A REMARK ON NON-INTEGRAL p-ADIC SLOPES FOR MODULAR FORMS

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ABSTRACT. We give a sufficient condition, namely "Buzzard irregularity", for there to exist an cuspidal eigenform which does not have integral p-adic slope.

RÉSUMÉ. *Une remarque sur les pentes p-adiques non-entières des formes modulaires.* On donne une condition suffisante, à savoir « irrégularité au sens de Buzzard », pour qu'il y existe une forme parabolique propre de pente p-adique non-entière.

1. STATEMENT OF RESULT

Let p be a prime number. If k and M are integers then we write $S_k(\Gamma_0(M))$ for the space of weight k cusp forms of level $\Gamma_0(M)$. The p-th Hecke operator acting on $S_k(\Gamma_0(M))$ is written T_p if $p \nmid M$ and U_p otherwise.

For $T = T_p$ or U_p, we define the slopes of T to be the slopes of p-adic Newton polygon of the inverse characteristic polynomial $\det(1 - TX)$. This is the same as the list of the p-adic valuations of the non-zero eigenvalues of T, counted with algebraic multiplicity.

To state our theorem we need a definition due to Buzzard [4].

Definition 1.1. *Let $N \geq 1$ be an integer with $p \nmid N$.*
  (a) *An odd prime p is $\Gamma_0(N)$-regular if the slopes of T_p acting on $S_k(\Gamma_0(N))$ are all zero for $2 \leq k \leq \frac{p+3}{2}$.*
  (b) *The prime $p = 2$ is $\Gamma_0(N)$-regular if the slopes of T_2 acting on $S_2(\Gamma_0(N))$ are all zero and the slopes of T_2 acting on $S_4(\Gamma_0(N))$ are all either zero or one.*

This definition first appeared in [4] where Buzzard gives an elementary algorithm, depending on p and N, which on input k will output a list of integers. He conjectures that if p is $\Gamma_0(N)$-regular then this list is exactly the list of slopes of T_p acting on $S_k(\Gamma_0(N))$. The authors of the present work also have made a separate conjecture ([3]) which predicts the U_p-slopes of all p-adic modular forms of tame level $\Gamma_0(N)$ still assuming that p is $\Gamma_0(N)$-regular. The two conjectures are consistent with each other experimentally, but have not yet been shown to be consistent in general.

Buzzard's conjecture clearly implies that every slope is an integer. (This implication is not at all clear from the conjectures in [3].) It is worth asking if the integrality of slopes is characteristic of $\Gamma_0(N)$-regularity. We show that it is. The proof occupies the second section.

Theorem 1.2. *If p is not $\Gamma_0(N)$-regular then there exists an even integer k such that U_p acting on $S_k(\Gamma_0(Np))$ has a slope strictly between zero and one.*

Coleman theory (which is used below) shows that no harm comes from assuming the witnessing weight in Theorem 1.2 is arbitrarily large. One could try to determine the minimum weight k which confirms Theorem 1.2. An effective bound should follow from [10], but it is likely suboptimal. For example, the authors have not seen an example where the optimal k was not of the form $k = j + (p-1)$ for some $2 \leq j \leq \frac{p+3}{2}$.

The theorem is also true for if we replace U_p and $S_k(\Gamma_0(Np))$ by T_p and $S_k(\Gamma_0(N))$. Indeed, if a_p is an eigenvalue for T_p acting on $S_k(\Gamma_0(N))$ then the polynomial $X^2 - a_pX + p^{k-1}$ divides the characteristic polynomial of U_p acting on $S_k(\Gamma_0(Np))$; the eigenvalues λ for U_p which are not roots of such polynomials are known to satisfy $\lambda^2 = p^{k-2}$. So, if $k > 2$ (which is sufficient by the previous paragraph) the slopes of U_p between zero and one are the same as the slopes of T_p between zero and one.

For p odd, the converse to Theorem 1.2 is also true. Namely, if there exists an even integer k such that $S_k(\Gamma_0(N))$ has a slope strictly between zero and one then p is not $\Gamma_0(N)$-regular. See [5, Theorem 1.6]. Its proof uses the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ and is thus significantly deeper than the present work. Combining the two results, the following two conditions are equivalent for an odd prime p:

(a) The prime p is not $\Gamma_0(N)$-regular.
(b) There exists an even integer k such that T_p acting on $S_k(\Gamma_0(N))$ has a slope strictly between zero and one.

There is a natural third condition, implied by (b):

(c) There exists an integer k such that T_p acting on $S_k(\Gamma_0(N))$ has a non-integral slope.

It is conjectured (see [6]) that all three conditions are equivalent, but this seems difficult.

2. The proof

We fix algebraic closures $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$ and write $v_p(-)$ for the induced p-adic valuation on $\overline{\mathbf{Q}}$ normalized so that $v_p(p) = 1$. We also fix an embedding $\overline{\mathbf{Q}} \subset \mathbf{C}$. We assume now that $N \geq 1$ is an integer co-prime to p.

If η is a Dirichlet character of modulus p we write $S_k(\Gamma_1(Np), \eta)$ for the subspace of forms in $S_k(\Gamma_1(Np))$ with character given by η (η promoted to a character of modulus Np). An eigenform f in particular means a normalized eigenform for the standard Hecke operators and the diamond operators. For such an f, its p-th Hecke eigenvalue is written $a_p(f)$.

Corresponding to the choice of embeddings, each eigenform has an associated two-dimensional p-adic Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$. Write $\overline{\rho}_f$ for its reduction modulo p and $\overline{\rho}_{f,p}$ (resp. $\rho_{f,p}$) for the restriction of $\overline{\rho}_f$ (resp. ρ_f) to the decomposition group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ induced from the embedding $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$. Note that the construction of $\overline{\rho}_f$ requires the choice of a Galois-stable lattice, but that the semi-simplification of $\overline{\rho}_f$ is independent of this choice. In particular, whether or not $\overline{\rho}_{f,p}$ is irreducible is also independent of the choice of a stable lattice.

Lemma 2.1. *Let η be a Dirichlet character of conductor p and f an eigenform in $S_2(\Gamma_1(Np), \eta)$. If $v_p(a_p(f))$ equals 0 or 1, then $\rho_{f,p}$ is reducible.*

Proof. If $v_p(a_p(f)) = 0$ then it is well known that $\rho_{f,p}$ is reducible. For example, see [11, Lemma 2.1.5] and the references therein. (This is also commonly attributed to a letter from Deligne to Serre in the 1970s which has never been made public.)

Now suppose that $v_p(a_p(f)) = 1$. Then, there is an eigenform f' in $S_2(\Gamma_1(Np), \eta^{-1})$ with $v_p(a_p(f')) = 0$ and ρ_f isomorphic to $\rho_{f'}$ up to a twist. (The form f' is sometimes called the Atkin–Lehner involute of f; see [2, Proposition 3.8].) Since the first argument applies to f', we deduce that $\rho_{f',p}$ and its twist $\rho_{f,p}$ are both reducible. $\qquad\square$

Proposition 2.2. *If p is odd and not $\Gamma_0(N)$-regular then there exists an even Dirichlet character η of modulus p such that U_p acting on $S_2(\Gamma_1(Np), \eta)$ has a slope strictly between zero and one.*

Proof. Choose an integer $2 \le k \le \frac{p+3}{2}$ and an eigenform $f \in S_k(\Gamma_0(N))$ with $v_p(a_p(f)) > 0$. By [9, Theorem 2.6], $\overline{\rho}_{f,p}$ is irreducible.

Suppose first that f has weight 2. Then, the polynomial $X^2 - a_p(f)X + p$ divides the characteristic polynomial of U_p acting on $S_k(\Gamma_0(Np))$ (as in the remarks after Theorem 1.2). The theory of the Newton polygon implies that the roots of this polynomial have valuation strictly between zero and one, so we can choose η to be the trivial character and we are done in this case.

Now assume that f has weight at least 4 and thus also $p \ge 5$. By [1, Theorem 3.5(a)], which assumes $p \ge 5$, there exists an even Dirichlet character η necessarily of conductor p (because f has weight at most $\frac{p+3}{2} < p + 1$) and an eigenform $g \in S_2(\Gamma_1(Np), \eta)$ such that $\overline{\rho}_g$ and $\overline{\rho}_f$ have isomorphic semi-simplifications. Since $\overline{\rho}_{f,p}$ is irreducible, $\overline{\rho}_{g,p}$ is as well. Thus, $\rho_{g,p}$ is irreducible, and Lemma 2.1 implies that $v_p(a_p(g))$ is strictly between zero and one. $\qquad\square$

Proposition 2.2 is an analog of Theorem 1.2 for weight two forms with character, and its proof confirms our theorem when there is a weight 2 form of level $\Gamma_0(N)$ with positive T_p-slope. To prove Theorem 1.2 in general, we use the theory of p-adic modular forms. We refer to [8] for the facts in the next two paragraphs.

If $\kappa : \mathbf{Z}_p^\times \to \overline{\mathbf{Q}}_p^\times$ is a continuous character (a "p-adic weight") then we write $S_\kappa^\dagger(N)$ for the space of overconvergent p-adic cusp forms of weight κ and tame level $\Gamma_0(N)$ equipped with its U_p-operator. If k is an integer and $\kappa(z) = z^k$ then we write this space as $S_k^\dagger(N)$; it contains $S_k(\Gamma_0(Np))$ as a U_p-compatible subspace. Likewise, if $\kappa(z) = z^k \eta(z)$ where η is a non-trivial finite order character of $\mathbf{Z}_p^\times$ then $S_{z^k\eta}^\dagger(N)$ contains $S_k(\Gamma_1(Np^{f_\eta}), \eta)$ as a U_p-compatible subspace (where p^{f_η} is the conductor of η).

By Coleman theory we mean the following: suppose that κ is a p-adic weight and h is the p-adic valuation of an eigenvalue for U_p appearing in $S_\kappa^\dagger(N)$. Then, for any sequence of p-adic weights $(\kappa_n)_{n\geq 0}$ such that κ_n and κ agree on the torsion subgroup of $\mathbf{Z}_p^\times$, and $\kappa_n(1+2p) \to \kappa(1+2p)$ as $n \to \infty$, we have that h is also a U_p-slope in $S_{\kappa_n}^\dagger(N)$ for $n \gg 0$.

We can now give the proof of the theorem.

Proof of Theorem 1.2. Assume first that p is odd. By Proposition 2.2 there exists an even Dirichlet character η of modulus p and rational number $0 < h < 1$ which appears as a U_p-slope in $S_2(\Gamma_1(Np), \eta)$. Thus, the slope h appears as a U_p-slope in $S_{z^2\eta}^\dagger(N)$. Choose $j \geq 0$ even so that $\eta|_{\mathbf{F}_p^\times}$ is of the form $z \mapsto z^j$. Then, for $n \gg 0$ and $k_n = 2 + j + (p-1)p^n$, the slope h is a U_p-slope in $S_{k_n}^\dagger(N)$ by Coleman theory described above. For such k we have $h < 1 < k - 1$ and so h is U_p-slope in $S_k(\Gamma_0(Np))$ by [7, Theorem 6.1].

The proof for $p = 2$ is similar to the argument in Proposition 2.2 when $k = 2$. If either $S_2(\Gamma_0(N))$ or $S_4(\Gamma_0(N))$ has a non-integral slope we are done. If not, then either $S_2(\Gamma_0(N))$ contains a slope one form, or $S_4(\Gamma_0(N))$ contains a form of slope two or three. In either case, the corresponding 2-adic refinements will have fractional slope. $\square$

References

[1] A. Ash and G. Stevens. Modular forms in characteristic l and special values of their L-functions. *Duke Math. J.*, 53(3):849–868, 1986.

[2] J. Bergdall and R. Pollack. Arithmetic properties of Fredholm series for p-adic modular forms. *Proc. Lond. Math. Soc.*, 113(3):419–444, 2016.

[3] J. Bergdall and R. Pollack. Slopes of modular forms and the ghost conjecture. *Preprint*, 2016. `arXiv:1607.04658`.

[4] K. Buzzard. Questions about slopes of modular forms. *Astérisque*, (298):1–15, 2005.

[5] K. Buzzard and T. Gee. Explicit reduction modulo p of certain two-dimensional crystalline representations. *Int. Math. Res. Not. IMRN*, (12):2303–2317, 2009.

[6] K. Buzzard and T. Gee. Slopes of modular forms. *To appear in Proceedings of the 2014 Simons symposium on the trace formula.*, 2015.

[7] R. F. Coleman. Classical and overconvergent modular forms. *Invent. Math.*, 124(1-3):215–241, 1996.
[8] R. F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.
[9] B. Edixhoven. The weight in Serre's conjectures on modular forms. *Invent. Math.*, 109(3):563–594, 1992.
[10] D. Wan. Dimension variation of classical and p-adic modular forms. *Invent. Math.*, 133(2):449–463, 1998.
[11] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.

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