1

CHERN–WEIL THEORY FOR CERTAIN INFINITE-DIMENSIONAL LIE GROUPS

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To Joe Wolf, with great appreciation

ABSTRACT. Chern–Weil and Chern–Simons theory extend to certain infinite-rank bundles that appear in mathematical physics. We discuss what is known of the invariant theory of the corresponding infinite-dimensional Lie groups. We use these techniques to detect cohomology classes for spaces of maps between manifolds and for diffeomorphism groups of manifolds.

Key words: Chern–Weil theory, Chern–Simons theory, infinite-rank bundles, mapping spaces, diffeomorphism groups, families index theorem.

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1. Introduction

The purpose of this survey is to emphasize algebraic aspects of Chern–Weil and Chern–Simons theory in infinite dimensions. The main open questions concern the classification or even existence of nontrivial Ad-invariant functions on the Lie algebra of some infinite-dimensional Lie groups that naturally appear in differential geometry and mathematical physics. These groups include gauge groups of vector bundles, diffeomorphism groups of manifolds, groups of bounded invertible pseudodifferential operators, and semidirect products of these groups. They typically appear as the structure groups of manifolds of maps between manifolds (e.g., in string theory) and in the setup of the Atiyah–Singer families index theorem. In effect, this article is a plea by a geometer for help from the experts in Lie groups.

As reviewed in Section 2, Chern–Weil theory is a well-established procedure to pass from an Ad-invariant polynomial p on the Lie algebra of a finite-dimensional Lie group G defined over a field k to an element $c_p \in H^*(BG, k)$, the cohomology of the classifying space BG. For the classical compact connected groups over $\mathbb{C}$, this correspondence is an isomorphism. For a G-bundle $P \longrightarrow B$ classified by a map $f : B \longrightarrow BG$, the class $c_p(P) = f^*c_p \in H^*(B, \mathbb{C})$ is by definition the characteristic class of P associated to p. For $G = U(n), SO(n)$, these are the Chern classes and Pontrjagin classes, respectively. They are used extensively in differential geometry, algebraic geometry and differential topology.

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Characteristic classes are obstructions to bundle triviality, i.e., they vanish on trivial bundles. When a Chern class of a bundle vanishes, the precise obstruction information for that class is unavailable, but there is a chance to obtain more refined geometric information. To begin, one can directly construct a de Rham representative $C_p(P)$ of $c_p(P)$ from a connection on P; this is often also called Chern–Weil theory. The advantage of the geometric approach is that one can in theory, and sometimes in practice, explicitly compute this de Rham representative from knowledge of the curvature of the connection.

If $C_p(P)$ vanishes pointwise, a very strong condition, then there is a secondary or Chern–Simons class $TC_p(P) \in H^*(B, \mathbb{C}/\mathbb{Z})$. As opposed to the topologically defined Chern classes, the Chern–Simons classes depend on the choice of connection, and so are inherently geometric objects. When defined, the Chern–Simons classes are obstructions to a trivial bundle admitting a trivialization by flat sections of a connection. Thus the secondary classes are more subtle and correspondingly harder to work with than with the primary/Chern classes. They notably appear as the generators of the integer cohomology of the classical groups.

Infinite-dimensional manifolds such as loop spaces LM of manifolds and mapping spaces $\mathrm{Maps}(N, M)$ between manifolds occur frequently in mathematical physics. Here the structure group of the tangent or frame bundle is an infinite-dimensional Lie group, the gauge group of a finite rank bundle. More generally, a finite rank bundle $E \longrightarrow M$ over the total space of a fibration $M \longrightarrow B$ of manifolds, the setup of the Atiyah–Singer families index theorem, naturally leads to an infinite rank bundle $\mathcal{E} \longrightarrow B$ with a more complicated structure group. In light of physicists' intriguing formal manipulations with path integrals, in particular their quick non-rigorous proofs of the Atiyah–Singer index theorem using loop spaces [2], it is natural to look for a good theory of characteristic classes of infinite rank vector bundles.

There are several immediate pitfalls. The fiber of such a bundle, an infinite-dimensional vector space, comes with many inequivalent norm topologies, in contrast to finite-dimensional vector spaces. As a result, the topology of the fiber has to be specified carefully. If the topology is compatible with a Hilbert space structure on the model fiber $\mathcal{H}$, it is tempting to take as structure group $GL(\mathcal{H})$, the group of bounded invertible operators with bounded inverse. However, unlike in finite dimensions, $GL(\mathcal{H})$ is contractible, so every $GL(\mathcal{H})$ bundle is trivial. This kills the theory of Chern classes in this generality.

Of course, $GL(\mathcal{H})$ contains many interesting subgroups with nontrivial topology. In particular, if a subgroup G consists of determinant class operators, then one can try to form the characteristic classes associated to the invariant polynomial $\mathrm{Tr}(A^k)$ for A in the Lie algebra of trace class operators; after all, for finite rank complex bundles, these polynomials form a generating set for the algebra of $U(n)$-invariant polynomials. However, in infinite dimensions, these classes are generally noncomputable, in the sense that operator traces are rarely given by e.g., integrals of pointwise calculable

expressions. In particular, it will usually be impossible to tell if these Chern classes vanish or not. Notice that we are not even considering the more difficult topological approach of working with BG.

In summary, in infinite dimensions we do not expect a version of Chern–Weil theory that applies to all bundles. Instead, we should look for naturally occurring structure groups with nontrivial topology, and we should look for computable Ad-invariant functions.

As with the example $\text{Tr}(A^k)$, traces on the Lie algebra of a group give rise to invariant functions. For gauge groups, a wide class of traces is known and these "tend to be" computable. However, the determination of all invariant functions is open. This gives us a theory of characteristic classes on mapping spaces, the subject of Section 3, and allows us to determine some nontrivial cohomology of mapping spaces.

This theory is not as geometric as desired, in the sense that natural connections on mapping spaces are not compatible with a gauge group, but instead are Ψ_0^*-connections for a larger group Ψ_0^* of pseudodifferential operators. This larger group has fewer traces, which in fact have been classified. (An excellent reference for pseudodifferential operators and traces is [38].) Again, it is not known if there are Ad-invariant functions not arising from traces.

It turns out that the Pontrjagin classes vanish for $\text{Maps}(N, M)$, so we are forced to consider secondary classes. In Section 4 we discuss Chern–Simons classes for loop spaces. We use these classes to show that $\pi_1(\text{Diff}(S^2 \times S^3))$ is infinite, where $\text{Diff}(S^2 \times S^3)$ is the diffeomrophism group of this 5-manifold. This result is new but not unexpected, and is given more as an illustration of potential applications of these techniques.

In Section 5 we discuss characteristic classes associated to $\text{Diff}(Z)$, the group of diffeomorphisms of a closed manifold Z. As pointed out by Singer, there is no known theory of characteristic classes for $\text{Diff}(Z)$-bundles. Specifically, there are no known nontrivial Ad-invariant functions on $\text{Lie}(\text{Diff}(Z))$. Instead, we outline a method to detect elements of $H^*(\text{Diff}(G), \mathbb{C})$ for classical Lie groups G. This is cheating somewhat, as in finite-dimensional bundle theory we want the cohomology of classifying spaces like $BU(n)$, not of $U(n)$ itself. Of course, $H^*(BU(n), \mathbb{C})$ is related to $H^*(U(n), \mathbb{C})$ by transgression arguments dating back to Borel. It is completely unclear if these arguments can be formulated in infinite dimensions, so the results of this section are baby steps towards understanding characteristic classes for diffeomorphism groups.

In Section 6, we discuss the setup of the families index theorem of Atiyah–Singer. As recognized by Atiyah and Singer and used by Bismut, this theorem can be restated in terms of an infinite rank superbundle $\mathcal{E}$. We discuss constructing a theory of characteristic classes on these bundles. Here the structure group $\mathbb{G}$ contains both a gauge group and the group $\text{Diff}(Z)$. Having a very large group makes it easier to find Ad-invariant functions in principle, but again we know of no nontrivial invariant

functions. Nevertheless, we can define characteristic classes of $\mathcal{E}$ for certain connections due to Bismut. We discuss an attempt to construct a proof of the families index theorem using characteristic classes on $\mathcal{E}$. While there are serious gaps in the argument, it is very intriguing that a semidirect product $\mathbb{G} \ltimes \Psi_0^*$ naturally appears as a structure group. Thus the work in this last section is in some sense is a culmination of the techniques in the previous sections.

The determination of the algebra of invariants for Lie groups is a classical topic with a very 19$^{\text{th}}$ century feel. In highlighting the obvious, namely the central role of these Lie-theoretic questions in Chern–Weil theory, I'm reminded of Moliere's M. Jourdain, who discovers that he has been speaking prose all his life without knowing it. In any case, I hope this article spurs interest in extending this classical theory to infinite-dimensional settings of current interest in geometry and physics.

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2. **General comments on Chern–Weil theory**

Let G be a finite-dimensional Lie group, and let $\mathcal{P}_G$ be algebra of Ad_G-invariant polynomials from $\mathfrak{g} = \mathrm{Lie}(G)$ to $\mathbb{C}$. In its more abstract form, Chern–Weil theory gives a map

$$\mathrm{cw} : \mathcal{P}_G \longrightarrow H^*(BG, \mathbb{C}).$$

Since G-bundles $E \longrightarrow B$ are classified by elements $f \in [B, BG]$, the set of homotopy classes of maps from B to BG, a polynomial $p \in \mathcal{P}_G$ gives rise to a characteristic class $c_p(E) = f^* \mathrm{cw}(p) \in H^*(B, \mathbb{C})$.

For compact connected groups, the suitably normalized map cw is a ring isomorphism to $H^*(BG, \mathbb{Z})$ [9] (see Ch. 3, Section 4 for references to Borel's original work), [10], with the corresponding characteristic classes called Chern classes for $G = U(n)$ and Pontrjagin classes for $G = SO(n)$. Since the adjoint action is given by conjugation for classical groups, for any $k \in \mathbb{Z}^+$ the polynomials $A \mapsto \mathrm{Tr}(A^k)$ are in $\mathcal{P}_G$. For $U(n)$, the corresponding characteristic classes are the k$^{\text{th}}$ components of the Chern character (up to normalization). It is a classical result of invariant theory that these polynomials generate $\mathcal{P}_{U(n)}$. We note that the k$^{\text{th}}$ Chern class is given by the trace of the transformation induced by A on $\Lambda^k(\mathbb{C}^n)$; since this transformation is usually also denoted by A^k, it is easy to confuse the two uses of $\mathrm{Tr}(A^k)$.

Remark 2.1. If a group G is linear, i.e., there is an embedding $i : G \longrightarrow GL(N, \mathbb{C})$ for some N (or equivalently, G admits a finite-dimensional faithful representation), then an Ad_G-invariant function on $\mathfrak{g}$ corresponds to a $i(G)$-conjugation invariant functional on $di(\mathfrak{g})$. The functionals $A \mapsto \mathrm{Tr}(A^k)$ certainly work, but there may be other invariant functions if $i(G)$ has "small enough" image in $GL(N, \mathbb{C})$. For example, the Pfaffian $\mathrm{Pf}(A)$ is an invariant polynomial for $A \in SO(n)$ which is not in the algebra generated by the $\mathrm{Tr}(A^k)$; while $\det(A)$ is in this algebra, the Pfaffian satisfies $(\mathrm{Pf}(A))^2 = \det(A)$.

This abstract approach to characteristic classes is not very useful in practice, both because BG tends to be far from a manifold, and because classifying maps are hard to find. However, the classifying space approach certainly is powerful. For example, if a bundle is trivial, then it is classified by a constant map, and so it immediately follows that its characteristic classes vanish. (For $U(n)$, the converse almost holds: if a hermitian vector bundle E has vanishing Chern classes, then some multiple kE is trivial. The proof is nontrivial.)

There are alternative approaches to constructing e.g., Chern classes, one topological and one geometric. The topological approach constructs the highest Chern class $c_n(E)$ of a rank-n complex vector bundle and then iteratively constructs the lower Chern classes by passing to a flag bundle [19, 28]. Since our bundles will have infinite rank, it's hard to get started on this approach.

At this point, we once and for all pass from principal G-bundles to vector bundles with G as structure group, although the former case is somewhat more general. Thus we are assuming that G is linear, and a G-bundle denotes a vector bundle with structure group G.

Since the topological approach seems unpromising, we follow the geometric method. If B is a paracompact manifold, then it admits a partition of unity, so G-bundles over B admit G-connections. (Finite-dimensional manifolds and even Banach manifolds are paracompact). $c_p(E)$ is then the de Rham cohomology class $[p(\Omega)]$ of $p(\Omega)$, where Ω is the $\mathfrak{g}$-valued curvature two-form of the connection, and $p(\Omega)$ involves wedging of forms and the Lie bracket in $\mathfrak{g}$ in a natural way. The Ad-invariance of p is used crucially to show that $p(\Omega)$ is closed and that its cohomology class is independent of the connection. This material is standard, and can be found in e.g., [4, 6, 37]. In summary,

Theorem 2.2 (Chern–Weil Theorem). Let p be an Ad_G-invariant $\mathbb{C}$-valued power series on $\mathfrak{g}$. Let $E \longrightarrow B$ be a bundle over a manifold B with structure group G, and let ∇ be a G-connection with curvature Ω. Then

(i) $p(\Omega)$ is a closed even degree form on B.

(ii) The de Rham cohomology class $[p(\Omega)] \in H^*(B, \mathbb{C})$ is independent of the choice of ∇.

Often, the power series of interest are in fact polynomials of some degree $k < \dim(B)/2$, in which case $[p(\Omega)] \in H^{2k}(B, \mathbb{C})$. In particular, for $U(n)$, the classes on a complex bundle E associated to $(2\pi i)^{-1} \mathrm{Tr}(A^k)$ are denoted by $c_k(E)$ and are called the k$^{\text{th}}$ *Chern classes*; the normalization ensures that they are in fact integral classes. The *Pontrjagin classes* of a real finite rank bundle F are by definition the Chern classes of the complexification $F \otimes \mathbb{C}$, corresponding to the embedding of $SO(n)$ into $U(n)$. On $U(n)$, the most important example of a power series is the exponential function e^A; the corresponding Chern class is called the *Chern character*. In index theory, other power series like the $\hat{A}$-genus naturally occur, although all these are truncated to polynomials at the dimension of the manifold. On infinite rank bundles

over infinite-dimensional manifolds, there is no reason to truncate, so the use of power series is more natural.

The advantage to the geometric approach is that the de Rham representative $p(\Omega)$ is pointwise computable on B. For example, a trivial complex bundle has vanishing Chern classes. From the geometric construction, we can conclude more: a bundle with a nonvanishing Chern class does not admit a flat connection. This is a stronger statement precisely because there are nontrivial bundles E with flat connections; these have a discrete structure group.

For noncompact finite-dimensional Lie groups, the situation is not so clean. For $G = GL(n, \mathbb{R})^+$, the connected group of orientation-preserving elements of $GL(n, \mathbb{R})$, cw is not surjective. For G is homotopy equivalent to $SO(n)$, so the universal Euler class, the element $e \in H^n(BSO(n), \mathbb{Z})$ corresponding to the Pfaffian in $\mathcal{P}_{SO(n)}$, is also an element of $H^n(BGL(n, \mathbb{R})^+, \mathbb{Z})$. However, since the Pfaffian is not $GL(n, \mathbb{R})^+$-invariant, e is not in the image of cw on $\mathcal{P}_{GL(n, \mathbb{R})^+}$.

It is clear that parts of geometric Chern–Weil theory carry over to infinite-dimensional Lie groups, especially since tricky questions about the topology of these Lie groups can often be avoided. For example, let G be the group of invertible transformations of a fixed Hilbert space of the form $I + A$, where A is trace class. Since all such operators are bounded, it is not hard to show that G is indeed a Lie group, with Lie algebra given by the set of trace class operators. Certainly the first Chern class $c_1(E) = [\mathrm{Tr}(\Omega)] \in H^2(B, \mathbb{C})$ exists for any connection on a G-bundle $E \longrightarrow B$. Here Tr refers to the operator trace. However, this Chern class is not computable except in special cases. In particular, we do not expect to be able to tell if this class is nonzero or not. By restricting A to lie in higher Shatten classes, we can construct higher but similarly noncomputable Chern classes, as discussed in [34].

Making sense of Lie groups of unbounded operators on a Hilbert space is difficult, particularly since the exponential map may have a sparse image. For finite rank bundles, one can take a default position by considering bundles for classical groups as $GL(n, \mathbb{C})$-bundles. Since $GL(n, \mathbb{C})$ deformation retracts onto $U(n)$, the topological theory of characteristic classes is the same for the two groups. In fact, the geometric theory is the same, as the Ad-invariant polynomials are the same for the two groups. Of course, for other linear groups the situation can be more complicated.

By analogy, in infinite dimensions we might begin with $GL(\mathcal{H})$, the group of bounded invertible operators with bounded inverses on a real or complex Hilbert space $\mathcal{H}$. As an open subset of the set of bounded endomorphisms of $\mathcal{H}$, $GL(\mathcal{H})$ is a Lie group [31]. However, this group has trivial Chern–Weil theory. For by Kuiper's Theorem, the unitary group $U(\mathcal{H})$ and hence $GL(\mathcal{H})$ is contractible in the norm topology. Thus $BGL(\mathcal{H})$ has the homotopy type of a point, so all $GL(\mathcal{H})$ bundles are trivial. ($U(\mathcal{H})$ should not be confused with the group $U(\infty) = \lim_{\longrightarrow} U(n)$, which has nontrivial topology by Bott periodicity.)

This problem of having a large contractible structure group is a key feature of infinite dimensions. As an example of its annoying presence, we can use $GL(\mathcal{H})$ to "ruin" Chern–Weil theory for finite rank bundles. For example, if we embed $GL(n, \mathbb{C})$ into $GL(N, \mathbb{C})$ as an upper left block for $N > n$, then we can extend a rank-n complex bundle E to a rank-N bundle with essentially the same transition functions. Since this amounts to adding a trivial $(N - n)$-rank bundle to E, the Chern classes are unchanged. However, if we embed $GL(n, \mathbb{C})$ into $GL(\mathcal{H})$, then the extended bundle and any characteristic classes become trivial. (We can cook up a similar example in finite dimensions: Let $E \longrightarrow S^1$ be the $\mathbb{Q}$-bundle which is trivial over $(0, 2\pi)$ and with $(0, q)$ glued to $(2\pi, 2q)$. Then E is nontrivial as a $\mathbb{Q}$-bundle, but becomes trivial when extended to an $\mathbb{R}$-bundle.)

From these examples, we see that we should consider infinite rank bundles whose structure group is a subgroup of $GL(\mathcal{H})$ with nontrivial topology. Fortunately, there are several well-known infinite-dimensional manifolds with a good Chern–Weil theory of characteristic classes.

3. Mapping Spaces and their characteristic classes

3.1. **The topological setup.** Let N^n, M^m be smooth, oriented, compact manifolds. Fix $s_0 \gg 0$, and let $\mathrm{Maps}(N, M)$ be the functions $f : N \longrightarrow M$ of Sobolev class s_0 (denoted $f \in H^{s_0}$). Here we fix covers $\{(U_\alpha, \phi_\alpha)\}, \{(V_\beta, \psi_\beta)\}$ of N, M, respectively, and we are imposing that $\psi_\beta f \phi_\alpha^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Sobolev class s_0 for all α, β. $\mathrm{Maps}(N, M)$ is a smooth Banach manifold [11], and the smooth structure is independent of the choice of covers.

We could work with the space of smooth maps from N to M as a Fréchet manifold, but this is technically more difficult. In particular, the implicit function theorem, which is used repeatedly in the foundations of manifold theory, is not guaranteed to hold for Fréchet manifolds. This is a little lazy, as the implicit function theorem holds for tame Fréchet manifolds [17] such as $\mathrm{Maps}(N, M)$, but we choose to work with Sobolev spaces just to keep the notation down.

The easiest examples of mapping spaces are free loop spaces LM ($N = S^1$) and in particular free loop groups LG. The most natural bundles are the tangent bundles TLM, TLG. Just as in finite dimensions, TLG is canonically trivial, so we do not expect characteristic classes for loop groups.

To develop a theory of characteristic classes of TLM, we should determine its structure group and look for Ad-invariant functions. A tangent vector in TLM at a loop γ should be the infinitesimal information in a family of loops $s \mapsto \gamma_s(\theta)$, for $\theta \in S^1$ and $s \in (-\epsilon, \epsilon)$. The infinitesimal information is $\{\dot{\gamma}(\theta) = (d/ds)|_{s=0} \gamma_s(\theta) : \theta \in S^1\}$. This is a vector field along γ, i.e., a section of $\gamma^* TM \longrightarrow S^1$; the pullback bundle has the effect of distinguishing tangent vectors $\dot{\gamma}(\theta_0), \dot{\gamma}(\theta_1)$ where $\gamma(\theta_0) = \gamma(\theta_1)$. Conversely, given a Riemannian metric on M, the exponential maps $\exp_{\gamma(\theta)} : T_{\gamma(\theta)} M \longrightarrow M$ combine to take a vector field along γ to a loop. Taking care of the

analytic details, we get $T_\gamma LM = \Gamma(\gamma^* TM)$, where we take H^{s_0} sections of $\gamma^* TM$. Since M is oriented, $\gamma^* TM$ is a trivial rank-m real bundle over S^1 denoted $\mathcal{R}^m = \mathcal{R}^m_\gamma$. The trivialization is not canonical, so TLM need not be trivial, and we have a hope of constructing characteristic classes.

We now show that the structure group of TLM is a group of gauge transformations. This structure group is determined by the differentials of the transition functions of LM. Fix a Riemannian metric on M. Call $s \in T_\gamma LM$ short if $\exp_{\gamma(\theta)} s(\theta)$ is inside the cut locus of $\gamma(\theta)$ for all θ. Let U_γ be the neighborhood of γ in LM consisting of the all exponentials along γ of short loops. These neighborhoods give an open cover of LM. On $U_{\gamma_0} \cap U_{\gamma_1}$, the transition functions are given by fiberwise invertible nonlinear maps $\Gamma \mathcal{R}^m \longrightarrow \Gamma \mathcal{R}^m$. Since $\Gamma \mathcal{R}^m$ is a vector space, the differentials of the transition maps at a $v \in \mathcal{R}^n$ can be naturally identified with invertible linear maps on $\mathcal{R}^m$ which act fiberwise. Thus the structure group is the group of H^{s_0} bundle automorphisms of $\mathcal{R}^m$, i.e., the gauge group $\mathcal{G}(\mathcal{R}^n)$. (Strictly speaking, we should take the gauge group of $\mathcal{G}(T\mathcal{R}^n)$, but this has the same homotopy type as $\mathcal{G}(\mathcal{R}^n)$.)

The general case of Maps(N, M) is similar. The path components of Maps(N, M) are in bijection with $[N, M]$. Pick a path component X_0 and $f : N \longrightarrow M$ in X_0. Then for all $g \in X_0$, $T_g \text{Maps}(N, M) \simeq \Gamma(f^* TM \longrightarrow N)$ noncanonically. $f^* TM$ need not be trivial, but as above the structure group on X_0 is $\mathcal{G}(f^* TM)$. For convenience, we always complexify real bundles, so the structure group is $\mathcal{G}(f^* TM \otimes \mathbb{C})$. From now on, we often omit the $\otimes \mathbb{C}$ term.

In summary, $T\text{Maps}(N, M)$ is a gauge bundle, or $\mathcal{G}$-bundle for short.

Now that the structure group of $T\text{Maps}(N, M)$ has been determined, we look for $\text{Ad}_\mathcal{G}$-invariant functions on $\mathfrak{g}$. Here $\mathfrak{g} = \text{Lie}(\mathcal{G}) = \text{End}(f^* TM)$ is the vector space of $H^{s_0 - 1}$ bundle endomorphisms of $f^* TM$. Since the Lie group and Lie algebra act fiberwise on $f^* TM$, the adjoint action of $\mathcal{G}$ on $\mathfrak{g}$ is fiberwise conjugation: $\text{Ad}(A)(b) = AbA^{-1}$. For fixed Riemannian metrics on N and M, $f^* TM$ inherits an inner product, and we can set

$$c_k : \mathcal{G}(f^* TM) \longrightarrow \mathbb{C}, \quad c_k(A) = \int_N \text{tr}(A^k) \, \text{dvol}_N.$$

Note that the trace depends on the metric on M. This is clearly Ad-invariant, so we can define

$$c_k(X_0) = [c_k(\Omega)] \in H^{2k}(X_0, \mathbb{C}) \tag{3.1}$$

for Ω the curvature of any gauge connection on $T\text{Maps}(N, M)$. We will usually just write $c_k(\text{Maps}(N, M)) \in H^{2k}(\text{Maps}(N, M), \mathbb{C})$. Some examples of these "gauge classes" will be computed in Section 3.3.

The reader familiar with characteristic classes may be appalled that we are omitting the usual normalizing constants which in finite dimensions guarantee that Chern classes have integral periods. In infinite dimensions, there is no known topological method of producing integral characteristic classes, so there is no natural normalization.

There are certainly many other Ad-invariant functions. For any smooth function $h : N \longrightarrow \mathbb{C}$, $c_{k,h}(A) = \int_N h \cdot \text{tr}(A^k) \, \text{dvol}_N$ is Ad-invariant. Letting h approach a delta function, we see that for every distribution $h \in \mathcal{D}(N)$, $h(\text{tr}(A^k))$ is Ad-invariant. Although it is overkill, this fits in with the finite-dimensional situation, where the structure group $GL(n,\mathbb{C})$ is the gauge group of the bundle $\mathbb{C}^n \longrightarrow *$, where $*$ is a point, and $h \in \mathcal{D}(*)$ must be multiplication by a constant.

Open question: Determine all $\text{Ad}_{\mathcal{G}}$-invariant analytic functions on $\mathfrak{g}$.

Since $\mathcal{G}$ is dense in $\mathfrak{g}$, solving this question includes finding all the traces on $\mathfrak{g}$, i.e., linear functions $t : \mathfrak{g} \longrightarrow \mathbb{C}$ with $t(ab) = t(ba)$ for all $a, b \in \mathfrak{g}$. This is in turn equivalent to computing the Hochschild cohomology group $HH^0(\mathfrak{g}, \mathbb{C})$, which should be feasible. This is interesting even in the loop group case, where we are asking for $HH^0(L\mathfrak{g}, \mathbb{C})$, where $\mathfrak{g}$ is now the Lie algebra of the compact group G.

For an overview of a large class of Ad-invariant functions on $L\mathfrak{g}$ with applications to integrable systems, see [35].

3.2. **The geometric setup.** Since we are taking a geometric approach to characteristic classes, we should see if the natural geometry on $\text{Maps}(N, M)$ is compatible with the structure group $\mathcal{G}$. In fact, it is not, as we now explain.

For simplicity, we will just consider loop spaces LM^m. The parameter θ always denotes the loop parameter, so $LM = \{\gamma(\theta) : \theta \in S^1, \gamma(\theta) \in M\}$ for γ of Sobolev class s_0.

Fix a Riemannian metric on M and fix $s \in [0, s_0]$. Define an H^s inner product on $T_\gamma LM$ by

$$\langle X, Y \rangle_{\gamma,s} = \int_{S^1} \langle X_{\gamma(\theta)}, (I + \Delta)^s Y_{\gamma(\theta)} \rangle_{\gamma(\theta)} \, d\theta.$$

Here $\Delta = D^*D$, with $D = \frac{D}{d\gamma} = \gamma^* \nabla^M$ the covariant derivative along γ, or equivalently the γ-pullback of the Levi-Civita connection ∇^M on M. The role of the positive elliptic operator $(I + \Delta)^s$ is to count roughly $s - (m/2)$ derivatives of the vector fields, by the so-called basic elliptic estimate. Thus the larger the s and s_0, the closer we are to modeling the smooth loop space. In particular, the L^2 metric (i.e., $s = 0$), while independent of a choice of s and hence natural, is too weak for many situations. (For example, the absolute version of the Chern–Simons classes discussed in Section 4 are multiples of s, and hence vanish at $s = 0$.)

This H^s metric gives rise to a Levi-Civita connection ∇^s on LM by the Koszul formula

$$2\langle \nabla^s_Y X, Z \rangle_s \quad = \quad X\langle Y, Z \rangle_s + Y\langle X, Z \rangle_s - Z\langle X, Y \rangle_s \quad\quad\quad (3.2)$$
$$+ \langle [X, Y], Z \rangle_s + \langle [Z, X], Y \rangle_s - \langle [Y, Z], X \rangle_s,$$

but only if the right-hand side is a continuous linear functional of $Z \in T_\gamma LM$. Note that this continuity is not an issue in finite dimensions.

We will consider loop groups as an example. First, recall that for a finite-dimensional Lie group G with a left-invariant metric, there is a global frame for TG consisting of left-invariant vector fields X_i. For such vector fields, the Kozul formula simplifies, since the first three terms on the right-hand side of (3.2) vanish. In particular, one can determine $\nabla_{X_j} X_i$ in terms of the structure constants of G. Since any vector field on G can be written as $X = f^i X_i$ for $f_i \in C^\infty(G)$, the Leibniz rule then determines $\nabla_Y X$ completely for any X, Y.

For loop groups LG, we can take an infinite basis $\{X_i\}$ of $T_e LG = L\mathfrak{g}$, by e.g., taking Fourier modes with respect to a chosen basis of $\mathfrak{g}$. We can extend these to left-invariant vector fields. A calculation first due to Freed [13] gives

$$2\nabla_X^s Y = [X, Y] + (I + \Delta)^{-s}[(I + \Delta)^s X, Y] + (I + \Delta)^{-s}[X, (I + \Delta)^s Y], \qquad (3.3)$$

for X, Y left-invariant. The reader is encouraged to rework this calculation, which just uses that $(I + \Delta)^s$ is selfadjoint for the L^2 inner product. There are technical issues here, such as checking that the right-hand side of (3.3) stays in H^{s_0}, and that applying the Leibniz rule to infinite sums $f^i X_i$ also stays in H^{s_0}. Since the X_i are so explicit, these issues can be resolved.

In (3.3), $(I + \Delta)^s$ is a differential operator if $s \in \mathbb{Z}^+$ and is a classical pseudodifferential (ΨDO) operator otherwise. In any case, $(I + \Delta)^{-s}$ is always pseudodifferential. The critical Sobolev dimension for LG is $1/2$, since loops need to be in $H^{(1/2)+\epsilon}$ to be continuous, so we will always assume $s > 1/2$. As an operator on Y for fixed X, $\nabla_X^s Y$ has order zero: the first order differentiations in the first and third terms on the right-hand side of (3.3) cancel (as seen by a symbol calculation), and the second term has order $-2s + 1$.

These technical calculations are really quite crucial. On general principles, the connection one-form and the curvature two-form take values in the Lie algebra of the structure group. So calculating these forms tells us for which structure group G our connection is a G-connection. For the curvature two-form

$$\Omega^s(X, Y) = \nabla_X^s \nabla_Y^s - \nabla_Y^s \nabla_X^s - \nabla_{[X,Y]}^s,$$

we can always say that $\Omega^s \in \Lambda^2(LG, \operatorname{End}(T_\gamma LG))$ by default, but the vector space $\operatorname{End}(T_\gamma LG)$ is too big to be useful. After all, $\operatorname{End}(T_\gamma LG)$ could only be the Lie algebra of $\operatorname{Aut}(T_\gamma LG)$, all technical issues aside. Without further restrictions, this group contains both bounded and unbounded operators, so its topology is unclear.

Thus without some detailed computations, the setup would be too formal. However, since ∇^s is built from zero order ΨDOs, Ω^s also takes values in zero order ΨDOs. That is good news, since order zero ΨDO are bounded operators on $T_\gamma LG$ with any H^s norm. Moreover, the vector space $\Psi_{\leq 0}$ of classical ΨDOs of integer order at most zero is the Lie algebra of Ψ_0^*, the Lie group of invertible classical zeroth order ΨDOs [32]. (Inverses of elements in Ψ_0^* are automatically bounded.) Note that $\Psi_0^* \supset \mathcal{G}(\mathcal{R}^n)$, since gauge transformations are (zero-th order) multiplication operators.

Thus by just working out the connection and curvature, we see that it is natural to extend the structure group from the gauge group $\mathcal{G}$, which was good enough for the topology of LG, to Ψ_0^*, which is needed to incorporate the Levi-Civita connection.

Before leaving the loop group case, we note that Freed proved that Ω^s actually takes values in ΨDOs of order at most -1. By some careful calculations, sharp results have been obtained:

Proposition 3.1. [20], [27] If G is abelian, the curvature two-form Ω^s for LG takes values in ΨDOs of order $-\infty$ for all $s \geq 1$. If G is nonabelian, the curvature two-form takes values in ΨDOs of order $-\infty$ for $s = 1$ and order -2 for $s > 1$. These results are also valid for the based loop groups ΩG.

The case $s = 1$ is known to be special: for complex groups G, ΩG is a Kähler manifold for the $s = 1$ metric [34]. The proposition again singles out this case, and applies to all finite-dimensional Lie groups. One proof that the curvature has order $-\infty$ involves showing that the map $\alpha : L\mathfrak{g} \mapsto L\mathfrak{g}[[\xi^{-1}]]$ (which appears in integrable systems as the space of formal nonpositive integer order ΨDOs on the trivial bundle $E = S^1 \times \mathfrak{g} \longrightarrow S^1$) given by $\alpha(X) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell}(\partial_\theta X)\xi^{-\ell}$ is a Lie algebra homomorphism. It would be interesting to know how this representation of $L\mathfrak{g}$ on the H^s sections of E fits into the general theory of loop group representations.

We now consider (3.2) for general loop spaces LM. Now all six terms on the right-hand side contribute unless M is parallelizable. For $s \in \mathbb{Z}^+$, after some simplifications, we end up with terms that take θ derivatives of Z (e.g., $(I + \Delta)^s Z$). This apparently obstructs the right-hand side from being a linear functional in Z in the H^s norm, but we can integrate by parts $2s$ times over S^1 to remove this problem. As a result, the Levi-Civita connection exists in this case. In contrast, for $s \notin \mathbb{Z}^+$, trying to integrate by parts involves the infinite symbol asymptotics $\sigma(I + \Delta)^s \sim \sum_{k=0}^{\infty} \sigma_{-k}(I + \Delta)^s$ of the ΨDO $(I + \Delta)^s$, and so does not terminate. To make a long story short, the Levi-Civita connection does not exist.

Theorem 3.2. [27] Let M be a Riemannian manifold. Then the Levi-Civita connection for the H^s metric on LM exists for $s \in \mathbb{Z}^+ \cup \{0\}$. If M is not parallelizable, then the Levi-Civita connection does not exist for $s \notin \mathbb{Z}^+ \cup \{0\}$.

This demonstrates the perils of geometry in infinite dimensions. A similar result should hold for $\mathrm{Maps}(N, M)$ but has not been worked out.

Remark 3.3. Now that the structure group $\Psi_0^* = \Psi_0^*(f^*TM \longrightarrow N)$ has naturally appeared, we can formulate the notion of principal Ψ_0^*-bundles and associated vector bundles for general $\Psi_0^*(E \longrightarrow N)$. To see that the theory of Ψ_0^*-bundles is topologically distinct from the theory of $\mathcal{G}$-bundles, we should show that $\mathcal{G}$ is not a deformation retract of Ψ_0^*. This is probably true, as for $A \in \Psi_0^*$, the top order symbol $\sigma_0(A)$ lives in $\mathcal{G}(\pi^*E \longrightarrow S^*N)$, which does not retract onto $\mathcal{G}(E \longrightarrow N)$. It would be good to work this out.

To summarize, the geometry of $\mathrm{Maps}(N, M)$ leads us to extend the structure group from a gauge group to a group of Ψ_0^*. In contrast, putting a metric on a complex finite rank bundle leads to a reduction of the structure group $GL(n, \mathbb{C})$ to $U(n)$.

3.3. **Characteristic classes for** $\mathrm{Maps}(N, M)$**.** The immediate issue is to find all Ad-invariant functions for $\Psi_{\leq 0} = \Psi_{\leq 0}(f^*TM \longrightarrow N)$. Since this algebra is larger than the gauge algebra $\mathfrak{g} = \mathrm{End}(f^*TM)$, we expect fewer invariants. As a start, we know we can build invariants from traces. In this setting the traces have been classified, as we explain.

Recall that the Wodzicki residue of a ΨDO A acting on sections of a bundle $E \longrightarrow N^n$ is defined by

$$\mathrm{res}^{\mathrm{w}} : \Psi_{\leq 0} \longrightarrow \mathbb{C}, \ \mathrm{res}^{\mathrm{w}}(A) = (2\pi)^{-n} \int_{S^*N} \mathrm{tr} \ \sigma_{-n}(A)(x, \xi) \ d\xi \ \mathrm{dvol}(x), \qquad (3.4)$$

where S^*N is the unit cosphere bundle. For $\dim(N) > 1$, the Wodzicki residue is the unique trace on the full algebra of ΨDOs up to scaling, although the facts that it is a trace and is unique are not obvious [12, 38]. (The issue for $N = S^1$, the loop space case, is that the unit cosphere bundle is not connected, but this case has been treated in [33].) In particular, the integrand $\mathrm{tr} \ \sigma_{-n}(A)(x, \xi)$ is not a trace, so we cannot apply distributions to the integrand to get other traces. The Wodzicki residue vanishes on ΨDOs which do not have a symbol term of order $-n$, so it vanishes on $\mathfrak{g}$, on all differential operators, and on all classical operators of noninteger order. The Wodzicki residue is orthogonal to the operator trace, in the sense that operators of order less than $-n$ are trace class but have vanishing Wodzicki residue. Just to reassure ourselves that this trace is nontrivial, for any first order elliptic operator D on sections of E, $\sigma_{-n}(I + D^*D)^{-n/2}(x, \xi) = |\xi|^{-n}$, so $\mathrm{res}^{\mathrm{w}}(I + D^*D)^{-n/2} = \mathrm{vol}(S^*N) \neq 0$.

Remark 3.4. The Wodzicki residue is the higher dimensional analogue of the residue considered by Adler, van Moerbeke and others in the study of the KdV equation and flows on coadjoint orbits of loop groups [1, 16]. The Ad-invariant functions becomes integrals of motion, and are used to study the complete integrability of this system.

On the subalgebra $\Psi_{\leq 0}$, there are more traces. The leading order symbol trace is defined by

$$\mathrm{Tr}^{\mathrm{lo}}(A) = (2\pi)^{-n} \int_{S^*N} \mathrm{tr} \ \sigma_0(A)(x, \xi) \ d\xi \ \mathrm{dvol}(x). \qquad (3.5)$$

Since $\sigma_0(AB) = \sigma_0(A)\sigma_0(B)$ for $A, B \in \Psi_{\leq 0}$, the integrand is a trace, and so any distribution on S^*N applied to the function $\sigma_0(A) \in C^\infty(S^*N)$ is a trace.

Theorem 3.5. [26] For $\dim(N) > 1$, all traces on $\Psi_{\leq 0}$ are of the form

$$A \mapsto c \cdot \mathrm{res}^{\mathrm{w}}(A) + C(\mathrm{tr} \, \sigma_0(A))$$

for some $c \in \mathbb{C}$ and $C \in \mathcal{D}(S^*N)$.

The proof is an impressive calculation in Hochschild cohomology.

Remark 3.6. For $\alpha < 0$, the set of ΨDOs of order at most α is a subalgebra of the full ΨDO algebra. In [25], the traces on these subalgebras are classified.

Now that we know what the traces are, we can define two types of characteristic classes for Ψ_0^*-bundles for $\Psi_0^* = \Psi_0^*(F \longrightarrow Z)$ with Z closed and F a complex bundle.

Definition 3.7. The kth Wodzicki–Chern class $c_k^{\mathrm{w}}(\mathcal{E})$ of the Ψ_0^*-bundle $\mathcal{E} \longrightarrow \mathcal{M}$ admitting a Ψ_0^*-connection ∇ is the de Rham cohomology class $[\mathrm{res}^{\mathrm{w}}(\Omega^k)] \in H^{2k}(\mathcal{M}, \mathbb{C})$, where Ω is the curvature of ∇. The kth leading order symbol class $c_k^{\mathrm{lo}}(\mathcal{E})$ is the de Rham class $[\mathrm{Tr}^{\mathrm{lo}}(\Omega^k)]$.

Note that if ∇ is in fact a gauge connection, then $c_k^{\mathrm{lo}}(\mathcal{E})$ is a multiple of the gauge classes in (3.1), since for gauge connections the symbol is independent of the cotangent variable ξ.

For $\mathrm{Maps}(N, M)$, we easily get $c_k^{\mathrm{w}}(\mathrm{Maps}(N, M)) \overset{\mathrm{def}}{=} c_k^{\mathrm{w}}(T\mathrm{Maps}(N, M) \otimes \mathbb{C}) = 0$. For $T\mathrm{Maps}(N, M)$ is a gauge bundle admitting a gauge connection. The curvature form of this connection takes values in $\mathrm{End}(f^*TM)$ and so has vanishing Wodzicki residue.

In contrast, it is shown in [23] that $c_k^{\mathrm{lo}}(\mathrm{Maps}(N, M)) = \mathrm{vol}(S^*N) \cdot \mathrm{ev}_n^* c_k(TM)$, where the evaluation map $\mathrm{ev}_n : \mathrm{Maps}(N, M) \longrightarrow M$ is given by $\mathrm{ev}_n(f) = f(n)$ for a fixed $n \in N$. With some work, this can be extended to:

Proposition 3.8. Let $\mathrm{Maps}_f(N, M)$ denote the connected component of an element $f \in \mathrm{Maps}(N, M)$ for M connected. Let $F \longrightarrow M$ be a rank-ℓ complex bundle with $c_k(F) \neq 0$. Then
$$0 \neq c_k^{\mathrm{lo}}(\pi_* \mathrm{ev}^* F) \in H^{2k}(\mathrm{Maps}_f(N, M), \mathbb{C}).$$

Thus we can use the leading order symbol classes to show that $\mathrm{Maps}(N, M)$ has roughly as much cohomology as M does. Of course, $\mathrm{Maps}(N, M)$ should have much more cohomology. Setting $M = BU(\ell)$, we get

Theorem 3.9. [23] Let $E \longrightarrow N$ be a rank-ℓ hermitian bundle. There are surjective ring homomorphisms from $H^*(B\mathcal{G}(E), \mathbb{C})$ and $H^*(B\Psi_0^*(E), \mathbb{C})$ to the polynomial algebra $H^*(BU(\ell), \mathbb{C}) = \mathbb{C}[c_1(EU(\ell)), \dots, c_\ell(EU(\ell))]$.

This is to our knowledge the first (incomplete) calculation of the cohomology of $B\Psi_0^*$, which is needed for a full understanding of the theory of characteristic classes. Even for the gauge group, these results seem to be new, although more precise results are known for specific 4-manifolds N^4 of interest in Donaldson theory. In contrast, the homotopy groups of (a certain stabilization of) Ψ_0^* and hence of $B\Psi_0^*$ have been completely computed in [36].

In summary, the study of traces on $\Psi_{\leq 0}$ yields two types of characteristic classes, the Wodzicki–Chern classes and the leading order symbol classes, but only the latter are nontrivial. Note that we have not addressed the question of finding Ad-invariant functions not associated to traces.

Open Question: Determine all $\mathrm{Ad}_{\Psi_0^*}$-invariant functions on $\Psi_{\leq 0}$.

4. SECONDARY CLASSES ON Ψ_0^*-BUNDLES

In this section we discuss secondary or Chern–Simons classes in infinite dimensions. This material is taken from [22, 23, 27].

It is useful to think of the Wodzicki–Chern classes as purely infinite-dimensional constructions: if $\mathcal{E} \longrightarrow \mathcal{M}$ is a $\Psi_0^*(E^\ell \longrightarrow *)$-bundle with $*$ just a point, then $\mathcal{E}$ is a finite-rank bundle and the only "ΨDOs" are elements of $GL(\ell, \mathbb{C})$, so there is no Wodzicki residue. In contrast, the leading order symbol Chern classes reduce to the usual Chern classes in this case.

We have already seen applications of the leading order symbol Chern classes. The Wodzicki–Chern classes are poised to detect the difference between Ψ_0^*- and $\mathcal{G}$-bundles. For as with $\mathrm{Maps}(N, M)$, $c_k^{\mathrm{w}}(\mathcal{E}) = 0$ if $\mathcal{E}$ admits a reduction to a $\mathcal{G}$-bundle. Thus if we can find a single Ψ_0^*-bundle $\mathcal{E}$ with $c_k^{\mathrm{w}}(\mathcal{E}) \neq 0$ for some k, then Ψ_0^* cannot have a deformation retraction to $\mathcal{G}$.

However, this approach has completely failed to date.

Conjecture 4.1. For any Ψ_0^*-bundle $\mathcal{E}$ over a paracompact base, $c_k^{\mathrm{w}}(\mathcal{E}) = 0$ for all k.

This conjecture holds if either (i) the structure group of $\mathcal{E}$ reduces from Ψ_0^* to the group Ell^* of invertible zeroth order ΨDOs with leading symbol the identity [23], or (ii) $\mathcal{E}$ admits a bundle map via fiberwise Fredholm zeroth order ΨDOs to a trivial bundle [22].

The proof of (i) uses the fact that Ell^* has the homotopy type of invertible operators of the form identity plus smoothing operator, and these operators have vanishing Wodzicki residue.

For (ii), we first note that this condition always holds for finite-rank bundles. The proof follows the structure of the heat equation proof of the families index theorem (FIT) for superbundles [5]. One takes a superconnection ∇ on $\mathcal{E}$ and modifies it to $B_t = \nabla + t^{1/2}A$, where A is a zero-th order odd operator on each fiber. The Wodzicki–Chern character of $\mathcal{E}$ has representative $\exp(-B_t^2)$ for any $t \geq 0$. As $t \longrightarrow \infty$, B_t becomes concentrated on the finite rank index bundle for A^*A, and so the Wodzicki–Chern character vanishes. This implies that all the Wodzicki–Chern classes vanish.

Remark 4.2. Although almost all details have been omitted, this proof is much quicker than the heat equation proofs of the FIT, and for good reason. In the FIT proofs, one is trying to compute the Chern character of the index bundle in terms of characteristic classes by comparing the $t \longrightarrow 0$ and $t \longrightarrow \infty$ limits of heat operators. The $t \longrightarrow \infty$ limit is relatively easy, and is mimicked in the proof outlined above. However, because one essentially wants to use the operator trace, the construction of the appropriate B_t is much more delicate, in order to have a well-defined limit at $t = 0$. Once again, we see the strong contrast between the operator trace and the Wodzicki residue.

The vanishing of the Wodzicki–Chern classes is not the end of the story, as it is in fact the prerequisite to defining secondary classes. Let ∇_0, ∇_1 be connections on a Ψ_0^*-bundle with local connection one-forms ω_0, ω_1 and curvature Ω_0, Ω_1. Then just as for finite-rank bundles, as even forms

$$c_k^{\mathrm{w}}(\Omega_1) - c_k^{\mathrm{w}}(\Omega_0) = dCS_k^{\mathrm{w}}(\nabla_1, \nabla_0), \tag{4.1}$$

with the odd form $CS_k^{\mathrm{w}}(\nabla_1, \nabla_0)$ given by

$$CS_k^{\mathrm{w}}(\nabla_1, \nabla_0) = \int_0^1 \mathrm{res}^{\mathrm{w}}[(\omega_1 - \omega_0) \wedge \overbrace{\Omega_t \wedge ... \wedge \Omega_t}^{k-1}] \, dt, \tag{4.2}$$

where

$$\omega_t = t\omega_0 + (1-t)\omega_1, \quad \Omega_t = d\omega_t + \omega_t \wedge \omega_t.$$

Here we are just lifting the finite-dimensional formula from [6, Appendix], replacing the matrix trace with the Wodzicki residue. (4.1) is precisely the explicit formula showing that the Wodzicki–Chern class is independent of connection, and so is the Ψ_0^* version of the proof of Theorem 2.2 (ii).

(4.1) shows that CS_k^{w} determines a $2k-1$ cohomology class if the Wodzicki–Chern forms for ∇_0, ∇_1 vanish pointwise. This holds in all cases we have been able to compute.

Open question: Do Wodzicki–Chern forms always vanish pointwise?

If this is the case, then the theory of secondary classes for Ψ_0^*-bundles based on the Wodzicki residue produces cohomology classes in odd degrees.

For finite-rank bundles $E \longrightarrow N$, the corresponding classes are called Chern–Simons classes $CS_k(\nabla_0, \nabla_1) \in H^{2k-1}(N, \mathbb{C})$, when they exist. In contrast to the Chern classes, which are defined via the geometric Chern–Weil theory but have topological content, these "relative" Chern–Simons classes really do depend on the choice of two connections, and so are geometric objects. There is an "absolute" version of Chern–Simons classes $CS_k(\nabla) \in H^{2k-1}(N, \mathbb{C}/\mathbb{Z})$ that only uses one connection but takes values in a weaker coefficient ring [7].

Definition 4.3. The k$^{\mathrm{th}}$ Wodzicki–Chern–Simons (WCS) class associated to connections ∇_0, ∇_1 on a Ψ_0^*-bundle $\mathcal{E} \longrightarrow \mathcal{M}$ is the cohomology class of $CS_k^{\mathrm{w}}(\nabla_1, \nabla_0)$ in $H^{2k-1}(\mathcal{M}, \mathbb{C})$, provided this form is closed.

In finite dimensions there are two ways to assure that characteristic forms for $E \longrightarrow N$ vanish: either use a flat connection, or pick a form whose degree is larger than the dimension of N or the rank of E. For example, if E is trivial, it admits a flat connection ∇, so we can define $CS_k(\nabla, g^{-1}\nabla g)$ for any gauge transformation g. The dimension restriction is only useful to define CS_r for $\dim(N) = 2r - 1$; this was used very effectively by Chern–Simons [7] and Witten [39] to produce invariants of 3-manifolds.

For $\mathrm{Maps}(N, M)$ with M parallelizable, a fixed trivialization of TM leads to a global trivialization of $T\mathrm{Maps}(N, M)$. A gauge transformation g of TM induces a gauge transformation of $T\mathrm{Maps}(N, M)$, so one has an element $CS_k^{\mathrm{w}}(\nabla, g^{-1}\nabla g)$ $\in H^{2k-1}(\mathrm{Maps}(N, M), \mathbb{C})$.

Open question: Is $CS_k^{\mathrm{w}}(\nabla, g^{-1}\nabla g)$ ever nonzero?

Although the dimension restriction on N looks incapable of generalization to $\mathrm{Maps}(N, M)$, an examination of the representative $\mathrm{res}^{\mathrm{w}}((\Omega^s)^k)$ of c_k^{w} for the H^s metric ($s \in \mathbb{N}$) shows that this form vanishes pointwise for $2k > \dim(M)$, as in (4.3) below. Thus we always get a secondary class $CS_k^{\mathrm{w}}(\nabla_1, \nabla_0) \in H^{2k-1}(\mathrm{Maps}(N, M^{2k-1}), \mathbb{C})$ associated to the $s = 1, 0$ metrics on $\mathrm{Maps}(N, M)$ determined by fixed metrics on N, M.

For simplicity, we go back to the loop space case LM. Because the L^2 (or $s = 0$) connection is so easy to treat – its connection one-form is just the one-form for the Levi-Civita connection on M – the local formula for $CS_k^{\mathrm{w}} = CS_k^{\mathrm{w}}(\nabla_1, \nabla_0)$ really is explicitly computable [27, Prop. 5.4]: as a $(2k - 1)$- form on $T_\gamma LM$,

$$CS_k^{\mathrm{w}}(X_1, ..., X_{2k-1}) \tag{4.3}$$
$$= \frac{2}{(2k-1)!} \sum_\sigma \mathrm{sgn}(\sigma) \int_{S^1} \mathrm{tr}[(-\Omega^M(X_{\sigma(1)}, \dot{\gamma}) - 2\Omega^M(\cdot, \dot{\gamma})X_{\sigma(1)})$$
$$\cdot (\Omega^M)^{k-1}(X_{\sigma(2)}, \ldots, X_{\sigma(2k-1)})],$$

where σ is a permutation of $\{1, \ldots, 2k-1\}$ and Ω^M is the curvature of the Levi-Civita connection on M. The tangent vectors X_i are vector fields in M along γ, so we see that this form will vanish if $2k - 1 > \dim(M)$.

We would like to use CS_k^{w} to detect odd cohomology in $H^*(LM, \mathbb{C})$. We need a test cycle in degree $2k - 1$. The natural candidate is M itself, thought of as the set of constant loops. However, for these loops $\dot{\gamma} = 0$, so (4.3) vanishes. Instead, assume that M admits an S^1-action $a : S^1 \times M \longrightarrow M$. This induces a map $\tilde{a} : M \longrightarrow LM$ by $\tilde{a}(m)(\theta) = a(\theta, m)$. Dropping the tilde, the action now produces a test cycle $a_*[M] \in H_{2k-1}(LM, \mathbb{Z})$, where $[M]$ is the fundamental class of M, which is assumed orientable. (For the trivial action, $a_*[M]$ is M as constant loops.)

If $\int_{a_*[M]} CS_k^{\mathrm{w}} = \int_M a^* CS_k^{\mathrm{w}}$ is nonzero, then $CS_k^{\mathrm{w}} \neq 0$ in the cohomology of LM. The computation of the integral is frustrating: it always vanishes in the easiest case $\dim(M) = 3$; in higher dimensions, most explicit Riemannian metrics come with large continuous symmetry groups, and in all examples we get $\int_{a_*[M]} CS_k^{\mathrm{w}} = 0$, although we cannot prove a general vanishing theorem. Fortunately, there is a family $g_t, t \in (0, 1)$, of Sasaki–Einstein metrics on $S^2 \times S^3$ due to [15] which is explicit enough and has enough symmetry to make the integral calculation feasible but which is not "too symmetric." In particular, this construction gives a metric fibration $S^1 \longrightarrow S^2 \times S^3 \longrightarrow S^2 \times S^2$ generalizing the Hopf fibration, so we get a circle action by rotating the fiber. The calculations can be done in closed form by Mathematica©. We get

$\int_{a_*[M]} CS_k^{\mathrm{w}} \neq 0$, and so with a little work we conclude that $H^5(L(S^2 \times S^3), \mathbb{C})$ is infinite.

A circle action a on M also induces $\tilde{a} : S^1 \longrightarrow \mathrm{Diff}(M)$ by $\tilde{a}(\theta)(m) = a(\theta, m)$, so we get an element of $\pi_1(\mathrm{Diff}(M))$. It is easy to check that $\int_{a_*[M]} CS_k^{\mathrm{w}} \neq 0$ implies $\pi_1(\mathrm{Diff}(M))$ is infinite. In particular, $\pi_1(\mathrm{Diff}(S^2 \times S^3))$ is infinite.

We tend to trust the computer calculations, as in the $t \longrightarrow 0$ limit the metric g_t becomes a metric on S^5 and the integral explicitly vanishes. This matches with the known result that $\pi_1(\mathrm{Diff}(S^5))$ is finite. On the other hand, up to factors of π, the integrals calculated are always rational; this needs further explanation.

Remark 4.4. (i) While there is nothing in theory to stop us from computing WCS classes for $\mathrm{Maps}(N, M)$, in practice the number of computations necessary to compute the Wodzicki residue of an operator on N^n increases exponentially in n. So while computations are feasible for loop spaces and $\mathrm{Maps}(\Sigma^2, M)$, the setting for string theory, one needs a very good reason to do computations on higher-dimensional source manifolds.

(ii) These results are too specific. If one can show that the Wodzicki–Chern forms always vanish pointwise, then a much more robust theory of WCS classes would be available.

5. Characteristic classes for diffeomorphism groups

The search for characteristic classes associated to the diffeomorphism group of a closed manifold X can be interpreted in two ways: (i) $\mathrm{Diff}(X)$ is an open subset of $\mathrm{Maps}(X, X)$, and so as in Section 3 characteristic classes can be used to detect elements of $H^*(\mathrm{Diff}(X), \mathbb{C})$; (ii) certain infinite rank bundles have $\mathrm{Diff}(X)$ as part of their structure group. In this section we consider the first question, and in Section 6 we treat (ii).

First, the proof that cohomology classes for $\mathrm{Maps}(X, X)$ are detected by characteristic classes of X (Prop. 3.8) does not carry over to $\mathrm{Diff}(X)$. Indeed, the proof should break down, since as in finite dimensions the Lie group $\mathrm{Diff}(X)$ admits a flat connection and so has vanishing leading order symbol classes. (As usual, there are technicalities about the Lie group structure on $\mathrm{Diff}(X)$, which are most easily treated by considering H^s diffeomorphisms.)

Thus we expect to find only secondary classes. We now outline a method that may produces odd degree classes in $H^*(\mathrm{Diff}(X), \mathbb{C})$.

The cohomology ring $H^*(U(n), \mathbb{Z})$ is generated by suitably normalized Chern–Simons classes, as the standard forms $\mathrm{tr}((g^{-1}dg)^{2k-1})$ built from the Maurer–Cartan form $g^{-1}dg$ are the Chern–Simons forms for c_k associated to the flat connection ∇ on the trivial bundle $U(n) \times \mathfrak{gl}(n, \mathbb{C})$ and to the gauge equivalent connection $g^{-1}\nabla g$ [29]. Here we think of g as the gauge transformation $M \mapsto g \cdot M$ for $M \in \mathfrak{gl}(n, \mathbb{C})$. There are similar results for other classical linear groups; see e.g., [34, Ch. 4.11] and the Bourbaki references therein, [8], particularly the references to the original work

of Hopf, and [18, Ch. 3D] for a modern treatment for $SO(n)$. Moreover, a certain average of these forms along loops in G give generators for $H^*(LG, \mathbb{Z})$ [34].

In finite dimensions, identifying the Maurer–Cartan form with a gauge transformation requires an embedding $G \longrightarrow GL(N, \mathbb{C})$. For $G = \mathrm{Diff}(X)$, we can embed $i : G \longrightarrow GL(\Gamma(\mathcal{C}^N))$, where $\mathcal{C}^N = X \times \mathbb{C}^N$ is the trivial bundle over X, GL refers to bounded operators with bounded inverses, and $\Gamma(\mathcal{C}^N)$ refers to H^s sections. The embedding is given by $i(\phi)(s)(x) = s(\phi^{-1}x)$. Now ϕ makes sense as a gauge transformation of the trivial bundle $\mathrm{Diff}(X) \times \Gamma(\mathcal{C}^N)$ via $s \mapsto i(\phi)(s)$.

We can now define the Maurer–Cartan form $\phi^{-1}d\phi$ for $\phi \in \mathrm{Diff}(X)$. As in Section 4, we will get secondary classes in $H^{\mathrm{odd}}(\mathrm{Diff}(X), \mathbb{C})$ associated to the trivial connection ∇ and $\phi^{-1}\nabla\phi$ once we pick an Ad_G-invariant function on $\mathrm{Lie}(\mathrm{Diff}(X))$. Since a family of diffeomorphisms ϕ_t starting at the identity has infinitesimal information $\dot{\phi}_0$, a vector field on X, we have $\mathrm{Lie}(\mathrm{Diff}(X)) = \Gamma(TX)$. The adjoint action of a diffeomorphism ϕ on a vector field V is easily seen to be $V \mapsto \phi_* V$.

Open Question: Find a nontrivial $\mathrm{Ad}_{\mathrm{Diff}(X)}$-invariant function on the set of (H^s) vector fields on a closed manifold X.

Remark 5.1. (i) Because the adjoint action is not by conjugation, finding a trace on $\Gamma(TX)$ does not produce secondary classes. It seems to be an open question whether there exist any nontrivial traces on $\Gamma(TX)$ for general X. If $X = G$ is itself a compact linear Lie group, there are many traces on $T_I\mathrm{Diff}(G) = \Gamma(TG)$. Namely, a vector field V on G is just a $\mathfrak{g}$-valued function on G, so for any distribution $f \in \mathcal{D}(G)$, $f(\mathrm{tr}(V))$ is a trace. This case needs further work.

(ii) There is another context in which this Open Question comes up. $\mathrm{Diff}(X)$ is the structure group for fibrations $X \longrightarrow M \longrightarrow B$ of manifolds with fibers X, so characteristic classes associated to $\mathrm{Diff}(X)$ would be obstructions to the triviality of a fibration, just as ordinary characteristic classes as obstructions to the triviality of principal G-bundles.

Thus this general approach to secondary classes for $\mathrm{Diff}(X)$ is unavailable at present. However, for $X = G$ a compact linear Lie group, we can detect odd degree classes in $H^*(\mathrm{Diff}(X), \mathbb{C})$ using finite rank bundles. Let $\alpha : G \longrightarrow \mathrm{Diff}(G)$ be the embedding $g \mapsto L_g$, for L_g left translation by g. The trivial rank bundle $\mathcal{C}_1^N = \mathrm{Diff}(G) \times \mathfrak{gl}(N, \mathbb{C})$ admits the gauge transformation $\phi \in \mathrm{Diff}(G) \mapsto (M \mapsto \phi(e)M)$. This gauge transformation, also denoted by ϕ, restricts to the gauge transformation g on $\alpha(G) \subset \mathrm{Diff}(G)$ since $L_g(e) = g$. The bundle $\mathcal{C}_1^N$ has the trivial connection ∇_1. The gauge transformed connection $\nabla_1^\phi = \phi^{-1}\nabla_1\phi$ has the global connection one-form $\phi^{-1}d\phi$, which restricts to $g^{-1}dg$ on $\alpha(G)$. On the finite-rank bundle $\alpha(G)$, we use the ordinary matrix trace to define Chern–Simons forms

$$CS^{2k-1}(\nabla_1, \nabla_1^\phi) = \mathrm{tr}((\phi^{-1}d\phi)^{2k-1}).$$

We can also define Chern–Simons classes for ∇^g on $\mathcal{C}^N$ by the same formula. A straightforward calculation gives

$$\int_{\alpha(z_{2k-1})} CS^{2k-1}(\nabla_1, \nabla_1^\phi) = \int_{z_{2k-1}} CS^{2k-1}(\nabla, \nabla^g)$$

for any $(2k-1)$-chain z_{2k-1} on G. This implies

Theorem 5.2. [29] For any compact linear group G, the map $\alpha : G \longrightarrow \mathrm{Diff}(G)$, $g \mapsto L_g$, induces a surjection $\alpha^* : H^*(\mathrm{Diff}(G), \mathbb{R}) \longrightarrow H^*(G, \mathbb{R})$.

Dualizing this result, we see that the real homology of G injects into the homology of $\mathrm{Diff}(G)$. As with mapping spaces, we expect nontrivial homology in infinitely many degrees, but these techniques only give information up to $\dim(G)$.

6. Characteristic classes and the Families Index Theorem

As explained below, the families index theorem (FIT) is a generalization of the Atiyah–Singer index theorem. Infinite rank superbundles $\mathcal{E}$ naturally appear in the setup of the FIT. In this section we discuss how a theory of characteristic classes on $\mathcal{E}$ may give insight into the FIT. In particular, the relevant structure group incorporates aspects of gauge groups, diffeomorphism groups, and the group Ψ_0^* of pseudodifferential operators discussed in previous sections. This section is based on [21].

The bundle $\mathcal{E}$ was explicitly mentioned by Atiyah and Singer [3] as a topological object, but was not used in their proof. Bismut [5] used $\mathcal{E}$ as a geometric object, in that he constructed what is now called the Bismut superconnection on $\mathcal{E}$. Bismut did not define characteristic classes for $\mathcal{E}$, because he did not need them: the fine details of his proof take place on the sections of a finite rank bundle, the model fiber of $\mathcal{E}$ (see Remark 6.3). In this section we try to define characteristic classes directly on $\mathcal{E}$. The hope, not yet realized, is that not only is $\mathcal{E}$ a proper setting for the FIT, but that a proof of the FIT can take place on $\mathcal{E}$.

We recall the basic setup, inevitably leaving out a slew of technicalities. Let $Z \longrightarrow M \xrightarrow{\pi} B$ be a fibration of closed connected manifolds, and let $E, F \longrightarrow M$ be finite rank bundles. Set $Z_b = \pi^{-1}(b)$, $E_b = E|_{Z_b}, F_b = F|_{Z_b}$. Assume that we have a smoothly varying family of elliptic operators $D_b : \Gamma(E_b) \longrightarrow \Gamma(F_b)$. Although the dimensions of the kernel and cokernel of D_b need not be continuous in b, the index $\mathrm{ind}(D_b) = \dim \ker(D_b) - \dim \mathrm{coker}(D_b)$ is constant. It is therefore plausible and indeed true that the virtual bundle $\mathrm{IND}(D) = [\ker(D_b)] - [\mathrm{coker}(D_b)] \in K(B)$ is well defined.

Although the FIT can be stated entirely within K-theory, it is easier to state it as an equality in cohomology. The Chern character $ch : K(B) \otimes \mathbb{Q} \longrightarrow H^{\mathrm{ev}}(B, \mathbb{Q})$ is an isomorphism, and the FIT identifies $ch(\mathrm{IND}(D))$ with an explicit characteristic class built from the symbols of the D_b.

Even the case where $B = \{b\}$ is a point is highly nontrivial. In this case,

$$ch(\mathrm{IND}(D)) = \mathrm{ind}(D_b) \in H^0(\{b\}, \mathbb{Q}) = \mathbb{Q}$$

(of course the index is an integer). Identifying the corresponding characteristic class gives the "ordinary" Atiyah–Singer index theorem, which generalizes Riemann–Roch type theorems for smooth varieties and the Chern–Gauss–Bonnet theorem. Thus in the appearance of a base parameter space, the FIT is a smooth version of Grothendieck–Riemann–Roch theorems.

Rather than discussing the characteristic classes built from the symbols of general D_b, we will discuss the particular case of families of coupled Dirac operators; in fact, a K-theory argument shows that proving the FIT for coupled Dirac operators implies the full FIT. (Coupled Dirac operators are discussed in e.g., [24].) So assume that M and every fiber Z_b are orientable and spin in a compatible way, and that $E = S^+ \otimes K, F = S^- \otimes K$, where $S^\pm$ are the spinor bundles for M and K is yet another bundle over M. Put a metric on M; the restriction of the metric to each Z_b defines a Dirac operator on $S_b^\pm$. Put a connection ∇^K on K. This induces connections ∇_b on each K_b, and gives a family of coupled Dirac operators $\partial\!\!\!/^{\nabla^K} = \partial\!\!\!/^{\nabla_b} : E_b \longrightarrow F_b$.

Let $\hat{A}(M)$ be the $\hat{A}$-genus of M, and let $\int_Z$ denote integration over the fiber (i.e., capping with Z as a class in $H_*(M, \mathbb{Q})$).

Theorem 6.1. (FIT) $ch(\mathrm{IND}(\partial\!\!\!/^\nabla)) = \int_Z \hat{A}(M) \cup ch(K)$ in $H^{\mathrm{ev}}(B, \mathbb{Q})$.

If $\pi_! : K(M) \longrightarrow K(B)$ is the analytic pushforward map, which by definition sends H to $\mathrm{IND}(\partial\!\!\!/^{\nabla^H})$, then the FIT can be restated as the commutativity of the diagram

$$
\begin{array}{ccc}
K(M) & \xrightarrow{\ ch\ } & H^{\mathrm{ev}}(M, \mathbb{Q}) \\
\pi_! \downarrow & & \downarrow \int_Z \hat{A} \cup (\cdot) \\
K(B) & \xrightarrow{\ ch\ } & H^{\mathrm{ev}}(B, \mathbb{Q}).
\end{array}
$$

As in the last section, the structure group of the fibration is $\mathrm{Diff}(Z)$, but there is more going on. The bundle $E \longrightarrow M$ pushes down to the infinite rank bundle $\pi_* E = \mathcal{E} \longrightarrow B$, where the fiber $\mathcal{E}_b$ is the smooth sections of E_b. (Thus $\mathcal{E}$ is the sheaf theoretic pushdown of the sheaf $\Gamma(E)$.) It is easy to check that $\mathcal{E}_b$ is modeled on $\Gamma(F)$ for some bundle $F \xrightarrow{p} Z$ of rank equal to $\mathrm{rank}(E)$.

The structure group of $\mathcal{E}$ is

$$
\mathbb{G} = \left\{
\begin{array}{ccc}
F & \xrightarrow{\ f\ } & F \\
p \downarrow & & p \downarrow \\
Z & \xrightarrow{\ \phi\ } & Z
\end{array}
: \phi \in \mathrm{Diff}(Z), f|_{F_z} \text{ a linear isomorphism}
\right\}.
$$

This just says that when fibers of M are glued by a diffeomorphism of Z over $b \in B$, E_b must be glued by a bundle isomorphism. $\mathbb{G}$ is called $\mathrm{Diff}(Z, F)$ in [3]. These transition maps act pointwise on $\Gamma(F)$ (within a fixed Sobolev class), the model space for the fibers of $\mathcal{E}$, by

$$
s \mapsto fs\phi^{-1}, \quad \text{i.e.,} \quad s \mapsto [z \mapsto f(z)(s(\phi^{-1}(z)))].
$$

We note that this equation defines a faithful action of $\mathbb{G}$ on $\Gamma(F)$, which is a Hilbert space once we fix a Sobolev class of sections. The tangent space to $\mathbb{G}$ at a pair (ϕ, f) is given by [31]:

$$T_{(\phi,f)}\mathbb{G} = \left\{ \begin{array}{ccc} F & \xrightarrow{\ s\ } & f^*TF \\ p\downarrow & f^*p_*\downarrow & : s|_{F_z} \text{ linear} \\ Z & \xrightarrow{\ V\ } & \phi^*TZ \end{array} \right\}.$$

This follows from thinking of ϕ as an element of $\text{Maps}(Z, Z)$ and similarly for f, and calculating as in previous sections. In particular, the Lie algebra $\mathfrak{g} = T_{(id,id)}\mathbb{G}$ is

$$\mathfrak{g} = \left\{ \begin{array}{ccc} F & \xrightarrow{\ s\ } & TF \\ p\downarrow & p_*\downarrow & : s|_{F_z} \text{ linear} \\ Z & \xrightarrow{\ V\ } & TZ \end{array} \right\}.$$

The difficulty of implementing Chern–Weil theory is the following:

Open question: Find a nontrivial $\text{Ad}_{\mathbb{G}}$-invariant function on $\mathfrak{g}$.

Remark 6.2. (i) The subgroup of $\mathbb{G}$ where ϕ is the identity is precisely the gauge group $\mathcal{G}(F)$ of F, so we are looking for generalizations of the invariant functions in Section 3. The structure group restricts to $\mathcal{G}(F)$ if the fibration $Z \longrightarrow M \longrightarrow B$ is trivial. In particular, if the fibration is $N \longrightarrow \text{Maps}(N, M) \times N \longrightarrow \text{Maps}(N, M)$ and $E = \text{ev}^*TM \longrightarrow \text{Maps}(N, M)$, then $\mathcal{E}$ is precisely $T\text{Maps}(N, M)$.

(ii) As a subcase of (i), fix a compact group G, let E be a trivial G-bundle, and let $S^1 \longrightarrow M = B \times S^1 \longrightarrow B$ be a trivial circle fibration. Then $\mathcal{G}(F)$ is the loop group LG. This so-called caloron correspondence between G-bundles over M and LG-bundles over B is discussed thoroughly in [30] and goes back to work of Garland and Murray [14]. In particular, characteristic classes for LG-bundles are constructed by Murray and Vozzo in [30]; the characteristic classes treated below reduce to the Murray–Vozzo classes in this case.

We can avoid answering the Open Question and still define characteristic classes in a restricted sense. We first recall the construction of a connection on $\mathcal{E}$ due to Bismut [5]. Let HM be a complement to the vertical bundle $VM = \ker \pi_*$ in TM. For example, if we have chosen a metric on M, we can take $HM = (VM)^\perp$. Recall that $E \longrightarrow M$ has a connection ∇^E; for a given hermitian metric on E, we may assume that ∇^E is a hermitian connection. The Bismut connection $\nabla = \nabla^B$ on $\mathcal{E} \longrightarrow B$ is defined by

$$\nabla_X r(b)(z) = \nabla^E_{X^H} \tilde{r}(b, z), \tag{6.1}$$

where $X \in T_b B, r \in \Gamma(\mathcal{E}), z \in \pi^{-1}(b)$, X^H is the horizontal lift of X to $HM_{(b,z)}$, and $\tilde{r} \in \Gamma(E)$ is defined by $\tilde{r}(b, z) = r(b)(z)$. (Here we abuse notation a little by writing (b, z), which assumes that a local trivialization of $Z \longrightarrow M \longrightarrow B$ has been given.)

Remark 6.3. We outline Bismut's heat equation proof of the FIT. Bismut first adjusts ∇^B to be unitary with respect to the L^2 hermitian metric on $\mathcal{E}$. He then modifies the new connection in a nontrivial way to form a superconnection ∇_t on $\mathcal{E}$ for $t > 0$. The "curvature" two-form ∇_t^2 acts fiberwise on $\mathcal{E}$, just as for finite rank bundles, and takes values in smoothing (and hence trace class) operators. As $t \longrightarrow \infty$, the form-valued operator trace $\mathrm{Tr}(\nabla_t^2)$ converges to a representative of the Chern character of the index bundle; this step is not too difficult in light of the original heat equation proof of the index theorem. As $t \longrightarrow 0$, $\mathrm{Tr}(\nabla_t^2)$ converges nontrivially to the differential form representative of the right hand side of the FIT. It is not hard to show that the two limits differ by an exact form, so their cohomology classes are the same.

This is called the local form of the FIT, since the proof generates the specific characteristic forms in the right cohomology class.

The (easy) Bismut connection fits into the Atiyah–Singer framework as follows:

Lemma 6.4. [21] The Bismut connection is a $\mathbb{G}$-connection. In a fixed local trivialization, the connection one-form assigns to $X \in T_b B$ the pair $(V, s) \in \mathfrak{g}$, where $V = \dot{\phi}_t(0) = X^H$ and $s(v)_{(b,z)} = (d/dt)_{t=0}\|_{0,t}(z)v$.

Here the parallel translation $\|_{0,t}$ for the Bismut connection is thought of as a bundle isomorphism of F via a local trivialization. The proof directly shows that the holonomy of the Bismut connection lies in $\mathbb{G}$.

This suggests that if we can define the Chern character for connections on $\mathcal{E}$ for the coupled Dirac operator case (i.e., $\mathcal{E}$ is the superbundle associated to $(S^+ \otimes K) \oplus (S^- \otimes K)$), then we could try to prove the local FIT by showing

(i) For the (easy) Bismut connection on $\mathcal{E}$, the representative differential form $ch(\Omega^B)$ of the Chern character $ch(\mathcal{E})$ equals $\int_Z \hat{A}(\Omega^M)ch(\Omega^K)$, where Ω^M is the curvature of the Levi-Civita connection for the metric on M, and Ω^K is the curvature of the connection on K.

(ii) There exists a connection on $\mathcal{E}$ for which $ch(\mathcal{E}) = ch(\mathrm{IND}(\bar{\partial}^{\nabla^K})) \in H^{\mathrm{ev}}(B, \mathbb{Q})$.

Since the Chern character should be independent of the connection on $\mathcal{E}$, this would give the FIT.

As we will now see, step (i) fails, but in a very precise way.

For a fixed hermitian connection ∇^E on E, the associated Bismut connection has curvature two-form Ω^B taking values in $\mathfrak{g}$, so we can write $\Omega^B(X, Y) = (V, s)$ for $X, Y \in T_b B$, $V \in \Gamma(TZ)$, and $s \in \Gamma(TF)$. With respect to a local trivialization, we can consider $s \in \Gamma(TE_b)$. The connection ∇^E induces a connection on E_b, or equivalently gives a splitting $TE_b = VE_b \oplus HE_b$. The vertical component $(\Omega^B)^v = s^v \in VE_b$ can naturally be identified with a map $s^v : E_b \longrightarrow E_b$. Since s covers V, it easily follows that $s^v \in \mathrm{End}(E_b)$, i.e., s^v is a fiberwise endomorphism of E_b. Thus we can create forms

$$b \mapsto c_k(\Omega_b^B) = \int_Z \mathrm{tr}((\Omega_b^B)^v)^k \in \Lambda^{2k-\dim(Z)}(B). \tag{6.2}$$

We claim that these forms are closed and have de Rham class independent of the choice of ∇ on E. To see this, first note that it is well known and not difficult to compute that the Bismut connection on $\mathcal{E}$ has curvature

$$\Omega^B(\xi_1, \xi_2) = \nabla^E_{T(\xi_1^H, \xi_2^H)} + R^E(\xi_1^H, \xi_2^H), \tag{6.3}$$

with $T(\xi_1^H, \xi_2^H) = [\xi_1, \xi_2]^H - [\xi_1^H, \xi_2^H]$ and R^E the curvature of ∇^E. Moreover, the first term on the right-hand side of (6.3) is horizontal and the second term is vertical with respect to the splitting of TE_b. Thus $(\Omega^B)^v = R^E$. Then for $k = 1$ for simplicity, we have

$$\begin{aligned}
d_B \int_Z \mathrm{tr}(\Omega^B)^v &= \int_Z d_M \, \mathrm{tr}(R^E) = \int_Z \mathrm{tr} \, \nabla^{\mathrm{Hom}}(R^E) \\
&= \int_Z \mathrm{tr}[\nabla, R^E] = 0.
\end{aligned}$$

The equality $d_M \, \mathrm{tr}(R^E) = \mathrm{tr} \, \nabla^{\mathrm{Hom}}(R^E)$ is an easy calculation using the fact that R^E is skew-hermitian, and the last line uses the Bianchi identity.

Thus we have constructed characteristic classes, and in particular a Chern character, for the restricted class of Bismut connections without finding $\mathrm{Ad}_{\mathbb{G}}$-invariant functions on $\mathfrak{g}$. Note that in the case of Remark 6.2, these classes reduce to the classes discussed in Section 3, since for gauge transformations $\Omega = \Omega^v$.

The Chern character of the infinite rank superbundle associated to a family of coupled Dirac operators is

$$ch(\mathcal{E}) = \int_Z ch(R^E) = \int_Z ch(\Omega^{S^+ - S^-}) \cup ch(\Omega^K). \tag{6.4}$$

Since we have $ch(\Omega^{S^+ - S^-})$ and not $\hat{A}(\Omega^M)$, this is not what we wanted!

Despite this failure, we now see what we can do with step (ii). We want to mimic the usual "cancellation of nonzero eigenspaces" in the heat equation proof of the index theorem, i.e., we want a connection that respects the splitting

$$(S^+ \otimes K)_b = \ker(\eth_b^{\nabla^K}) \oplus \ker^\perp_+,$$

where $\ker^\perp_+$ equals $(\ker(\eth_b^{\nabla^K}))^\perp$, and similarly for $S^- \otimes K$. This perpendicular component is spanned by the eigensections of $\eth^{\nabla^K} = \eth_+$ with nonzero eigenvalues, and $\eth^{\nabla^K}$ is an isomorphism between these eigenspaces. For this connection, we expect that $ch(\ker^\perp_+) = ch(\ker^\perp_-)$ as forms computed with respect to this split connection. This would imply

$$ch(\mathrm{IND}(\eth^{\nabla^K})) = ch(\ker(\eth_+^{\nabla^K})) + ch(\ker^\perp_+) - ch(\ker(\eth_-^{\nabla^K})) - ch(\ker^\perp_-) = ch(\mathcal{E}). \tag{6.5}$$

This would finish (ii).

The natural choice for such a connection is given by orthogonally projecting the Bismut connection to the kernel and its perpendicular complement. The problem is that the isomophism $\eth_+^{\nabla^K} : (\ker \eth_+)^\perp \longrightarrow (\ker \eth_-)^\perp$ is not a $\mathbb{G}$-isomorphism, since

$\bar{\partial}_+^{\nabla^K}$ is far from an element of $\mathbb{G}$. However, we can replace $D = \bar{\partial}_+^{\nabla^K}$ with its unitarization $D^u = D/|D^*D|^{1/2}$ on these complements. D^u is a zero order invertible pseudodifferential operator, precisely the type of operator treated in Section 4.

This motivates extending the structure group to a semidirect product $\tilde{\mathbb{G}} = \mathbb{G} \ltimes \Psi_0^*$, with $\mathbb{G}$ acting on Ψ_0^* by conjugation. We have to extend our definition of characteristic classes from $\mathbb{G}$ to $\tilde{\mathbb{G}}$, but this is straightforward based on the earlier work: for a $\tilde{\mathbb{G}}$-connection with curvature $\tilde{\Omega} = (V, s, B) \in \tilde{\mathfrak{g}}$ (so $B \in \Psi_{\leq 0}$), we consider expressions like

$$\int_{Z_b} \mathrm{tr}((s^{v^{\nabla_b}})^k) \mathrm{dvol}_{M/B} + \int_{Z_b} \mathrm{tr}((\sigma_0(B))^k).$$

It is now straightforward to check that (6.5) holds. However, we are not claiming that the Chern character form for this projected connection in (6.5) is closed. Even if it is, we are definitely not claiming that it is cohomologous to the Chern character form in (6.5), as this would give the wrong formula for the FIT.

Despite the glaring problems with these arguments, we see that when we try to prove the FIT directly on $\mathcal{E}$, the extended structure group $\tilde{\mathbb{G}}$ naturally occurs.

Remark 6.5. Recall that the starting point for Donaldson theory is the moduli space $\mathcal{A}/\mathcal{G}$, where $\mathcal{A} = \mathcal{A}(F)$ is the space of connections on F and $\mathcal{G} = \mathcal{G}(F)$ is the gauge group. The action of $\mathcal{G}$ on $\mathcal{A}$, $g \cdot \nabla = g\nabla g^{-1}$, extends to an action of $\mathbb{G}$ by $(\phi, f) \cdot \nabla = f(\phi^{-1})^* \nabla f^{-1}$, since for $\phi = Id$, f is a gauge transformation.

Thus it is natural to consider the moduli space $\mathcal{A}/\mathbb{G}$. However, this space seems to be a fat point in the following sense.

Conjecture 6.6. [21] The orbit of a generic connection is dense.

An example of a nongeneric connection is a flat connection. As justification for the conjecture, it can be shown that the normal space to the $\mathbb{G}$-orbit $\mathcal{O}_\nabla$ in the L^2 metric on $T_\nabla \mathcal{A} = \Lambda^1(Z, \mathrm{End}(F))$ is

$$\{\alpha \in \Lambda^1(Z, \mathrm{End}(F)) : \nabla^*\alpha = 0 \text{ and } R^F(\cdot, V)_z \perp \alpha_z, \forall z \in Z, \forall V \in T_z Z\}.$$

(The equation $\nabla^*\alpha = 0$ is the equation for the normal space to the gauge orbit of ∇.) It is plausible that for a generic connection, the curvature equation $R^F(\cdot, V)_z \perp \alpha_z$ and its higher covariant derivatives form an overdetermined system of equations, and so has only the zero solution. We expect standard gauge theory techniques to help prove the conjecture.

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