\sqrt{r^2-x^2}$$

So the integral is

$$\int_{-r}^{r} \sqrt{r^2-x^2} - \left(\sqrt{R^2-x^2} - \sqrt{R^2-r^2}\right) dx$$

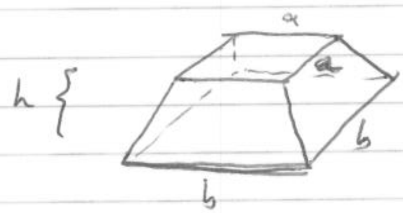
or

$$2\int_{0}^{r} \left(\sqrt{r^2-x^2} - \sqrt{R^2-x^2} + \sqrt{R^2-r^2}\right) dx.$$

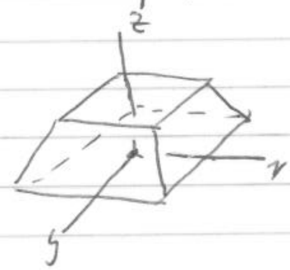
(Remember $\int \sqrt{r^2-x^2}\,dx = \frac{1}{2}x\sqrt{r^2-x^2} + \frac{1}{2}\,\text{Arctan}\left[\frac{x}{\sqrt{r^2-x^2}}\right]$.).

...

6.2 -
28. Volume of the frustrum of a pyramid pictured below.

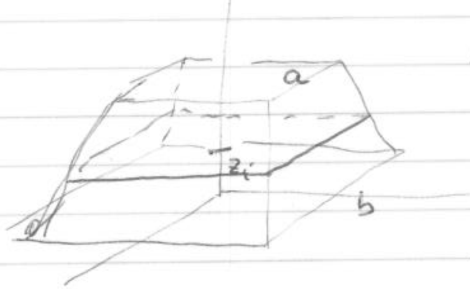


If we put coordinates on this with the z-axis up the middle



Then we can "slice" perpendicular to the z axis.

The width of the slice at $z = z_i$ varies linearly with z, the width is b at $z=0$ and a at $z=h$.



so width $= \frac{h-z}{h}\cdot b + \frac{z}{h}\cdot a$
(when $z=0$, width $= b$
when $z=h$, width $= a$ and this is linear)

So width $= (1-\frac{z}{h})b + \frac{z}{h}a$

$$\text{width} = b + \frac{(a-b)}{h}z$$

So the area of the slice at z_i is $\left(b + \frac{(a-b)}{h}z_i\right)^2$

So the volume of the piece from z_i to z_{i+1} is $\Delta z\left(b + \frac{(a-b)}{h}z_i\right)^2$

So the total volume is approximately

$$\sum_{i=0}^{n-1} \left(b + \frac{a-b}{h} z_i\right)^2 \Delta z$$

so the exact volume is

$$\int_0^h \left(b + \frac{a-b}{h} z\right)^2 dz$$

$$= \int_0^h \left(b^2 + 2b\frac{a-b}{h} z + \frac{(a-b)^2}{h^2} z^2\right) dz$$

$$= b^2 z + 2b\frac{a-b}{h}\frac{z^2}{2} + \frac{(a-b)^2}{h^2}\frac{z^3}{3}\Big|_0^h$$

$$= b^2 h + 2b\frac{a-b}{h}\cdot\frac{h^2}{2} + \frac{(a-b)^2}{h^2}\frac{h^3}{3}$$

$$= b^2 h + b(a-b)\cdot h + \tfrac{1}{3}(a^2 - 2ab + b^2)h$$

$$= b^2 h + abh - b^2 h + \tfrac{1}{3}a^2 h - \tfrac{2}{3}abh + \tfrac{1}{3}b^2 h$$

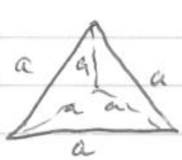
$$= \tfrac{1}{3}abh + \tfrac{1}{3}a^2 h + \tfrac{1}{3}b^2 h.$$

If $a = b$ then this is $a^2 h$ as expected

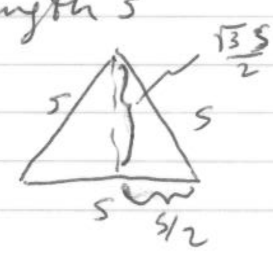
If $a = 0$ then this is $\frac{1}{3}b^2 h$ as expected.

6.2)

\#30 Volume of the equilateral tetrahedron



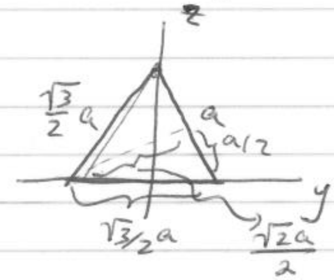
First the area of a tetrahedron of side length s



$\text{Area} = \frac{1}{2}s\cdot\frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2.$

Next put coordinates on the tetrahedron

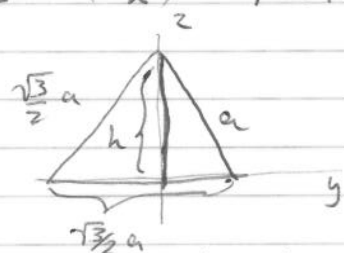
To determine the height h we use the fact that the ~~[illegible]~~ ~~the~~ Triangle in the y, z plane is



i.e. isosceles with sides $\frac{\sqrt{3}}{2}a$, $\frac{\sqrt{3}}{2}a$ and a.

The area of this triangle (using the side of length a as the base) is

$$\frac{1}{2}\, a \cdot \left(\frac{\sqrt{2}\,a}{2}\right) = \frac{\sqrt{2}}{4}a^2.$$



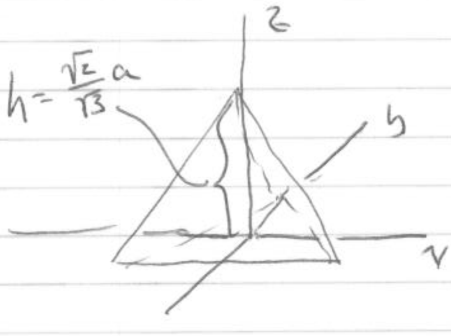
hence the height h must satisfy

$$\frac{1}{2}\,\frac{\sqrt{3}}{2}a \cdot h = \frac{\sqrt{2}}{4}a^2$$

$$\frac{\sqrt{3}}{4}a \cdot h = \frac{\sqrt{2}}{4}a^2$$

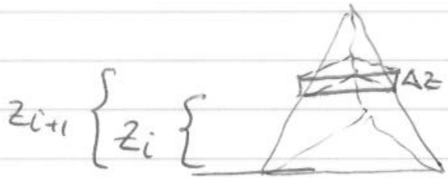
or

$$h = \frac{\sqrt{2}}{\sqrt{3}}a.$$



Now we follow the same procedure as before -

Slice $\perp$ to the z axis...



Each slice has height Δz

to find the cross sectional area, we need to find the length of the side on the $z = z_i$ cross section:

We know when $z = 0$, the side length is a
when $z = h = \frac{\sqrt{2}}{\sqrt{3}} a$ the side length is 0

and the side length changes linearly

So the side length $= a - \frac{z}{h} \cdot a$ (i.e. $= a$ when $z = 0$, $= 0$ when $z = h$, is linear.)

Hence the volume of the slice from i to $i+1$

is $\frac{\sqrt{3}}{4}\left(a - \frac{z_i}{h} a\right)^2 \cdot \Delta z$ So the total volume is approximately

$$\sum_{i=0}^{n-1} \frac{\sqrt{3}}{4}\left(a - \frac{z_i}{h} a\right)^2 \Delta z.$$

As $\Delta z \to 0$ this becomes $\displaystyle\int_0^h \frac{\sqrt{3}}{4}\left(a - \frac{z}{h} a\right)^2 dz$

$$= \frac{\sqrt{3}}{4} \int_0^h a^2 \left(1 - \frac{z}{h}\right)^2 dz = \frac{\sqrt{3}}{4} a^2 \int_0^h \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) dz$$

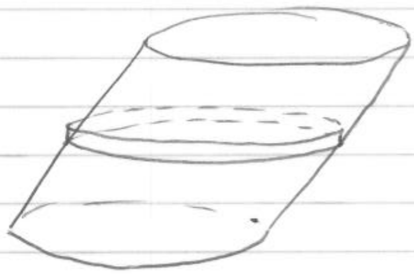
$$= \frac{\sqrt{3}}{4} a^2 \left(z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right)\Big|_0^h = \frac{\sqrt{3}}{4} a^2 \left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{\sqrt{3}}{4} a^2 \cdot \frac{h}{3}$$

With $h = \frac{\sqrt{2}}{\sqrt{3}}a$ we have

$$\text{Volume} = \frac{\frac{\sqrt{3}}{4}a^2 \frac{\sqrt{2}}{\sqrt{3}}a}{3} = \frac{\sqrt{2}}{12}a^3.$$

6.2

#41. To compute the volume of the slanted cylinder

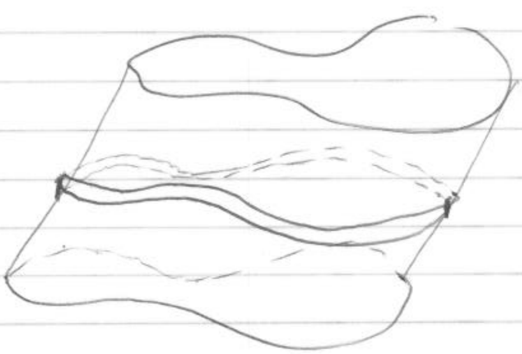


We use approximating disks (small cylinders), as shown

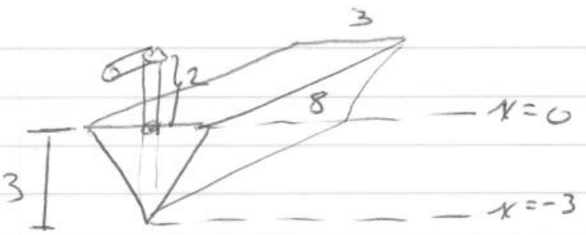
But the sum of these cylinders is always the same as the volume of the straight cylinder.

Hence the slanted and straight objects have the same volume.

The same principle works with any shape cross section.

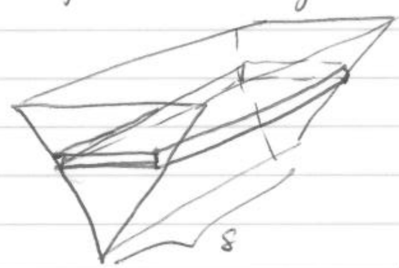


6.5 # 17:



a) Because of part b, we use $x=0$ as the top of ~~tank~~ water in the tank so the tank extends from $x=0$ to $x=-3$

As usual, we slice the tank, here $\perp$ to the x direction, into piece height Δx



The piece from x_i to x_{i+1} has height Δx length 8

To get the width, we note that $x=-3$ width $=0$

$x=0$ width $=3$

and width is linear in x, so

width $= x+3$

So the volume of the slice from x_i to x_{i+1} is

$$\Delta x \cdot 8 \cdot (x+3).$$

The mass is $\Delta x \cdot 8(x+3) \cdot 1000$

The distance this water is lifted is $(-x_i)$ (recall $x_i<0$ in tank) to get to the top of the tank, then $+2$ to get out the pipe, so lifted $-x_i+2$.

So the work removing the water from x_i to x_{i+1} is

$$\Delta x \cdot 8\,(x_i+3) \cdot 1000\,(2-x_i) \cdot 9.8.$$

$$=(8000 \times 9.8)(-x_i^2 - x_i + 6)\,\Delta x$$

Hence the total work is $\approx \sum_{i=0}^{n-1} 8000 \times 9.8\,(-x_i^2 - x_i + 6)\,\Delta x$

Turning this into an integral (take $\Delta x \to 0$)

gives $\int_{-3}^{0} 8000 \times 9.8 (-x^2 - x + 6) \, dx.$

...

b) To do part b, we solve

$$4.7 \times 10^5 = \int_h^0 8000 \times 9.8 \left(-x^2 - x + 6\right) dx$$

for h.