The main conjecture for CM elliptic curves at supersingular primes

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Abstract

At a prime of ordinary reduction, the Iwasawa "main conjecture" for elliptic curves relates a Selmer group to a p-adic L-function. In the supersingular case, the statement of the main conjecture is more complicated as neither the Selmer group nor the p-adic L-function is well-behaved. Recently Kobayashi discovered an equivalent formulation of the main conjecture at supersingular primes that is similar in structure to the ordinary case. Namely, Kobayashi's conjecture relates modified Selmer groups, which he defined, with modified p-adic L-functions defined by the first author. In this paper we prove Kobayashi's conjecture for elliptic curves with complex multiplication.

Introduction

Iwasawa theory was introduced into the study of the arithmetic of elliptic curves by Mazur in the 1970's. Given an elliptic curve E over $\mathbf{Q}$ and a prime p there are two parts to such a program: an Iwasawa-Selmer module containing information about the arithmetic of E over subfields of the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty$ of $\mathbf{Q}$, and a p-adic L-function attached to E, belonging to a suitable Iwasawa algebra. The goal, or "main conjecture", is to relate these two objects by proving that the p-adic L-function controls (in precise terms, is a characteristic power series of the Pontrjagin dual of) the Iwasawa-Selmer module. The main conjecture has important consequences for the Birch and Swinnerton-Dyer conjecture for E.

For primes p where E has ordinary reduction,

- Mazur introduced and studied the Iwasawa-Selmer module [Ma],

- Mazur and Swinnerton-Dyer constructed the p-adic L-function [MSD],

- the main conjecture was proved by the second author for elliptic curves with complex multiplication [Ru3],

- Kato proved that the characteristic power series of the Pontrjagin dual of the Iwasawa-Selmer module divides the p-adic L-function [Ka].

The latter two results are proved using Kolyvagin's Euler system machinery.

For primes p where E has supersingular reduction, progress has been much slower. Using the same definitions as for the ordinary case gives a Selmer module that is not a torsion Iwasawa module [Ru1], and a p-adic L-function that does not belong to the Iwasawa algebra [MTT], [AV]. Perrin-Riou and Kato made important progress in understanding the case of supersingular primes, and independently proposed a main conjecture [PR3], [Ka].

More recently, the first author [Po] proved that when p is a prime of supersingular reduction (and either $p > 3$ or $a_p = 0$) the "classical" p-adic L-function corresponds in a precise way to *two* elements $\mathcal{L}_E^+, \mathcal{L}_E^-$ of the Iwasawa algebra. Shortly thereafter Kobayashi [Ko] defined two submodules $\mathrm{Sel}_p^+(E/\mathbf{Q}_\infty)$, $\mathrm{Sel}_p^-(E/\mathbf{Q}_\infty)$ of the "classical" Selmer module, and proposed a main conjecture: that $\mathcal{L}_E^\pm$ is a characteristic power series of the Pontrjagin dual of $\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty)$. Kobayashi proved that this conjecture is equivalent to the Kato and Perrin-Riou conjecture, and (as an application of Kato's results [Ka]) that the characteristic power series of the Pontrjagin dual of $\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty)$ divides $\mathcal{L}_E^\pm$.

The purpose of the present paper is to prove Kobayashi's main conjecture when the elliptic curve E has complex multiplication:

THEOREM. *If E is an elliptic curve over $\mathbf{Q}$ with complex multiplication, and $p > 2$ is a prime where E has good supersingular reduction, then $\mathcal{L}_E^\pm$ is a characteristic power series of the Iwasawa module* $\mathrm{Hom}(\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p)$.

See Definition 3.3 for the definition of Kobayashi's Selmer groups $\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty)$, and Section 7 for the definition of $\mathcal{L}_E^\pm$. With the same proof (and a little extra notation) one can prove an analogous result for $\mathrm{Sel}_p^\pm(E/\mathbf{Q}(\boldsymbol{\mu}_{p^\infty}))$, the Selmer groups over the full p-cyclotomic field $\mathbf{Q}(\boldsymbol{\mu}_{p^\infty})$.

The proof relies on the Euler system of elliptic units, and the results and methods of [Ru3] which also went into the proof of the main conjecture for CM elliptic curves at ordinary primes. We sketch the ideas briefly here, but we defer the precise definitions, statements, and references to the main text below.

Fix an elliptic curve E defined over $\mathbf{Q}$ with complex multiplication by an imaginary quadratic field K, and a prime $p > 2$ where E has good reduction (ordinary or supersingular, for the moment). Let $\mathfrak{p}$ be a prime of K above p, and let $\mathcal{K} = K(E[p^\infty])$, the (abelian) extension of K generated by all p-power torsion points on E. Class field theory gives an exact sequence

(1) $$0 \longrightarrow \mathcal{E}/\mathcal{C} \longrightarrow \mathcal{U}/\mathcal{C} \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \longrightarrow 0$$

where $\mathcal{U}$, $\mathcal{E}$, and $\mathcal{C}$ are the inverse limits of the local units, global units, and elliptic units, respectively, up the tower of abelian extensions $K(E[p^n])$ of K, and $\mathcal{X}$ (resp. $\mathcal{A}$) is the Galois group over $K(E[p^\infty])$ of the maximal unramified outside $\mathfrak{p}$ (resp. everywhere unramified) abelian p-extension of $K(E[p^\infty])$. Further

(a) the classical Selmer group $\mathrm{Sel}_{\mathfrak{p}}(E/\mathcal{K}) = \mathrm{Hom}(\mathcal{X}, E[\mathfrak{p}^\infty])$,

(b) the "Coates-Wiles logarithmic derivatives" of the elliptic units are special values of Hecke L-functions attached to E,

(c) the Euler system of elliptic units can be used to show that the (torsion) Iwasawa modules $\mathcal{E}/\mathcal{C}$ and $\mathcal{A}$ have the same characteristic ideal.

If E has ordinary reduction at p, then $\mathcal{U}/\mathcal{C}$ and $\mathcal{X}$ are torsion Iwasawa modules. It then follows from (1) and (c) that $\mathcal{U}/\mathcal{C}$ and $\mathcal{X}$ have the same characteristic ideal, and from (b) that the characteristic ideal of $\mathcal{U}/\mathcal{C}$ is a ("two-variable") p-adic L-function. Now using (a) and restricting to $\mathbf{Q}_\infty \subset \mathcal{K}$ one can prove the main conjecture in this case.

When E has supersingular reduction at p, the Iwasawa modules $\mathcal{U}/\mathcal{C}$ and $\mathcal{X}$ are *not* torsion (they have rank one), so the argument above breaks down. However, Kobayashi's construction suggests a way to remedy this. Namely, one can define submodules $\mathcal{V}^+, \mathcal{V}^- \subset \mathcal{U}$ such that in the exact sequence induced from (1)

$$0 \longrightarrow \mathcal{E}/\mathcal{C} \longrightarrow \mathcal{U}/(\mathcal{C} + \mathcal{V}^\pm) \longrightarrow \mathcal{X}/\mathrm{image}(\mathcal{V}^\pm) \longrightarrow \mathcal{A} \longrightarrow 0$$

we have torsion modules $\mathcal{U}/(\mathcal{C} + \mathcal{V}^\pm)$ and $\mathcal{X}/\mathrm{image}(\mathcal{V}^\pm)$, and the Kobayashi Selmer groups satisfy

(a′) $\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty) = \mathrm{Hom}(\mathcal{X}/\mathrm{image}(\mathcal{V}^\pm), E[p^\infty])^{G_{\mathbf{Q}_\infty}}$.

Using (b) (to relate $\mathcal{U}/(\mathcal{C} + \mathcal{V}^\pm)$ with $\mathcal{L}_E^\pm$) and (c) as above this will enable us to prove the main conjecture in this case as well.

The layout of the paper is as follows. The general setting and notation are laid out in Section 1. Sections 2 and 3 describe the classical and Kobayashi Selmer groups, and Sections 4 and 5 relate Kobayashi's construction to local units, elliptic units, and L-values. Section 6 applies the results of [Ru3] to our situation. The proof of the main theorem (restated as Theorem 7.3 below) is given in Section 7, and in Section 8 we give some arithmetic applications.

1. The setup

Throughout this paper we fix an elliptic curve E defined over $\mathbf{Q}$, with complex multiplication by the ring of integers $\mathcal{O}$ of an imaginary quadratic field K. (No generality is lost by assuming that $\mathrm{End}(E)$ is the maximal order in K, since we could always replace E by an isogenous curve with this property.) Fix also a rational prime $p > 2$ where E has good supersingular reduction. As is well known, it follows that p remains prime in K. It also follows that $a_p = p + 1 - |E(\mathbf{F}_p)| = 0$, so we can apply the results of the first author [Po] and Kobayashi [Ko]. Let K_p and $\mathcal{O}_p$ denote the completions of K and $\mathcal{O}$ at p.

For every k let $E[p^k]$ denote kernel of p^k in $E(\bar{\mathbf{Q}})$, $E[p^\infty] = \cup_k E[p^k]$, and $T_p(E) = \varprojlim E[p^k]$. Let $\mathcal{K} = K(E[p^\infty])$, let K_∞ denote the (unique) $\mathbf{Z}_p^2$-extension of K, let $\mathbf{Q}_\infty \subset K_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and let $K_{\mathrm{cyc}} = K\mathbf{Q}_\infty \subset K_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of K. Let ρ denote the character

$$\rho : G_K \longrightarrow \mathrm{Aut}_{\mathcal{O}_p}(E[p^\infty]) \cong \mathcal{O}_p^\times.$$

Let $\hat{E}$ denote the formal group giving the kernel of reduction modulo p on E. The theory of complex multiplication shows that $\hat{E}$ is a Lubin-Tate formal group of height two over $\mathcal{O}_p$ for the uniformizing parameter $-p$. It follows that ρ is surjective, even when restricted to an inertia group of p in G_K. Therefore p is totally ramified in $\mathcal{K}/K$ and ρ induces an isomorphism $\mathrm{Gal}(\mathcal{K}/K) \cong \mathcal{O}_p^\times$. We can decompose

$$\mathrm{Gal}(\mathcal{K}/K) = \Delta \times \Gamma_+ \times \Gamma_-$$

where $\Delta = \mathrm{Gal}(\mathcal{K}/K_\infty) \cong \mathrm{Gal}(K(E[p])/K)$ is the non-p part of $\mathrm{Gal}(\mathcal{K}/K)$, which is cyclic of order $p^2 - 1$, and $\Gamma_\pm$ is the largest subgroup of $\mathrm{Gal}(\mathcal{K}/K(E[p]))$ on which the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ acts by ± 1. Then Γ_+ and Γ_- are both free of rank one over $\mathbf{Z}_p$.

Let M (resp. L) denote the maximal abelian p-extension of $K(E[p^\infty])$ that is unramified outside of the unique prime above p (resp. unramified everywhere), and let $\mathcal{X} = \mathrm{Gal}(M/\mathcal{K})$ and $\mathcal{A} = \mathrm{Gal}(L/\mathcal{K})$. If F is a finite extension of K in $\mathcal{K}$ let $\mathcal{O}_F$ denote the ring of integers of F, and define subgroups $C_F \subset E_F \subset U_F \subset (\mathcal{O}_F \otimes \mathbf{Z}_p)^\times$ as follows. The group U_F is the pro-p-part of the local unit group $(\mathcal{O}_F \otimes \mathbf{Z}_p)^\times$, E_F is the closure of the projection of the global units $\mathcal{O}_F^\times$ into U_F, and C_F is the closure of the projection of the subgroup of elliptic units (as defined for example in §1 of [Ru3]) into U_F. Finally, define

$$\mathcal{C} = \varprojlim C_F \subset \mathcal{E} = \varprojlim E_F \subset \mathcal{U} = \varprojlim U_F,$$

inverse limit with respect to the norm map over finite extensions of K in $\mathcal{K}$.

Class field theory gives an isomorphism $\operatorname{Gal}(M/L) \cong \mathcal{U}/\mathcal{E}$. We summarize this setting in Figure 1 below.
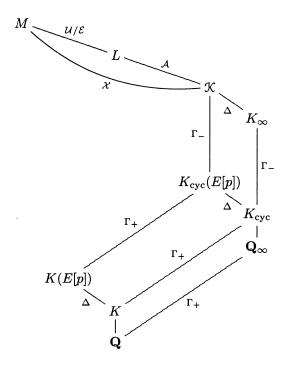

Figure 1.

If $K \subset F \subset \mathcal{K}$ we define the Iwasawa algebra $\Lambda(F) = \mathbf{Z}_p[[\operatorname{Gal}(F/K)]]$. In particular we have

$$
\begin{aligned}
\Lambda(\mathcal{K}) &= \mathbf{Z}_p[[\operatorname{Gal}(\mathcal{K}/K)]] = \mathbf{Z}_p[[\Delta \times \Gamma_+ \times \Gamma_-]], \\
\Lambda(K_\infty) &= \mathbf{Z}_p[[\operatorname{Gal}(K_\infty/K)]] = \mathbf{Z}_p[[\Gamma_+ \times \Gamma_-]], \\
\Lambda(K_{\mathrm{cyc}}) &= \mathbf{Z}_p[[\operatorname{Gal}(K_{\mathrm{cyc}}/K)]] \cong \mathbf{Z}_p[[\Gamma_+]] \cong \mathbf{Z}_p[[\operatorname{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]].
\end{aligned}
$$

We write simply Λ for $\Lambda(K_{\mathrm{cyc}})$, and we write $\Lambda_{\mathcal{O}}(F) = \Lambda(F) \otimes \mathcal{O}_p$ and $\Lambda_{\mathcal{O}} = \Lambda \otimes \mathcal{O}_p$.

Definition 1.1.   Suppose Y is a $\Lambda(\mathcal{K})$-module. We define the twist

$$
Y(\rho^{-1}) = Y \otimes \operatorname{Hom}_{\mathcal{O}}(E[p^\infty], K_p/\mathcal{O}_p).
$$

The module $\operatorname{Hom}_{\mathcal{O}}(E[p^\infty], K_p/\mathcal{O}_p)$ is free of rank one over $\mathcal{O}_p$, and G_K acts on it via ρ^{-1}. Thus we have $T_p(E)(\rho^{-1}) \cong \mathcal{O}_p$ and $E[p^\infty](\rho^{-1}) \cong K_p/\mathcal{O}_p$.

If $K \subset F \subset \mathcal{K}$ we define

$$
Y_F^\rho = Y(\rho^{-1}) \otimes_{\Lambda(\mathcal{K})} \Lambda(F) = Y(\rho^{-1})/\langle \gamma - 1 : \gamma \in \operatorname{Gal}(\mathcal{K}/F)\rangle,
$$

the F-coinvariants of $Y(\rho^{-1})$. We will be interested in $Y^\rho_{K_\infty}$ and $Y^\rho_{K_{\mathrm{cyc}}}$. Concretely, if we write Z for the $\Lambda_{\mathcal{O}}(K_\infty)$-submodule of $Y \otimes \mathcal{O}_p$ on which Δ acts via ρ, then $Y^\rho_{K_\infty}$ can be identified with $Z(\rho^{-1})$ and $Y^\rho_{K_{\mathrm{cyc}}}$ can be identified with $(Z/(\gamma_* - \rho(\gamma_*))Z)(\rho^{-1})$ where γ_* is a topological generator of Γ_-.

2. The classical Selmer group

For every number field F we have the classical p-power Selmer group $\mathrm{Sel}_p(E/F) \subset H^1(F, E[p^\infty])$, which sits in an exact sequence

$$0 \longrightarrow E(F) \otimes (\mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow \mathrm{Sel}_p(E/F) \longrightarrow \text{Ш}(E/F)[p^\infty] \longrightarrow 0$$

where $\text{Ш}(E/F)[p^\infty]$ is the p-part of the Tate-Shafarevich group of E over F. Taking direct limits allows us to define $\mathrm{Sel}_p(E/F)$ for every algebraic extension F of $\mathbf{Q}$.

THEOREM 2.1. $\mathrm{Sel}_p(E/K_{\mathrm{cyc}}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}^\rho_{K_{\mathrm{cyc}}}, K_p/\mathcal{O}_p)$.

Proof. Combining Theorem 2.1, Proposition 1.1, and Proposition 1.2 of [Ru1] shows that

$$\begin{aligned}
\mathrm{Sel}_p(E/K_{\mathrm{cyc}}) &\cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}, E[p^\infty])^{\mathrm{Gal}(\mathcal{K}/K_{\mathrm{cyc}})} \\
&= \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}(\rho^{-1}), K_p/\mathcal{O}_p)^{\mathrm{Gal}(\mathcal{K}/K_{\mathrm{cyc}})} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}^\rho_{K_{\mathrm{cyc}}}, K_p/\mathcal{O}_p).
\end{aligned}$$

$\square$

Remark 2.2.   We have $\mathrm{rank}_{\Lambda_{\mathcal{O}}(K_\infty)} \mathcal{X}^\rho_{K_\infty} = 1$ (see for example [Ru3, Th. 5.3(iii)]), so $\mathrm{rank}_{\Lambda_{\mathcal{O}}} \mathcal{X}^\rho_{K_{\mathrm{cyc}}} \geq 1$. Thus, unlike the case of ordinary primes, the Selmer group $\mathrm{Sel}_p(E/K_{\mathrm{cyc}})$ is not a co-torsion $\Lambda_{\mathcal{O}}$-module. This makes the Iwasawa theory for supersingular primes more difficult than the ordinary case. In the next section, following Kobayashi [Ko], we will remedy this by defining two smaller Selmer groups which will both be co-torsion $\Lambda_{\mathcal{O}}$-modules.

3. Kobayashi's restricted Selmer groups

If F is a finite extension of K in $\mathcal{K}$ let F_p denote the completion of F at the unique prime above p, and for an arbitrary F with $K \subset F \subset \mathcal{K}$ let $F_p = \cup_N N_p$, union over finite extensions of K in F. For every such F let $\mathfrak{m}_F$ denote the maximal ideal of F_p and let $E_1(F_p) \subset E(F_p)$ be the kernel of reduction. Then $E_1(F_p)$ is the pro-p part of $E(F_p)$ and we define the logarithm map λ_E to be the composition

$$\lambda_E : E(F_p) \twoheadrightarrow E_1(F_p) \xrightarrow{\sim} \hat{E}(\mathfrak{m}_F) \longrightarrow F_p$$

where the first map is projection onto the pro-p part, the second is the canonical isomorphism between the kernel of reduction and the formal group $\hat{E}$, and the third is the formal group logarithm map.

Definition 3.1. For $n \geq 0$ let $\mathbf{Q}_n$ denote the extension of $\mathbf{Q}$ of degree p^n in $\mathbf{Q}_\infty$, and if $n \geq m$ let $\mathbf{Tr}_{n/m}$ denote the trace map from $E(\mathbf{Q}_{n,p})$ to $E(\mathbf{Q}_{m,p})$. For each n define two subgroups $E^+(\mathbf{Q}_{n,p}), E^-(\mathbf{Q}_{n,p}) \subset E(\mathbf{Q}_{n,p})$ by

$$E^+(\mathbf{Q}_{n,p}) = \{x \in E(\mathbf{Q}_{n,p}) : \mathbf{Tr}_{n/m}x \in E(\mathbf{Q}_{m-1,p}) \text{ if } 0 < m \leq n, m \text{ odd}\}$$
$$E^-(\mathbf{Q}_{n,p}) = \{x \in E(\mathbf{Q}_{n,p}) : \mathbf{Tr}_{n/m}x \in E(\mathbf{Q}_{m-1,p}) \text{ if } 0 < m \leq n, m \text{ even}\}$$

and let $E_1^\pm(\mathbf{Q}_{n,p}) = E^\pm(\mathbf{Q}_{n,p}) \cap E_1(\mathbf{Q}_{n,p})$. Equivalently, let Ξ_n^+ (resp. Ξ_n^-) denote the set of nontrivial characters $\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q}) \to \boldsymbol{\mu}_{p^n}$ whose order is an odd (resp. even) power of p, and then

$$E^\pm(\mathbf{Q}_{n,p}) = \{x \in E(\mathbf{Q}_{n,p}) : \textstyle\sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} \chi(\sigma)x^\sigma = 0 \text{ for every } \chi \in \Xi_n^\pm\}$$

where the sum takes place in $E(\mathbf{Q}_{n,p}) \otimes \mathbf{Z}[\boldsymbol{\mu}_{p^n}]$. Note that when $n = 1$ we get $E^+(\mathbf{Q}_p) = E^-(\mathbf{Q}_p) = E(\mathbf{Q}_p)$. When $n = \infty$ we define

$$E^\pm(\mathbf{Q}_{\infty,p}) = \cup_n E^\pm(\mathbf{Q}_{n,p}).$$

We also define $E^\pm(K\mathbf{Q}_{n,p})$ exactly as above with $\mathbf{Q}_n$ replaced by $K\mathbf{Q}_n$. The complex multiplication map $E(\mathbf{Q}_{n,p}) \otimes \mathcal{O}_p \to E(K\mathbf{Q}_{n,p})$ induces isomorphisms

$$(2) \quad E_1(\mathbf{Q}_{n,p}) \otimes \mathcal{O}_p \xrightarrow{\sim} E_1(K\mathbf{Q}_{n,p}), \quad E_1^\pm(\mathbf{Q}_{n,p}) \otimes \mathcal{O}_p \xrightarrow{\sim} E_1^\pm(K\mathbf{Q}_{n,p})$$

for every $n \leq \infty$.

Fix once and for all a generator $\{\zeta_{p^n}\}$ of $\mathbf{Z}_p(1)$, so ζ_{p^n} is a primitive p^n-th root of unity and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. If $\chi : \Gamma_+ \twoheadrightarrow \boldsymbol{\mu}_{p^k}$ define the Gauss sum $\tau(\chi) = \sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{p^k})/\mathbf{Q})} \chi(\sigma)\zeta_{p^k}^\sigma$.

THEOREM 3.2 (Kobayashi [Ko]).

(i) $E^+(\mathbf{Q}_{n,p}) + E^-(\mathbf{Q}_{n,p}) = E(\mathbf{Q}_{n,p})$.

(ii) $E^+(\mathbf{Q}_{n,p}) \cap E^-(\mathbf{Q}_{n,p}) = E(\mathbf{Q}_p)$.

Further, there is a sequence of points $d_n \in E_1(\mathbf{Q}_{n,p})$ (depending on the choice of $\{\zeta_{p^n}\}$ above) with the following properties.

(iii) $\mathbf{Tr}_{n/n-1}d_n = \begin{cases} d_{n-2} & \text{if } n \geq 2, \\ \frac{1-p}{2}d_0 & \text{if } n = 1. \end{cases}$

(iv) *If $\chi : \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q}) \xrightarrow{\sim} \boldsymbol{\mu}_{p^n}$ then*

$$\sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} \chi(\sigma)\lambda_E(d_n^\sigma) = \begin{cases} (-1)^{[\frac{n}{2}]}\tau(\chi) & \text{if } n > 0, \\ \frac{p}{p+1} & \text{if } n = 0. \end{cases}$$

(v) *If $\varepsilon = (-1)^n$ then*

$$E_1^\varepsilon(\mathbf{Q}_{n,p}) = \mathbf{Z}_p[\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})]d_n \quad and \quad E_1^{-\varepsilon}(\mathbf{Q}_{n,p}) = \mathbf{Z}_p[\mathrm{Gal}(\mathbf{Q}_{n-1}/\mathbf{Q})]d_{n-1}.$$

Proof. The first two assertions are Proposition 8.12(ii) of [Ko].

Let $d_n = (-1)^{[\frac{n+1}{2}]}\mathbf{Tr}_{\mathbf{Q}(\boldsymbol{\mu}_{p^{n+1}})/\mathbf{Q}_n}c'_{n+1}$ where $c'_{n+1} \in E_1(\mathbf{Q}(\boldsymbol{\mu}_{p^{n+1}})_p)$ corresponds to the point $c_{n+1} \in \hat{E}(\mathbf{Q}(\boldsymbol{\mu}_{p^{n+1}})_p)$ defined by Kobayashi in Section 4 of [Ko]. Then the last three assertions of the theorem follow from Lemma 8.9, Proposition 8.26, and Proposition 8.12(i), respectively, of [Ko]. $\qquad\square$

Definition 3.3. If $0 \leq n \leq \infty$ we define Kobayashi's restricted Selmer groups $\mathrm{Sel}_p^\pm(E/\mathbf{Q}_n) \subset \mathrm{Sel}_p(E/\mathbf{Q}_n)$ by

$$\mathrm{Sel}_p^\pm(E/\mathbf{Q}_n) = \ker\Big(\mathrm{Sel}_p(E/\mathbf{Q}_n) \to H^1(\mathbf{Q}_{n,p}, E[p^\infty])/(E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p)\Big).$$

Since $E(\mathbf{Q}_{n,v}) \otimes \mathbf{Q}_p/\mathbf{Z}_p = 0$ when $v \nmid p$, a class $c \in H^1(\mathbf{Q}_n, E[p^\infty])$ belongs to $\mathrm{Sel}_p^\pm(E/\mathbf{Q}_n)$ if and only if its localizations $c_v \in H^1(\mathbf{Q}_{n,v}, E[p^\infty])$ satisfy $c_v = 0$ if $v \nmid p$ and

$$c_p \in \mathrm{image}\Big(E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to H^1(\mathbf{Q}_{n,p}, E[p^\infty])\Big).$$

(If we replace $E^\pm(\mathbf{Q}_{n,p})$ by $E(\mathbf{Q}_{n,p})$ we get the definition of $\mathrm{Sel}_p(E/\mathbf{Q}_n)$.)

We define $\mathrm{Sel}_p^\pm(E/K_{\mathrm{cyc}})$ in exactly the same way with $\mathbf{Q}_n$ replaced by $K\mathbf{Q}_n$, using $E^\pm(K\mathbf{Q}_{n,p})$, and then

$$\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty) \otimes \mathcal{O}_p \cong \mathrm{Sel}_p^\pm(E/K_{\mathrm{cyc}}).$$

4. The Kummer pairing

The composition

$$E(\mathcal{K}_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow H^1(\mathcal{K}_p, E[p^\infty]) \xrightarrow{\sim} \mathrm{Hom}(G_{\mathcal{K}_p}, E[p^\infty])$$
$$\longrightarrow \mathrm{Hom}(\mathcal{U}, E[p^\infty]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathcal{U}(\rho^{-1}), K_p/\mathcal{O}_p),$$

where the third map is induced by the inclusion $\mathcal{U} \hookrightarrow G_{\mathcal{K}_p}$ of local class field theory, induces an $\mathcal{O}_p$-linear Kummer pairing

$$(3) \qquad\qquad (E(\mathcal{K}_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \times \mathcal{U}(\rho^{-1}) \to K_p/\mathcal{O}_p.$$

PROPOSITION 4.1. *The Kummer pairing of* (3) *induces an isomorphism*

$$\mathcal{U}_{K_{\mathrm{cyc}}}^\rho \cong \mathrm{Hom}_{\mathcal{O}}(E(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, K_p/\mathcal{O}_p).$$

Proof. This is equivalent to Proposition 5.4 of [Ru2]. $\qquad\square$

Definition 4.2.   Define $\widetilde{\mathcal{V}}^{\pm} \subset \mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ to be the subgroup of $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ corresponding to $\mathrm{Hom}_{\mathcal{O}}(E(K_{\mathrm{cyc},p})/E^{\pm}(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, K_p/\mathcal{O}_p)$ under the isomorphism of Proposition 4.1. Since $\mathrm{Hom}_{\mathcal{O}}(\ \cdot\ , K_p/\mathcal{O}_p)$ is an exact functor on $\mathcal{O}_p$-modules we have

$$(4) \qquad E^{\pm}(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}/\widetilde{\mathcal{V}}^{\pm}, K_p/\mathcal{O}_p),$$

$$(5) \qquad \mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}/\widetilde{\mathcal{V}}^{\pm} \cong \mathrm{Hom}_{\mathcal{O}}(E^{\pm}(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, K_p/\mathcal{O}_p).$$

Let $\alpha : \mathcal{U} \to \mathcal{X}$ be the Artin map of global class field theory. The following theorem is the step labeled (a′) in the introduction.

THEOREM 4.3. $\mathrm{Sel}_p^{\pm}(E/K_{\mathrm{cyc}}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{X}_{K_{\mathrm{cyc}}}^{\rho}/\alpha(\widetilde{\mathcal{V}}^{\pm}), K_p/\mathcal{O}_p).$

Proof. This is Theorem 2.1 combined with Definition 3.3 of $\mathrm{Sel}_p^{\pm}(E/K_{\mathrm{cyc}})$ and (4). $\qquad\square$

PROPOSITION 4.4.   (i) $\mathcal{U}_{K_{\infty}}^{\rho}$ *is free of rank two over* $\Lambda_{\mathcal{O}}(K_{\infty})$ *and* $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ *is free of rank two over* $\Lambda_{\mathcal{O}}$.

 (ii) $\widetilde{\mathcal{V}}^{\pm}$ *and* $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}/\widetilde{\mathcal{V}}^{\pm}$ *are free of rank one over* $\Lambda_{\mathcal{O}}$.

 (iii) *There is a* (*noncanonical*) *submodule* $\mathcal{V}^{\pm} \subset \mathcal{U}_{K_{\infty}}^{\rho}$ *whose image in* $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ *is* $\widetilde{\mathcal{V}}^{\pm}$ *and such that* $\mathcal{V}^{\pm}$ *and* $\mathcal{U}_{K_{\infty}}^{\rho}/\mathcal{V}^{\pm}$ *are free of rank one over* $\Lambda_{\mathcal{O}}(K_{\infty})$.

Proof.   By [Gr], $\mathcal{U}_{K_{\infty}}^{\rho}$ is free of rank two over $\Lambda_{\mathcal{O}}(K_{\infty})$, and then the definition of $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ shows that $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ is free of rank two over $\Lambda_{\mathcal{O}}$. Theorem 6.2 of [Ko] (see also Theorem 7.1 below) and (5) show that $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}/\widetilde{\mathcal{V}}^{\pm}$ is free of rank one over $\Lambda_{\mathcal{O}}$, so the exact sequence $0 \to \widetilde{\mathcal{V}}^{\pm} \to \mathcal{U}_{K_{\mathrm{cyc}}}^{\rho} \to \mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}/\widetilde{\mathcal{V}}^{\pm} \to 0$ splits. Thus $\widetilde{\mathcal{V}}^{\pm}$ is a projective $\Lambda_{\mathcal{O}}$-module, and Nakayama's lemma shows that every projective $\Lambda_{\mathcal{O}}$-module is free. This proves (ii).

Let u be any element of $\mathcal{U}_{K_{\infty}}^{\rho}$ whose image in $\mathcal{U}_{K_{\mathrm{cyc}}}^{\rho}$ generates $\widetilde{\mathcal{V}}^{\pm}$, and let $\mathcal{V}^{\pm} = \Lambda_{\mathcal{O}}(K_{\infty})u$. Then $\mathcal{V}^{\pm}$ is free of rank one, and it follows from (ii) and Nakayama's lemma that $\mathcal{U}_{K_{\infty}}^{\rho}/\mathcal{V}^{\pm}$ is free of rank one over $\Lambda_{\mathcal{O}}(K_{\infty})$ as well. $\qquad\square$

5. Elliptic units and the explicit reciprocity law

Let ψ_E denote the Hecke character of K attached to E, and for every character χ of finite order of G_K let $L(\psi_E\chi, s)$ denote the Hecke L-function. If χ is the restriction of a character of $G_{\mathbf{Q}}$ then $L(\psi_E\chi, s) = L(E, \chi, s)$, the usual L-function of E twisted by the Dirichlet character χ. Let $\Omega_E \in \mathbf{R}^+$ denote the real period of a minimal model of E.

The explicit reciprocity law of Wiles [Wi] together with a computation of Coates and Wiles [CW] leads to the following theorem, which is the step labeled (b) in the introduction.

THEOREM 5.1. *The $\Lambda_{\mathcal{O}}(K_\infty)$-module $\mathcal{C}^\rho_{K_\infty}$ of elliptic units is free of rank one over $\Lambda_{\mathcal{O}}(K_\infty)$. It has a generator ξ with the property that if $K \subset F \subset K_\infty$, $x \in E(F_p)$, and $\chi : \mathrm{Gal}(F/K) \to \boldsymbol{\mu}_{p^\infty}$, then the Kummer pairing $\langle\ ,\ \rangle$ of (3) satisfies*

$$\sum_{\sigma \in \mathrm{Gal}(F/K)} \chi^{-1}(\sigma)\langle x^\sigma \otimes p^{-k}, \xi \rangle = p^{-k}\frac{L(\psi_E\chi, 1)}{\Omega_E} \sum_{\sigma \in \mathrm{Gal}(F/K)} \chi^{-1}(\sigma)\lambda_E(x^\sigma).$$

Proof. See [Wi] and [CW, §5], or Theorem 7.7(i) of [Ru3] and Theorem 3.2 and the proof of Proposition 5.6 of [Ru2]. $\qquad\square$

COROLLARY 5.2. (i) *The map $\mathcal{C}^\rho_{K_{\mathrm{cyc}}} \to \mathcal{U}^\rho_{K_{\mathrm{cyc}}}$ is injective.*

(ii) *$\mathcal{C}^\rho_{K_{\mathrm{cyc}}}$ is free of rank one over $\Lambda_{\mathcal{O}}$ and $\mathcal{C}^\rho_{K_{\mathrm{cyc}}} \cap \widetilde{\mathcal{V}}^+ = \mathcal{C}^\rho_{K_{\mathrm{cyc}}} \cap \widetilde{\mathcal{V}}^- = 0$.*

(iii) *$\mathrm{rank}_{\Lambda_{\mathcal{O}}(K_\infty)}\mathcal{E}^\rho_{K_\infty} = 1$ and $\mathcal{E}^\rho_{K_\infty} \cap \mathcal{V}^+ = \mathcal{E}^\rho_{K_\infty} \cap \mathcal{V}^- = 0$.*

Proof. Since $\mathcal{C}^\rho_{K_{\mathrm{cyc}}}$ and $\mathcal{U}^\rho_{K_{\mathrm{cyc}}}/\widetilde{\mathcal{V}}^\pm$ are free of rank one over $\Lambda_{\mathcal{O}}$ (Theorem 5.1 and Proposition 4.4(ii)), the map $\mathcal{C}^\rho_{K_{\mathrm{cyc}}} \to \mathcal{U}^\rho_{K_{\mathrm{cyc}}}/\widetilde{\mathcal{V}}^\pm$ is either injective or identically zero. Thus to prove both (i) and (ii) it will suffice to show that the image $\widetilde{\xi} \in \mathcal{U}^\rho_{K_{\mathrm{cyc}}}$ of the generator $\xi \in \mathcal{C}^\rho_{K_\infty}$ of Theorem 5.1 satisfies $\widetilde{\xi} \notin \widetilde{\mathcal{V}}^+$ and $\widetilde{\xi} \notin \widetilde{\mathcal{V}}^-$.

Rohrlich [Ro] proved that $L(E, \chi, 1) \neq 0$ for all but finitely many characters χ of $\mathrm{Gal}(K_{\mathrm{cyc}}/K)$. Applying Theorem 5.1 with $x = d_{2n}$ for large n and using Theorem 3.2(iv) it follows that the image of ξ in $\mathrm{Hom}_{\mathcal{O}}(E^+(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, K_p/\mathcal{O}_p)$ is nonzero. Hence $\widetilde{\xi} \notin \widetilde{\mathcal{V}}^+$. Similarly, using the points d_{2n+1} for large n shows that $\widetilde{\xi} \notin \widetilde{\mathcal{V}}^-$. This proves (i) and (ii).

By Corollary 7.8 of [Ru3], $\mathcal{E}^\rho_{K_\infty}$ is a torsion-free, rank-one $\Lambda_{\mathcal{O}}(K_\infty)$-module. Just as in (i), since $\mathcal{U}^\rho_{K_\infty}/\mathcal{V}^\pm$ is torsion-free (Proposition 4.4(iii)) the map $\mathcal{E}^\rho_{K_\infty} \to \mathcal{U}^\rho_{K_\infty}/\mathcal{V}^\pm$ is either injective or identically zero. But we saw above that $\widetilde{\xi} \notin \widetilde{\mathcal{V}}^\pm$, so $\xi \notin \mathcal{V}^\pm$ and $\mathcal{E}^\rho_{K_\infty} \to \mathcal{U}^\rho_{K_\infty}/\mathcal{V}^\pm$ is not identically zero. This proves (iii). $\qquad\square$

6. The characteristic ideals

If B is a finitely generated torsion module over $\Lambda_{\mathcal{O}}(K_\infty)$ (resp. $\Lambda_{\mathcal{O}}$, resp. Λ), we will write $\mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(B)$ (resp. $\mathrm{char}_{\Lambda_{\mathcal{O}}}(B)$, resp. $\mathrm{char}_{\Lambda}(B)$) for its characteristic ideal.

The following theorem is Theorem 4.1(ii) of [Ru3], twisted by ρ^{-1}. It is the step labeled (c) in the introduction.

THEOREM 6.1 ([Ru3]).   The $\Lambda_{\mathcal{O}}(K_\infty)$-modules $\mathcal{A}^\rho_{K_\infty}$ and $\mathcal{E}^\rho_{K_\infty}/\mathcal{C}^\rho_{K_\infty}$ are finitely generated and torsion, and

$$\mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{A}^\rho_{K_\infty}) = \mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{E}^\rho_{K_\infty}/\mathcal{C}^\rho_{K_\infty}).$$

COROLLARY 6.2.   Let $\alpha : \mathcal{U} \to \mathcal{X}$ denote the Artin map of global class field theory. Then $\mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm)$ and $\mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty})$ are finitely generated torsion $\Lambda_{\mathcal{O}}(K_\infty)$-modules and

$$\mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm)) = \mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty})).$$

Proof. Class field theory gives an exact sequence

$$0 \longrightarrow \mathcal{E}/\mathcal{C} \longrightarrow \mathcal{U}/\mathcal{C} \overset{\alpha}{\longrightarrow} \mathcal{X} \longrightarrow \mathcal{A} \to 0.$$

Twisting by ρ^{-1} and using the fact that Δ has order prime to p gives another exact sequence

$$0 \longrightarrow \mathcal{E}^\rho_{K_\infty}/\mathcal{C}^\rho_{K_\infty} \longrightarrow \mathcal{U}^\rho_{K_\infty}/\mathcal{C}^\rho_{K_\infty} \overset{\alpha}{\longrightarrow} \mathcal{X}^\rho_{K_\infty} \longrightarrow \mathcal{A}^\rho_{K_\infty} \longrightarrow 0.$$

Since $\mathcal{E}^\rho_{K_\infty} \cap \mathcal{V}^\pm = 0$ by Corollary 5.2, we get finally an exact sequence

$$(6) \qquad 0 \to \mathcal{E}^\rho_{K_\infty}/\mathcal{C}^\rho_{K_\infty} \to \mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty}) \overset{\alpha}{\longrightarrow} \mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm) \to \mathcal{A}^\rho_{K_\infty} \to 0.$$

Since $\mathcal{C}^\rho_{K_\infty} \cap \mathcal{V}^\pm = 0$, it follows from Theorem 5.1 and Proposition 4.4 that the quotient $\mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty})$ is a finitely generated torsion $\Lambda_{\mathcal{O}}(K_\infty)$-module. Now (6) and Theorem 6.1 show that $\mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm)$ is a finitely generated torsion $\Lambda_{\mathcal{O}}(K_\infty)$-module as well, and that the two characteristic ideals are equal. $\square$

THEOREM 6.3.   The $\Lambda_{\mathcal{O}}$-modules $\mathcal{X}^\rho_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^\pm)$ and $\mathcal{U}^\rho_{K_{\mathrm{cyc}}}/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}^\rho_{K_{\mathrm{cyc}}})$ are finitely generated torsion modules and

$$\mathrm{char}_{\Lambda_{\mathcal{O}}}(\mathcal{X}^\rho_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^\pm)) = \mathrm{char}_{\Lambda_{\mathcal{O}}}(\mathcal{U}^\rho_{K_{\mathrm{cyc}}}/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}^\rho_{K_{\mathrm{cyc}}})).$$

Further, $\mathcal{X}^\rho_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^\pm)$ *has no finite* $\Lambda_{\mathcal{O}}$-*submodules.*

The proof of Theorem 6.3 is given below, after a few lemmas. The proof is essentially contained in Section 11 of [Ru3], but since it is crucial for our main result we recall some of the details.

If $\mathfrak{A}$ is an ideal of $\Lambda_{\mathcal{O}}(K_\infty)$, let $\overline{\mathfrak{A}} \subset \Lambda_{\mathcal{O}}$ denote the image of $\mathfrak{A}$ under the projection map $\Lambda_{\mathcal{O}}(K_\infty) \twoheadrightarrow \Lambda_{\mathcal{O}}$. Fix a topological generator γ_* of $\Gamma_- = \mathrm{Gal}(K_\infty/K_{\mathrm{cyc}})$.

LEMMA 6.4.   *Suppose* B *is a finitely generated torsion* $\Lambda_{\mathcal{O}}(K_\infty)$-*module with no nonzero pseudo-null submodules. Then*

$$\overline{\mathrm{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(B)} \neq 0 \text{ if and only if } B/(\gamma_* - 1)B \text{ is a torsion } \Lambda_{\mathcal{O}}\text{-module,}$$

and in that case

$$\operatorname{char}_{\Lambda_{\mathcal{O}}}(B/(\gamma_* - 1)B) = \overline{\operatorname{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(B)}.$$

Proof. See Lemma 4 of [PR1, §I.1.3] or Lemma 6.2 of [Ru3]. □

LEMMA 6.5. *Suppose B is a finitely generated $\Lambda_{\mathcal{O}}(K_\infty)$-module with no nonzero pseudo-null submodules. If B' is a free $\Lambda_{\mathcal{O}}(K_\infty)$-submodule of B then B/B' has no nonzero pseudo-null submodules.*

Proof. By induction we may reduce to the case that B' is free of rank one, and may reduce further to the case that B/B' is pseudo-null. Since $\Lambda_{\mathcal{O}}(K_\infty)$ is a unique factorization domain it follows that $B = B'$. □

LEMMA 6.6. *Suppose B is a finitely generated torsion $\Lambda_{\mathcal{O}}(K_\infty)$-module with no nonzero pseudo-null submodules, and both $B/(\gamma_* - 1)B$ and $B/(\gamma_* - \rho^{-1}(\gamma_*))B$ are torsion $\Lambda_{\mathcal{O}}$-modules. Then $B/(\gamma_* - 1)B$ has a nonzero finite submodule if and only if $B/(\gamma_* - \rho^{-1}(\gamma_*))B$ has.*

Proof. This is Lemma 11.15 of [Ru3] □

Proof of Theorem 6.3. By Proposition 4.4 and Corollary 5.2, $\mathcal{U}^\rho_{K_{\mathrm{cyc}}}$ and $\widetilde{\mathcal{V}}^\pm + \mathcal{C}^\rho_{K_{\mathrm{cyc}}}$ are free of rank two over $\Lambda_{\mathcal{O}}$, and $\mathcal{U}^\rho_{K_\infty}$ and $\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty}$ are free of rank two over $\Lambda_{\mathcal{O}}(K_\infty)$. Therefore (using Lemma 6.5) $\mathcal{U}^\rho_{K_{\mathrm{cyc}}}/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}^\rho_{K_{\mathrm{cyc}}})$ and $\mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty})$ are torsion modules with no nonzero pseudo-null submodules. By Lemma 6.4 it follows that

$$(7) \qquad \operatorname{char}_{\Lambda_{\mathcal{O}}}(\mathcal{U}^\rho_{K_{\mathrm{cyc}}}/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}^\rho_{K_{\mathrm{cyc}}})) = \overline{\operatorname{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty}))} \neq 0.$$

Class field theory shows that the kernel of $\alpha : \mathcal{U}^\rho_{K_\infty} \to \mathcal{X}^\rho_{K_\infty}$ is $\mathcal{E}^\rho_{K_\infty}$. Therefore by Corollary 5.2 α is injective on $\mathcal{V}^\pm$, so $\alpha(\mathcal{V}^\pm)$ is a free, rank-one $\Lambda_{\mathcal{O}}(K_\infty)$-submodule of $\mathcal{X}^\rho_{K_\infty}$. By [Gr], $\operatorname{rank}_{\Lambda_{\mathcal{O}}(K_\infty)}\mathcal{X}^\rho_{K_\infty} = 1$ and $\mathcal{X}^\rho_{K_\infty}$ has no nonzero pseudo-null submodules, so (using Lemma 6.5) $\mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm)$ is a torsion $\Lambda_{\mathcal{O}}(K_\infty)$-module with no nonzero pseudo-null submodules. Further, Corollary 6.2 and (7) show that

$$(8) \qquad \overline{\operatorname{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm))} = \overline{\operatorname{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{U}^\rho_{K_\infty}/(\mathcal{V}^\pm + \mathcal{C}^\rho_{K_\infty}))} \neq 0.$$

Thus we can apply Lemma 6.4 to conclude that

$$\operatorname{char}_{\Lambda_{\mathcal{O}}}(\mathcal{X}^\rho_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^\pm)) = \overline{\operatorname{char}_{\Lambda_{\mathcal{O}}(K_\infty)}(\mathcal{X}^\rho_{K_\infty}/\alpha(\mathcal{V}^\pm))},$$

and together with (7) and (8) this proves

$$\operatorname{char}_{\Lambda_{\mathcal{O}}}(\mathcal{X}^\rho_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^\pm)) = \operatorname{char}_{\Lambda_{\mathcal{O}}}(\mathcal{U}^\rho_{K_{\mathrm{cyc}}}/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}^\rho_{K_{\mathrm{cyc}}})).$$

It remains to prove that $\mathcal{X}^{\rho}_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^{\pm})$ has no nonzero finite submodules. This will follow from Lemma 6.6. We give the argument briefly here; see the proof of Theorem 11.16 of [Ru3] for more details.

We can identify $\mathcal{X}^{\rho}_{K_{\infty}}/(\gamma_* - \rho^{-1}(\gamma_*))\mathcal{X}^{\rho}_{K_{\infty}}$ with a subgroup of

$$(\mathcal{X}/(\gamma_* - 1)\mathcal{X})(\rho^{-1}).$$

Standard techniques (for example [Gr, §2]) identify $\mathcal{X}/(\gamma_* - 1)\mathcal{X}$ with a subgroup of $\mathrm{Gal}(M_0/K_{\mathrm{cyc}}(E[p]))$ where M_0 is the maximal abelian p-extension of $K_{\mathrm{cyc}}(E[p])$ unramified outside p, and by [Gr], $\mathrm{Gal}(M_0/K_{\mathrm{cyc}}(E[p]))$ has no nonzero finite submodules. Hence $\mathcal{X}^{\rho}_{K_{\infty}}/(\gamma_* - \rho^{-1}(\gamma_*))\mathcal{X}^{\rho}_{K_{\infty}}$ has no nonzero finite submodules.

Let $B = \mathcal{X}^{\rho}_{K_{\infty}}/\alpha(\mathcal{V}_{\pm})$. Lemma 6.5 now shows that $B/(\gamma_* - \rho^{-1}(\gamma_*))B$ has no nonzero finite submodules, and we observed above that B has no nonzero pseudo-null submodules, so Lemma 6.6 shows that $B/(\gamma_* - 1)B = \mathcal{X}^{\rho}_{K_{\mathrm{cyc}}}/\alpha(\widetilde{\mathcal{V}}^{\pm})$ has no nonzero finite submodules. $\square$

7. Local units, elliptic units, and the p-adic L-functions

Fix a topological generator γ of $\Gamma_+ \cong \mathrm{Gal}(K_{\mathrm{cyc}}/K) \cong \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. For every $n \geq 1$ define

$$\nu_n = \sum_{i=0}^{p-1} \gamma^{ip^{n-1}} \in \Lambda$$

and define $\omega_n^{\pm} \in \Lambda$ by

$$\omega_n^+ = \prod_{1 \leq i \leq n, 2|i} \nu_i, \quad \omega_n^- = \prod_{1 \leq i \leq n, 2 \nmid i} \nu_i.$$

THEOREM 7.1 (Kobayashi [Ko]).   *The $\Lambda_{\mathcal{O}}$-module*

$$\mathrm{Hom}(E^{\pm}(\mathbf{Q}_{\infty,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \mathbf{Q}_p/\mathbf{Z}_p)$$

is free of rank one, with a generator $\mu^{\pm}$ such that for every $k, n \in \mathbf{Z}^+$, and every character $\chi : \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q}) \to \boldsymbol{\mu}_{p^n}$,

$$\sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} \chi(\sigma)\mu^{\pm}(d_n^{\sigma} \otimes p^{-k}) = \chi(\omega_n^{\mp})p^{-k}.$$

Proof. An easy exercise shows that for $0 \leq n \leq \infty$

(9) $\mathrm{Hom}(E^{\pm}(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{Hom}(E^{\pm}(\mathbf{Q}_{n,p}), \mathbf{Z}_p).$

In Section 8 of [Ko], especially Proposition 8.18 and Theorem 6.2, Kobayashi shows that for every n and $\varepsilon = \pm 1$, the map

$$f \mapsto \begin{cases} \sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} f(d_n^{\sigma})\sigma & \text{if } (-1)^n = \varepsilon \\ \sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} f(d_{n-1}^{\sigma})\sigma & \text{if } (-1)^n = -\varepsilon \end{cases}$$

is an isomorphism from $\mathrm{Hom}(E^\varepsilon(\mathbf{Q}_{n,p}), \mathbf{Z}_p)$ to $\omega_n^{-\varepsilon}\mathbf{Z}_p[\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})]$, and that for $m \geq n \geq 1$ these maps are compatible in the sense that the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Hom}(E^\pm(\mathbf{Q}_{m,p}), \mathbf{Z}_p) & \xrightarrow{\sim} & \omega_m^\mp \mathbf{Z}_p[\mathrm{Gal}(\mathbf{Q}_m/\mathbf{Q})] \\
\downarrow & & \downarrow \\
\mathrm{Hom}(E^\pm(\mathbf{Q}_{n,p}), \mathbf{Z}_p) & \xrightarrow{\sim} & \omega_n^\mp \mathbf{Z}_p[\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})].
\end{array}
$$

Here the left-hand vertical map is restriction, and the right-hand vertical map sends $\omega_m^\mp$ to $\omega_n^\mp$.

In the limit it follows ([Ko] Theorem 6.2) that $\mathrm{Hom}(E^\pm(\mathbf{Q}_{\infty,p}), \mathbf{Z}_p)$ is free of rank one over Λ with a generator $f_\pm$ satisfying $\sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} f_\pm(d_n^\sigma)\sigma = \omega_n^\mp$. If we take $\mu^\pm$ to be the map corresponding to $f_\pm$ under (9), then $\mu^\pm$ satisfies the conclusions of the theorem. $\square$

Let $\mathcal{L}_E^\pm \in \Lambda$ denote the p-adic L-functions defined by the first author in Section 6.2.2 of [Po]. These are characterized by the formulas

(10) $\chi(\mathcal{L}_E^+) = (-1)^{(n+1)/2}\dfrac{\tau(\chi)}{\chi(\omega_n^+)}\dfrac{L(E, \bar{\chi}, 1)}{\Omega_E}$ if χ has order p^n with n odd,

(11) $\chi(\mathcal{L}_E^-) = (-1)^{n/2+1}\dfrac{\tau(\chi)}{\chi(\omega_n^-)}\dfrac{L(E, \bar{\chi}, 1)}{\Omega_E}$ if χ has order $p^n > 1$ with n even.

In addition, if χ_0 is the trivial character then

(12) $\qquad \chi_0(\mathcal{L}_E^+) = (p-1)\dfrac{L(E,1)}{\Omega_E}, \quad \chi_0(\mathcal{L}_E^-) = 2\dfrac{L(E,1)}{\Omega_E}.$

THEOREM 7.2.   *There is an isomorphism* $\mathcal{U}_{K_\mathrm{cyc}}^\rho/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}_{K_\mathrm{cyc}}^\rho) \xrightarrow{\sim} \Lambda_\mathcal{O}/\mathcal{L}_E^\pm\Lambda_\mathcal{O}.$

Proof. By (5) and (2) we have

$$
\begin{array}{rcl}
\mathcal{U}_{K_\mathrm{cyc}}^\rho/\widetilde{\mathcal{V}}^\pm & \cong & \mathrm{Hom}_\mathcal{O}(E^\pm(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, K_p/\mathcal{O}_p) \\
& \cong & \mathrm{Hom}(E^\pm(\mathbf{Q}_{\infty,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, K_p/\mathcal{O}_p) \\
& \cong & \mathrm{Hom}(E^\pm(\mathbf{Q}_{\infty,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, \mathbf{Q}_p/\mathbf{Z}_p) \otimes \mathcal{O}_p.
\end{array}
$$

Let $\mu^\pm$ be as in Theorem 7.1, let ξ be the generator of $\mathcal{C}_{K_\infty}^\rho$ from Theorem 5.1, and let $\varphi^\pm$ be the image of ξ in $\mathrm{Hom}_\mathcal{O}(E^\pm(K_{\mathrm{cyc},p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p, E[p^\infty])$. For some $h^\pm \in \Lambda_\mathcal{O}$ we have

(13) $\qquad\qquad\qquad\qquad \varphi^\pm = h^\pm\mu^\pm,$

and then $\mathcal{U}_{K_\mathrm{cyc}}^\rho/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}_{K_\mathrm{cyc}}^\rho) \cong \Lambda_\mathcal{O}/h^\pm\Lambda_\mathcal{O}.$

It follows from (13) that for every $k, n \geq 1$ and every nontrivial character $\chi : \Gamma^+ \to \boldsymbol{\mu}_{p^n}$,

$$
\sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} \chi(\sigma)\varphi^\pm(d_n^\sigma \otimes p^{-k}) = \chi(h^\pm) \sum_{\sigma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} \chi(\sigma)\mu^\pm(d_n^\sigma \otimes p^{-k}).
$$

Using the formulas of Theorems 3.2(iv) and 5.1 to compute the left-hand side, and Theorem 7.1 for the right-hand side, we deduce that if the order of χ is $p^n > 1$ and $\varepsilon = (-1)^{n+1}$ then

$$\frac{L(E, \bar{\chi}, 1)}{\Omega_E}(-1)^{[\frac{n}{2}]}\tau(\chi) \equiv \chi(h^\varepsilon)\chi(\omega_n^\varepsilon) \pmod{p^k}$$

for every k. It follows from (10) and (11) that $h^\pm = -\mathcal{L}_E^\pm$. $\square$

The following theorem is our main result.

THEOREM 7.3. $\mathrm{char}_\Lambda(\mathrm{Hom}(\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p)) = \mathcal{L}_E^\pm\Lambda.$

Proof. We have

$$\begin{aligned}
\mathrm{char}_{\Lambda_\mathcal{O}}(\mathrm{Hom}_\mathcal{O}(\mathrm{Sel}_p^\pm(E/K_{\mathrm{cyc}}), K_p/\mathcal{O}_p)) &= \mathrm{char}_{\Lambda_\mathcal{O}}(\mathcal{X}_{K_{\mathrm{cyc}}}^\rho/\alpha(\widetilde{\mathcal{V}}^\pm)) \\
&= \mathrm{char}_{\Lambda_\mathcal{O}}(\mathcal{U}_{K_{\mathrm{cyc}}}^\rho/(\widetilde{\mathcal{V}}^\pm + \mathcal{C}_{K_{\mathrm{cyc}}}^\rho)) \\
&= \mathcal{L}_E^\pm\Lambda_\mathcal{O}
\end{aligned}$$

by Theorems 4.3, 6.3, and 7.2, respectively. Since

$$\mathrm{Sel}_p^\pm(E/K_{\mathrm{cyc}}) = \mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty) \otimes \mathcal{O}_p,$$

we also have

$$\begin{aligned}
\mathrm{Hom}_\mathcal{O}(\mathrm{Sel}_p^\pm(E/K_{\mathrm{cyc}}), K_p/\mathcal{O}_p) &= \mathrm{Hom}(\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty), K_p/\mathcal{O}_p) \\
&= \mathrm{Hom}(\mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p) \otimes \mathcal{O}_p
\end{aligned}$$

and the theorem follows. $\square$

8. Applications

We describe briefly the basic applications of the supersingular main conjecture. As in the previous sections, we assume that E is an elliptic curve defined over $\mathbf{Q}$, with complex multiplication by the ring of integers of an imaginary quadratic field K, and p is an odd prime where E has good supersingular reduction. For this section we write $\Gamma = \Gamma^+$, so $\Lambda = \mathbf{Z}_p[[\Gamma]]$.

Remark 8.1.   The results below also hold for primes of ordinary reduction, and can be proved using the main conjecture for ordinary primes.

The following application was already proved in [Ru3], as an application of Theorem 6.1.

THEOREM 8.2 ([Ru3, Th. 11.4]).   *If $L(E, 1) \neq 0$, then $E(\mathbf{Q})$ is finite and*

$$|\mathrm{III}(E)| = r\frac{L(E, 1)}{\Omega_E}$$

where $r \in \mathbf{Q}^\times$ satisfies $\mathrm{ord}_p(r) = 0$, as predicted by the Birch and Swinnerton-Dyer conjecture.

If $L(E, 1) = 0$, then either $E(\mathbf{Q})$ is infinite or $\mathrm{III}(E)[p^\infty]$ is infinite.

Before proving Theorem 8.2 we need the following lemma.

LEMMA 8.3.   *The natural restriction map $\mathrm{Sel}_p(E/\mathbf{Q}) \to \mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty)^\Gamma$ is an isomorphism.*

Proof. For every number field F let $\mathrm{Sel}'_p(E/F)$ denote the Selmer group of E over F with no local condition at p:

$$\mathrm{Sel}'_p(E/F) = \ker : H^1(F, E[p^\infty]) \to \oplus_{v \nmid p} H^1(F_v, E[p^\infty])$$

(note that $E(F_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p = 0$ when $v \nmid p$). Thus we have a commutative diagram

(14)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Sel}_p(E/\mathbf{Q}) & \longrightarrow & \mathrm{Sel}'_p(E/\mathbf{Q}) & \longrightarrow & H^1(\mathbf{Q}_p, E[p^\infty])/A \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Sel}_p^\pm(E/\mathbf{Q}_\infty)^\Gamma & \longrightarrow & \mathrm{Sel}'_p(E/\mathbf{Q}_\infty)^\Gamma & \longrightarrow & H^1(\mathbf{Q}_{\infty,p}, E[p^\infty])/A_\infty^\pm
\end{array}
$$

where A and $A_\infty^\pm$ are the images of $E(\mathbf{Q}_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ and $E^\pm(\mathbf{Q}_{\infty,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p$, respectively, and the vertical maps are restriction maps. It follows from the theory of complex multiplication that $E(\mathbf{Q}_{\infty,p})$ has no p-torsion, and then standard methods (see for example Proposition 1.2 of [Ru1]) show that the restriction maps

$$H^1(\mathbf{Q}_p, E[p^\infty]) \to H^1(\mathbf{Q}_{\infty,p}, E[p^\infty])^\Gamma, \quad \mathrm{Sel}'_p(E/\mathbf{Q}) \to \mathrm{Sel}'_p(E/\mathbf{Q}_\infty)^\Gamma$$

are isomorphisms.

We will show that for every n the map $E(\mathbf{Q}_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to (E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\Gamma$ is surjective. It will then follow that the right-hand vertical map in (14) is injective, and then (using the remarks above and the snake lemma) that the left-hand vertical map in (14) is an isomorphism, which is the assertion of the lemma.

To show that $E(\mathbf{Q}_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p \to (E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\Gamma$ is surjective it suffices to check that $\dim_{\mathbf{F}_p}(E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{F}_p)^\Gamma = 1$, since $E(\mathbf{Q}_p) \otimes \mathbf{Q}_p/\mathbf{Z}_p \cong \mathbf{Q}_p/\mathbf{Z}_p$. Identify $\mathbf{F}_p[\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})]$ with $\mathbf{F}_p[X]/(X^{p^n} - 1) = \mathbf{F}_p[X]/(X - 1)^{p^n}$. Since $E^\pm(\mathbf{Q}_{n,p})$ is cyclic over $\mathbf{Z}_p[\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})]$ (Theorem 3.2(v)),

$$E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{F}_p \cong \mathbf{F}_p[X]/(X - 1)^a$$

for some $a \geq 0$. Under this identification $(E^\pm(\mathbf{Q}_{n,p}) \otimes \mathbf{F}_p)^\Gamma$ is the kernel of $X - 1$, which is visibly one-dimensional. $\qquad\square$

Proof of Theorem 8.2. By Lemma 8.3 we have

$$
\begin{aligned}
|\mathrm{Sel}_p(E/\mathbf{Q})| &= |\mathrm{Hom}(\mathrm{Sel}_p(E/\mathbf{Q}), \mathbf{Q}_p/\mathbf{Z}_p)| \\
&= |\mathrm{Hom}(\mathrm{Sel}_p^{\pm}(E/\mathbf{Q}_\infty)^\Gamma, \mathbf{Q}_p/\mathbf{Z}_p)| \\
&= |\mathrm{Hom}(\mathrm{Sel}_p^{\pm}(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p) \otimes_\Lambda \mathbf{Z}_p|.
\end{aligned}
$$

By Theorems 4.3 and 6.3, $\mathrm{Hom}(\mathrm{Sel}_p^{\pm}(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p)$ has no nonzero finite submodules, and by Theorem 7.3 its characteristic ideal is $\mathcal{L}_E^{\pm}\Lambda$. Writing χ_0 for the trivial character of Γ, standard techniques (for example [PR1, Lemma 4 of §I.1.3]) show that

$$|\mathrm{Hom}(\mathrm{Sel}_p^{\pm}(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p) \otimes_\Lambda \mathbf{Z}_p| = |\mathbf{Z}_p/\chi_0(\mathcal{L}_E^{\pm})\mathbf{Z}_p| = |\mathbf{Z}_p/(L(E,1)/\Omega_E)\mathbf{Z}_p|$$

using (12) for the last equality. This proves the theorem. $\qquad\square$

Fix a generator γ of Γ. Define $\nu_0 = \gamma - 1$ and for every $n \geq 1$ let $\nu_n = \sum_{i=0}^{p-1} \gamma^{ip^{n-1}}$. If χ is a character of Γ of finite order, let $\mathbf{Z}_p[\chi]$ denote the ring obtained by adjoining the values of χ to $\mathbf{Z}_p$. We view $\mathbf{Z}_p[\chi]$ as a Λ-module with Γ acting via χ, and if M is a Λ-module we define $M^\chi = M \otimes_\Lambda \mathbf{Z}_p[\chi]$. Then $\chi(\nu_m) = 0$ if and only if the order of χ is p^m, and if M is finitely generated or co-finitely generated over $\mathbf{Z}_p$ and χ has order p^m, then M^χ is infinite if and only if $M^{\nu_m=0}$ is infinite, where $M^{\nu_m=0}$ is the kernel of ν_m on M.

For every n write $G_n = \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})$.

THEOREM 8.4. *Suppose χ is a character of G_n. If $L(E, \chi, 1) \neq 0$ then $E(\mathbf{Q}_n)^\chi$ and $\mathrm{III}(E/\mathbf{Q}_n)^\chi$ are finite. If $L(E, \chi, 1) = 0$ then either $E(\mathbf{Q}_n)^\chi$ is infinite or $\mathrm{III}(E/\mathbf{Q}_n)^\chi$ is infinite.*

Before proving Theorem 8.4 we need the following lemma.

LEMMA 8.5. *Suppose χ is a character of G_n of order $p^m > 1$, and let $\varepsilon = (-1)^m$. Then $\mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_n)^{\nu_m=0}$ is infinite if and only if $\mathrm{Sel}_p(E/\mathbf{Q}_n)^{\nu_m=0}$ is infinite.*

Proof. We have $\mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_n) \subset \mathrm{Sel}_p(E/\mathbf{Q}_n)$, so one implication is clear. Suppose now that $\mathrm{Sel}_p(E/\mathbf{Q}_n)^{\nu_m=0}$ is infinite. By Proposition 10.1 of [Ko], either $\mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_n)^{\nu_m=0}$ or $\mathrm{Sel}_p^{-\varepsilon}(E/\mathbf{Q}_n)^{\nu_m=0}$ must be infinite. But localization at p sends $\mathrm{Sel}_p^{-\varepsilon}(E/\mathbf{Q}_n)^{\nu_m=0}$ into $E^{-\varepsilon}(\mathbf{Q}_{n,p})^{\nu_m=0}$ which is zero, and so $\mathrm{Sel}_p^{-\varepsilon}(E/\mathbf{Q}_n)^{\nu_m=0} \subset \mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_n)^{\nu_m=0}$. Hence $\mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_n)^{\nu_m=0}$ is infinite. $\quad\square$

Proof of Theorem 8.4. Let p^m be the order of χ. If $m = 0$ then the theorem is a consequence of Theorem 8.2. So we may suppose $m \geq 1$, and we let $\varepsilon = (-1)^m$.

$$
\begin{aligned}
\mathrm{Sel}_p(E/\mathbf{Q}_n)^\chi \text{ is infinite} &\iff \mathrm{Sel}_p(E/\mathbf{Q}_n)^{\nu_m=0} \text{ is infinite} \\
&\iff \mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_n)^{\nu_m=0} \text{ is infinite}
\end{aligned}
$$

$$\iff \quad \mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_\infty)^{\nu_m=0} \text{ is infinite}$$

$$\iff \quad \mathrm{Hom}(\mathrm{Sel}_p^\varepsilon(E/\mathbf{Q}_\infty), \mathbf{Q}_p/\mathbf{Z}_p) \otimes \Lambda/\nu_m \text{ is infinite}$$

$$\iff \quad \Lambda/(\mathcal{L}_E^\varepsilon, \nu_m) \text{ is infinite}$$

$$\iff \quad \bar\chi(\mathcal{L}_E^\varepsilon) = 0$$

$$\iff \quad L(E, \chi, 1) = 0$$

using Lemma 8.5, Theorem 9.3 of [Ko], Theorem 7.3, and (10) and (11). $\qquad\square$

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