

CANARDS AND BIFURCATION DELAYS OF SPATIALLY HOMOGENEOUS AND INHOMOGENEOUS TYPES IN REACTION-DIFFUSION EQUATIONS

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Abstract. In ordinary differential equations of singular perturbation type, the dynamics of solutions near saddle-node bifurcations of equilibria are rich. Canard solutions can arise, which, after spending time near an attracting equilibrium, stay near a repelling branch of equilibria for long intervals of time before finally returning to a neighborhood of the attracting equilibrium (or of another attracting state). As a result, canard solutions exhibit bifurcation delay. In this article, we analyze some linear and nonlinear reaction-diffusion equations of singular perturbation type, showing that solutions of these systems also exhibit bifurcation delay and are, hence, canards. Moreover, it is shown for both the linear and the nonlinear equations that the exit time may be either spatially homogeneous or spatially inhomogeneous, depending on the magnitude of the diffusivity.

1. INTRODUCTION

1.1. Canards in singularly perturbed ODEs. Canards play a central role near bifurcations in singularly perturbed ordinary differential equations

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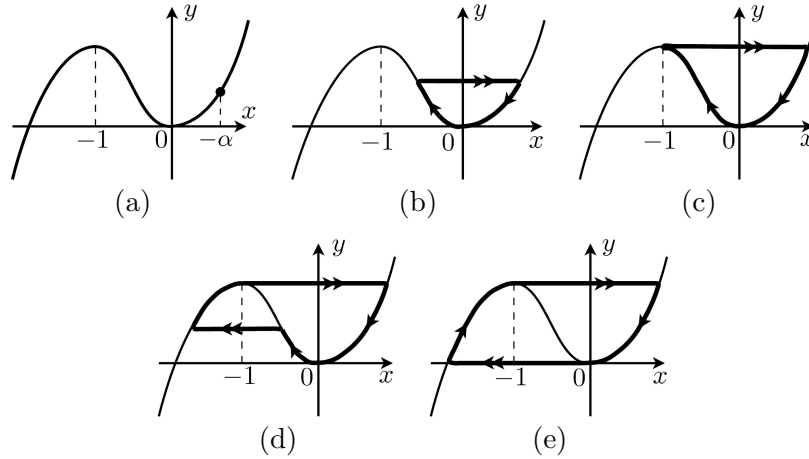


FIGURE 1. Hopf bifurcation in the van der Pol equation (1.1).

(ODEs). A prototypical example is given by the singularly perturbed van der Pol equation

$$\begin{cases} \epsilon \dot{x} = y - f(x), \\ \dot{y} = -(x + \alpha), \end{cases} \quad (1.1)$$

where x and y are real variables, the overdot indicates the derivative with respect to time t , $0 < \epsilon \ll 1$, α is a real number, and $f(x)$ is a cubic function. In particular, we use $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$ for illustration in this discussion and in Figure 1, which is of the class of van der Pol type oscillators studied in Section 1.2 of [10]. While Hopf bifurcations occur at $\alpha = 0$ and $\alpha = 1$ in (1.1), we focus on the former case here, noting that similar results hold in the latter. For $\alpha < 0$, the fixed point at $x = -\alpha$ is attracting, see Figure 1(a), while it is repelling for $0 < \alpha < 1$. For $0 < \epsilon \ll 1$, limit cycles exist. The limit cycle in frame (e) is a classical relaxation oscillation. In the frames in between, the limit cycles have increasingly larger amplitude and, most interestingly, spend a significant, $\mathcal{O}(1)$ amount of slow time near the middle, repelling (unstable) branch of the fast nullcline. These limit cycles were discovered in [7] and were labeled canards, due to their resemblance to ducks, without and with heads, see frames (b)–(d) in Figure 1.

The family of canards in (1.1) exists in an exponentially narrow interval of parameter values. In frame (c), $\alpha = \alpha_c(\epsilon)$, where $\alpha_c(\epsilon)$ is a critical value which vanishes as $\epsilon \rightarrow 0^+$, and which corresponds to the maximal headless canard. In frames (b) and (d), $\alpha = \alpha_c(\epsilon) + \sigma e^{-k^2/\epsilon}$ for some $\sigma < 0$ and

$\sigma > 0$, respectively, and some $k > 0$. A partial listing of references in which these canards have been studied includes [1, 8, 9, 10, 11, 12].

More generally, a canard solution of a singularly perturbed ODE is a solution that stays near a repelling slow manifold for an $\mathcal{O}(1)$ amount of slow time [7, 3]. Other examples of planar singularly perturbed ODEs that exhibit canard dynamics include the FitzHugh-Nagumo system [2], the generalized Bonhoeffer-van der Pol equations [4], and the generalized Rayleigh equations [4]. Singularly perturbed ODEs in \mathbb{R}^n with $n \geq 3$ can also possess canards, see for example [3, 5, 13, 16, 17, 18]. Moreover, in these systems, the canards are generic phenomena, in that they exist in $\mathcal{O}(\epsilon^\beta)$ intervals of parameter values, for some $\beta > 0$, rather than in the exponentially narrow intervals characteristic for planar systems.

Closely related to the notion of canards is the phenomenon of bifurcation delay. In the case of the van der Pol equation (1.1), as we just described, for values of α exponentially close to $\alpha_c(\epsilon)$, the canard solution stays close to the repelling manifold for an $\mathcal{O}(1)$ amount of slow time. In other words, the solution does not immediately feel the loss of stability of the fixed point at the bifurcation. Rather, it takes an $\mathcal{O}(1)$ amount of slow time before the solution jumps across to an attracting slow manifold; recall Figure 1(d), for example.

1.2. Canards in linear PDEs: first result. In this article, we study the phenomena of canards and bifurcation delay in partial differential equations (PDEs) of reaction-diffusion (RD) type. In order to fix ideas, we concentrate first and foremost on an RD equation with linear kinetics, much simpler than those of the van der Pol system (1.1), but still chosen so that the corresponding ODE possesses canard solutions exhibiting bifurcation delay. In particular, the first set of main results (see Theorem 1 below) concerns the scalar RD equation

$$\begin{cases} \epsilon(u_t - u_{xx}) = a(x, t, \epsilon)u, \\ u_x(0, t) = u_x(1, t) = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.2)$$

where $a = a(x, t, \epsilon)$ is chosen so that there is a smooth curve of turning points $t = t_*(x) > 0$, with

$$a(x, t, 0) < 0, \quad \forall t < t_*(x), \quad (1.3)$$

$$a(x, t, 0) > 0, \quad \forall t > t_*(x), \quad (1.4)$$

and the initial data $u_0(x)$ is assumed to be bounded and strictly positive on $[0, 1]$.

Remark. The restriction to nonzero u_0 implies the presence of a boundary layer in (1.2) at $t = 0$ for small ϵ , which is necessary for our argument, see e.g. Section 2 below. In particular, our results do not cover the important case of compactly supported initial data, which is left for future study.

The ODE for the kinetics associated to the PDE (1.2) is

$$\epsilon \frac{du}{dt} = a(t, \epsilon)u, \quad u(0) = u_0 > 0. \quad (1.5)$$

The conditions we impose on a are similar to those above. In particular, we assume that a is a smooth function (C^k in t and ϵ for $k \geq 1$). Moreover, we assume that a changes sign at some $t = t_*$, with $a(t, 0) < 0$ for $t \in [0, t_*)$ and $a(t, 0) > 0$ for $t \in (t_*, \infty)$, respectively (t_* is called the *turning point*). Under these conditions, the ODE (1.5) exhibits bifurcation delay, which can be seen as follows: due to the simplicity of this model, we can calculate the solution explicitly,

$$u(t) = u_0 \exp\left(\frac{1}{\epsilon} \int_0^t a(s, \epsilon) ds\right).$$

Letting t_{exit} denote the unique, strictly positive time for which

$$\int_0^{t_{\text{exit}}} a(s, 0) ds = 0,$$

we have the following interesting property:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} u(t, \epsilon) &= 0, & \forall t \in (0, t_{\text{exit}}), \\ \lim_{\epsilon \rightarrow 0} u(t, \epsilon) &= \infty, & \forall t \in (t_{\text{exit}}, \infty). \end{aligned} \quad (1.6)$$

Hence, the time at which the solution is repelled from the equilibrium at $u = 0$ is given by t_{exit} , and is greater than the time t_* at which the equilibrium loses its stability.

Looking back at the PDE (1.2), we are interested in the effect the diffusion term has on the equations. To that end, it is instructive to see first what the effect of an additional term in the ODE is. Consider the equation

$$\epsilon \frac{du}{dt} = a(t, \epsilon)u + b(t, \epsilon), \quad u(0) = u_0 > 0, \quad (1.7)$$

with similar conditions imposed on a and assuming that b is equally smooth. It is well-known that there is again a well-defined exit time t_{exit} for which the

solution u of (1.7) satisfies (1.6), see e.g. [6]. However, the exit time in this case will be earlier than the exit time in the case where $b = 0$. Indeed, for typical nonzero b , one has $t_{\text{exit}} = t_*$, and there is no delay in the bifurcation. In case b is exponentially small, the exit is delayed, just as in the case when $b = 0$. To illustrate this point further, we calculate explicitly the solution of (1.7),

$$u(t, \epsilon) = u_0 e^{\frac{1}{\epsilon} \int_0^t a(s, \epsilon) ds} + \frac{1}{\epsilon} \int_0^t e^{\frac{1}{\epsilon} \int_r^t a(s, \epsilon) ds} b(r, \epsilon) dr.$$

If, for example, $b = e^{-B/\epsilon}$ for some $B > 0$ independent of ϵ , i.e., if b is exponentially small in ϵ , we find

$$\begin{aligned} u(t, \epsilon) &= u_0 e^{\frac{1}{\epsilon} \int_0^t a(s, \epsilon) ds} + \mathcal{O}\left(e^{\frac{1}{\epsilon} \max_{r \in [0, t]} \int_r^t a(s, \epsilon) ds - B}\right) \\ &= u_0 e^{\frac{1}{\epsilon} \int_0^t a(s, \epsilon) ds} + \mathcal{O}\left(e^{\frac{1}{\epsilon} \int_{t_*}^t a(s, \epsilon) ds - B}\right), \end{aligned}$$

and the exit time t_{exit} is given by the smaller of the times satisfying

$$\int_{t_*}^{t_{\text{exit}}} a(t, 0) dt = B \quad \text{or} \quad \int_0^{t_{\text{exit}}} a(t, 0) dt = 0.$$

Clearly, $t_{\text{exit}} > t_*$ marks the time at which one of the terms in the solution becomes exponentially large in the limit as $\epsilon \rightarrow 0$, i.e., the time at which u escapes to infinity. Following the terminology introduced e.g. in [7, 3] and the discussion above, the corresponding solution will again be termed a canard solution.

Remark. In the case where $\int_0^{t_*} a(t, 0) dt = B$, the exit time cannot be decided from the above calculations.

Given the above discussion of canards and bifurcation delay in the van der Pol equation (1.1) and in the simple ODE (1.7), we now turn to the PDE (1.2) and ask whether or not solutions of (1.2) display the same exit time behavior. This question was first considered by Nefedov and Schneider in [14], where they prove that bifurcation delay occurs, but without deriving expressions for the exit time. In this article, we establish exact expressions for the exit time in RD equations, including for those equations considered in [14].

Theorem 1. *Given the smooth RD equation (1.2) with bounded, strictly positive initial data $u_0(x)$, assume that the smooth curve $t = t_*(x) > 0$ satisfies conditions (1.3) and (1.4).*

Then, for any $\delta > 0$, there exist $\epsilon_0 > 0$ and positive constants L_0 and U_0 such that for all $\epsilon \in]0, \epsilon_0]$, there is a solution $u(x, t; \epsilon)$ of (1.2) satisfying

$$L_0 \exp \frac{A(t) - \delta}{\epsilon} \leq u(x, t; \epsilon) \leq U_0 \exp \frac{A(t) + \delta}{\epsilon} \tag{1.8}$$

for all $(x, t) \in [0, 1] \times [0, t_{\max}]$, where

$$A(t) = \int_0^t \left[\max_x a(x, s, 0) \right] ds.$$

As a corollary to this theorem, we identify the exit time t_{PDEexit} as a (nontrivial) zero of $A(t)$. Moreover, we see that this time is independent of x . Indeed, from the above theorem it follows that t_{PDEexit} is located between the zero of $A(t) + \delta$ and that of $A(t) - \delta$. Since $\delta > 0$ is arbitrary, this shows that

$$\int_0^{t_{\text{PDEexit}}} \left[\max_x a(x, t, 0) \right] dt = 0, \tag{1.9}$$

and in particular also that the exit time is independent of x . Intuitively, one can think of the diffusion as being fast enough to homogenize the solution with respect to x .

1.3. Inhomogeneity of exit times in linear PDEs: second result. The second main set of results we present quantifies more precisely the impact of the magnitude of the diffusivities on the exit times. Specifically, we analyze RD equations with diffusivity $D\epsilon^\alpha$, where $D > 0$ and $\alpha > 0$:

$$\begin{cases} \epsilon u_t - D\epsilon^\alpha u_{xx} = a(x, t, \epsilon)u, \\ u_x(0, t) = u_x(1, t) = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.10}$$

again with conditions (1.3) and (1.4) on the function a . Clearly, equation (1.2) corresponds to the special case of $\alpha = 1$ in (1.10). In the following theorem, we show that the conclusions of Theorem 1 continue to hold for $\alpha < 2$, i.e., that the effect of diffusion is to homogenize the exit times. By contrast, we show that for smaller diffusivities, i.e., if $\alpha > 2$, the exit time is no longer homogeneous. Specifically, we prove

Theorem 2. *Given the smooth RD equation (1.10) with $D > 0$ and $\alpha > 0$ and bounded, strictly positive initial data $u_0(x)$, assume that conditions (1.3) and (1.4) are satisfied.*

Then, for $\alpha < 2$, the solution of (1.10) has a homogeneous exit time determined by (1.9). For $\alpha > 2$, the solution of (1.10) has a non-homogeneous

exit time equal to the exit time of the associated ODE, i.e., $t_{\text{PDEexit}}(x) = t_{\text{exit}}(x)$, where $t_{\text{exit}}(x) > t_*(x)$ is defined by the relation

$$\int_0^{t_{\text{exit}}(x)} a(x, t, 0) dt = 0. \quad (1.11)$$

Remark. The homogeneous exit time in equation (1.2) (or in (1.10) with $\alpha < 2$) may be earlier than the earliest exit time $t_{\text{exit}}(x)$, i.e., one does not necessarily have $t_{\text{PDEexit}} = \min_x t_{\text{exit}}(x)$. The reason is of course that, in general,

$$\int_0^t \left[\max_x a(x, t, 0) \right] dt \neq \max_x \int_0^t a(x, t, 0) dt.$$

As an example, consider equation (1.2) with $a(x, t, \epsilon) = t^2 - x^2t + x - 2$. One calculates that

$$\int_0^t a(x, s, 0) ds = t \left(\frac{t^2}{3} - \frac{x^2}{2}t + x - 2 \right),$$

and $t_{\text{exit}}(x) = \frac{3}{4}x^2 + \frac{1}{4}\sqrt{9x^4 - 48x + 96}$. The minimum value numerically evaluates to $\min_x t_{\text{exit}}(x) \approx 2.31$. On the other hand, the maximum value of $a(x, t, 0)$ is reached at $x = 1$ for $t \leq \frac{1}{2}$ and at $x = \frac{1}{2t}$ for $t > \frac{1}{2}$. A simple integration and numerical solve leads to $t_{\text{PDEexit}} \approx 2.23$.

In the boundary case where $\alpha = 2$ in (1.10), it is to be expected that the exit point gradually changes from the inhomogeneous time $t_{\text{exit}}(x)$ to the homogeneous exit point defined in (1.9) as the coefficient D in the diffusivity is increased. We leave the investigation of this transition case (which corresponds to the parabolic scaling with ϵ in time and ϵ^2 in space) for future study.

1.4. Canards in a class of nonlinear PDEs. Finally, in this article, we show the existence of canards with spatially homogeneous and inhomogeneous exit times, respectively, in the following class of scalar, nonlinear RD equations:

$$\begin{cases} \epsilon u_t - \epsilon^\alpha u_{xx} = f(u, x, t, \epsilon)u, \\ u_x(0, t) = u_x(1, t) = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.12)$$

where f is C^2 and defined for all $u \in [-K, K]$ with $K > 0$ fixed, $x \in [0, 1]$, $t \in [0, t_{\text{max}}]$, $\alpha > 0$, and $\epsilon > 0$. The linear part

$$a(x, t, \epsilon) := f(0, x, t, \epsilon) \quad (1.13)$$

of f is assumed to satisfy conditions (1.3) and (1.4), i.e., a turning point curve $t = t_*(x)$ exists, separating attracting from repelling behavior. Let us first describe the dynamics in the case of large diffusivities:

Theorem 3. *Given the RD equation (1.12) with $\alpha < 2$ and bounded, strictly positive initial data $u_0(x)$, assume that the linear part in (1.13) satisfies conditions (1.3) and (1.4). Also, assume that*

$$f(u, x, 0, 0) < 0, \quad \forall u \in [0, u_0(x)], \quad \forall x \in [0, 1], \quad (1.14)$$

i.e., that the initial condition lies in the basin of attraction of the attractor $u = 0$. Then, equation (1.12) has a positive solution $u(x, t; \epsilon)$, with a homogeneous exit time t_{PDEexit} given by

$$\int_0^{t_{\text{PDEexit}}} \left[\max_x f(0, x, t, 0) \right] dt = 0.$$

In the case of small diffusivities, with $\alpha > 2$, we need one additional condition on f . In fact, in this case we assume that there is bistability, in the sense that the solution u , after being repelled away from zero, has to tend to a secondary equilibrium instead of blowing up to infinity.

Theorem 4. *Given the RD equation (1.12) with $\alpha > 2$ and bounded, strictly positive initial data $u_0(x)$, assume that the linear part in (1.13) satisfies conditions (1.3) and (1.4). Also, assume that*

$$f(u, x, 0, 0) < 0, \quad \forall u \in [0, u_0(x)], \quad \forall x \in [0, 1],$$

i.e., that the initial condition lies in the basin of attraction of the attractor $u = 0$. Finally, assume that beyond the turning point, there is a simple positive zero of $f(u, x, t, 0)$, denoted $\tilde{u}(x, t)$, and that

$$0 \leq f(u, x, t, 0) \leq f(0, x, t, 0), \quad \forall u \in [0, \tilde{u}(x, t)], \quad \forall t > t_*(x), \quad \forall x \in [0, 1]. \quad (1.15)$$

Then, equation (1.12) has a positive solution $u(x, t; \epsilon)$, with an inhomogeneous exit time $t_{\text{exit}}(x)$ given by

$$\int_0^{t_{\text{exit}}(x)} f(0, x, t, 0) dt = 0.$$

2. PROOF OF THEOREM 1

The technique we use for proving Theorem 1 is the so-called method of lower and upper solutions, see also [15]. We recall the relevant definitions here:

Definition 1. Let $\epsilon > 0$ be fixed. Let $L(x, t)$ be a continuous function, twice continuously differentiable with respect to x and once with respect to t . The function $L(x, t)$ is called a lower solution of (1.2) if

$$\begin{aligned}\epsilon(L_t - L_{xx}) - a(x, t, \epsilon)L &\leq 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq t_{\max}, \\ L(x, 0) &\leq u_0(x), & 0 \leq x \leq 1, \\ L_x(0, t) \geq 0 &\geq L_x(1, t), & 0 \leq t \leq t_{\max}.\end{aligned}$$

Let $U(x, t)$ be a continuous function, twice continuously differentiable with respect to x and once with respect to t . The function $U(x, t)$ is called an upper solution of (1.2) if

$$\begin{aligned}\epsilon(U_t - U_{xx}) - a(x, t, \epsilon)U &\geq 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq t_{\max}, \\ U(x, 0) &\geq u_0(x), & 0 \leq x \leq 1, \\ U_x(0, t) \leq 0 &\leq U_x(1, t), & 0 \leq t \leq t_{\max}.\end{aligned}$$

The PDE (1.2) is of the type analyzed in [15]. Hence, we make use of the following result to construct solutions for (1.2) and to prove Theorem 1 :

Theorem 5. [15] Let $\epsilon > 0$ be fixed. Let (L, U) be a pair of lower and upper solutions of (1.2), and assume that $L(x, t) \leq U(x, t)$ on $[0, 1] \times [0, t_{\max}]$. Then, (1.2) has a unique solution $u(x, t; \epsilon)$ that lies between $L(x, t)$ and $U(x, t)$.

In the proof of Theorem 1, we will first apply Theorem 5 to show the existence and smoothness of a solution for (1.2). Then, in a second step, we will establish sharp bounds on that solution. To that end, we will apply Theorem 5 again, but this time with lower and upper solutions that are merely piecewise smooth. A precise definition of the latter is given as follows.

Definition 2. Let $\epsilon > 0$ be fixed, and let a finite number of time values $t_0 = 0, t_1, t_2, \dots, t_N = t_{\max}$ be given. For $i = 0, \dots, N$, let $L^{(i)}(x, t)$ be continuous functions, twice continuously differentiable with respect to x and once with respect to t on $[0, 1] \times [t_i, t_{i+1}]$. The function

$$L(x, t) = \begin{cases} L^{(0)}, & t \in [t_0, t_1], \\ L^{(1)}, & t \in]t_1, t_2], \\ \vdots \\ L^{(N-1)}, & t \in]t_{N-1}, t_{\max}] \end{cases}$$

is called a piecewise lower solution of (1.2) if, for each i ,

$$\epsilon(L_t^{(i)} - L_{xx}^{(i)}) - a(x, t, \epsilon)L^{(i)} \leq 0, \quad 0 \leq x \leq 1, \quad t_i \leq t \leq t_{i+1},$$

$$\begin{aligned}
 L^{(i)}(x, t_i) &\leq L^{(i-1)}(x, t_i), & 0 \leq x \leq 1, \quad i \neq 0, \\
 L_x^{(i)}(0, t) &\geq 0 \geq L_x^{(i)}(1, t), & t_i \leq t \leq t_{i+1}
 \end{aligned}$$

and if $L^{(0)}(x, 0) \leq u_0(x)$. A piecewise upper solution $U(x, t)$ can be defined in a similar fashion.

The following result is a straightforward consequence of Theorem 5 which shows that Pao’s theory also applies in this piecewise smooth case:

Corollary 1. *Let $\epsilon > 0$ be fixed. Let (L, U) be a pair of piecewise smooth lower and upper solutions of (1.2), and assume that $L(x, t) \leq U(x, t)$ on $[0, 1] \times [0, t_{\max}]$. If (1.2) has a unique smooth solution $u(x, t; \epsilon)$, then u lies between $L(x, t)$ and $U(x, t)$.*

Finding an upper solution of (1.2) is elementary. Let

$$U(x, t) = U_0 \exp\left(\frac{1}{\epsilon} A_\epsilon(t)\right),$$

with $U_0 = \max_x u_0(x)$ and $A_\epsilon(t) = \int_0^t [\max_x a(x, s, \epsilon)] ds$. It is readily checked that U is an upper solution. Since $A_\epsilon(t) = A(t) + \mathcal{O}(\epsilon)$, we can find a $\delta > 0$ so that $A_\epsilon(t) \leq A(t) + \delta$, which implies the upper bound given in the statement of Theorem 1, see (1.8).

The difficult part of the proof is to find a good lower solution. Of course, $u = 0$ is a trivial lower solution of (1.2), from which the existence of a solution can be derived. To obtain a better lower estimate, which is necessary for establishing the bounds in Theorem 1, we make the Ansatz

$$L(x, t) = L_0 \exp\left(\frac{1}{\epsilon} \int_0^t w(x, s) ds\right),$$

where $L_0 = \min_x u_0(x) > 0$. In the sequel, we keep ϵ fixed but small, and we sometimes suppress the dependence on ϵ in both L and w . In fact, w will depend on ϵ in a linear fashion. Note that

$$\begin{aligned}
 L_t(x, t) &= \frac{1}{\epsilon} L(x, t) \cdot w(x, t), \\
 L_x(x, t) &= \frac{1}{\epsilon} L(x, t) \cdot \int_0^t w_x(x, s) ds, \\
 L_{xx}(x, t) &= \frac{1}{\epsilon} L(x, t) \cdot \int_0^t w_{xx}(x, s) ds + \frac{1}{\epsilon^2} L(x, t) \cdot \left(\int_0^t w_x(x, s) ds\right)^2.
 \end{aligned}$$

In order to satisfy the definition of a lower solution, we need to find a function w so that

$$\begin{cases} w(x, t) - a(x, t, \epsilon) - \int_0^t w_{xx}(x, s) ds - \frac{1}{\epsilon} \left(\int_0^t w_x(x, s) ds \right)^2 \leq 0, \\ w_x(0, t) \geq 0 \geq w_x(1, t). \end{cases} \quad (2.1)$$

The construction of the function $w(x, t)$ constitutes the technical part of the proof of Theorem 1:

2.1. Construction of $w(x, t)$. For the construction, we require a uniform bound on a_x and a_ϵ . Hence, we define

$$M = \max \{ \sup |a_x(x, t, \epsilon)|, \sup |a_\epsilon(x, t, \epsilon)| \}, \quad (2.2)$$

where the supremum is taken for $x \in [0, 1], t \in [0, t_{\max}],$ and $\epsilon \in [0, \epsilon_0]$. Throughout, we also fix a choice of $\delta > 0$.

Step 1: Subdivision of the time interval. Applying uniform continuity, we find a sequence $t_0 = 0, t_1, \dots, t_N = t_{\max}$ so that for each of the intervals $[t_i, t_{i+1}]$, there is an $x_i \in [0, 1]$ with

$$a(x_i, t, 0) \geq \max_{x \in [0, 1]} a(x, t, 0) - \delta, \quad \forall t \in [t_i, t_{i+1}]. \quad (2.3)$$

(The sequence length $N + 1$ depends of course on δ .) In Steps 2 and 3 of the proof, we construct a function $w(x, t)$ that satisfies (2.1), restricted to the interval $[t_i, t_{i+1}]$:

$$\begin{cases} w(x, t) - a(x, t, \epsilon) - \int_{t_i}^t w_{xx}(x, s) ds - \frac{1}{\epsilon} \left(\int_{t_i}^t w_x(x, s) ds \right)^2 \leq 0, \\ w_x(0, t) \geq 0 \geq w_x(1, t). \end{cases} \quad t \in [t_i, t_{i+1}] \quad (2.4)$$

Step 2: Definition of $w(x, t)$ on the interval $[t_i, t_{i+1}]$. Consider the C^2 auxiliary function

$$\eta(x) = \begin{cases} -\frac{2}{3}\delta^3 - \delta^2(x - \delta), & x > \delta; \\ -\delta x^2 + \frac{1}{3}x^3, & 0 \leq x \leq \delta; \\ \eta(-x), & x < 0. \end{cases}$$

The graph defined by η connects a straight line with slope δ^2 (ending at $x = -\delta$) in a C^2 -fashion to a line with slope $-\delta^2$ (starting at $x = \delta$). It does this so that $\eta'' = \mathcal{O}(\delta)$. Now, define

$$w(x, t) = a(x_i, t, 0) + \eta(x - x_i) - C_i - D_i(t) - M\epsilon, \quad (2.5)$$

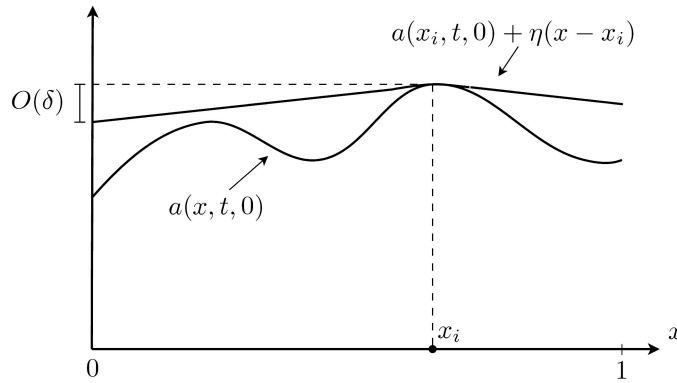


FIGURE 2. Construction of $w(x, t)$ on $[t_i, t_{i+1}]$.

where C_i and D_i are positive but yet to be determined. The first two terms in (2.5) give a curve that lies very close to $\max_x a(x, t, 0)$ near $x = x_i$ and that decreases a bit away from x_i in both directions (see Figure 2). The third and fourth terms lower the curve further in a spatially uniform manner. The fifth term finally introduces an ϵ -dependence, which is necessary because all previous terms only deal with $a(x, t, 0)$ (as we will see). From the definition of w , it follows immediately that $w_x(0, t) \geq 0 \geq w_x(1, t)$, which constitutes the second part of (2.4). We now establish that the first inequality also holds for a proper choice of C_i and $D_i(t)$.

Step 3: Verification of the first part of (2.4). For values of x near x_i , i.e., when $|x - x_i| \leq \delta$, we have

$$\int_{t_i}^t w_{xx}(x, s) ds = (t - t_i) \eta''(x - x_i).$$

Hence, using the definition of η , we find that this term in (2.4) is bounded by $(t_{i+1} - t_i)2\delta$. At the same time,

$$\begin{aligned} w(x, t) - a(x, t, \epsilon) &= a(x_i, t, 0) - a(x, t, \epsilon) + \eta(x - x_i) - C_i - D_i(t) - M\epsilon \\ &= a(x_i, t, 0) - a(x, t, 0) + \eta(x - x_i) - C_i - D_i(t) \\ &\quad - [a(x, t, \epsilon) - a(x, t, 0) + M\epsilon]. \end{aligned}$$

Using the Mean Value Theorem and the bound on a_ϵ in (2.2), one can see that the expression inside the square brackets is nonnegative. Hence,

$$w(x, t) - a(x, t, \epsilon) \leq a(x_i, t, 0) - a(x, t, 0) + \eta(x - x_i) - C_i - D_i(t)$$

$$\leq M\delta - C_i,$$

keeping in mind that both η and $-D_i(t)$ are negative and using the Mean Value Theorem on $a(\cdot, t, 0)$ in combination with $|x_i - x| \leq \delta$ and the bound on a_x in (2.2). Therefore, (2.4) is satisfied as soon as $M\delta - C_i + (t_{i+1} - t_i)2\delta \leq 0$. It now suffices to take $C_i = [M + 2(t_{i+1} - t_i)]\delta$ to achieve this. For future reference, we remember that

$$C_i \leq (M + 2t_{\max})\delta. \quad (2.6)$$

For values of x away from x_i , i.e., when $|x - x_i| \geq \delta$, we have $w_{xx} = 0$, which implies that one term in (2.4) drops out. Furthermore, we have $w_x = \pm\delta^2$, so

$$\frac{1}{\epsilon} \left(\int_{t_i}^t w_x(x, s) ds \right)^2 = \frac{(t - t_i)^2 \delta^4}{\epsilon}.$$

For the remaining terms in (2.4), we have

$$w(x, t) - a(x, t, \epsilon) \leq a(x_i, t, 0) - a(x, t, 0) - D_i(t) - [a(x, t, \epsilon) - a(x, t, 0) + M\epsilon].$$

As before, the expression inside the brackets is nonnegative. Hence, using the Mean Value Theorem on $a(\cdot, t, 0)$ in combination with $|x_i - x| \leq 1$ and the bound on a_x in (2.2), we conclude that $w(x, t) - a(x, t, \epsilon) \leq M - D_i(t)$. The left-hand side of (2.4) is thus bounded by

$$M - D_i(t) - \frac{(t - t_i)^2 \delta^4}{\epsilon}.$$

Let us now finish Step 3. We define

$$D_i(t) = \begin{cases} 0, & t \geq t_i + \frac{2}{\delta^2} \sqrt{M\epsilon}, \\ M, & t \leq t_i + \frac{1}{\delta^2} \sqrt{M\epsilon}, \end{cases} \quad (2.7)$$

and we extend the definition of $D_i(t)$ by filling the gap in an obvious C^1 -fashion. For this choice of $D_i(t)$, inequality (2.4) is satisfied for all $t \in [t_i, t_{i+1}]$. This completes the construction of $w(x, t)$ on $[t_i, t_{i+1}]$.

2.2. End of the proof of Theorem 1. Using the definition of w on the interval $[t_i, t_{i+1}]$, we can recursively define a lower bound for the solution of (1.2) on successive intervals $[t_0, t_1]$, $[t_1, t_2]$, etc., so that $L(x, t)$ is defined on all of $[0, t_{\max}]$. On the interval $[0, t_1] = [t_0, t_1]$, we take

$$L(x, t) = L_0 \exp \left(\frac{1}{\epsilon} \int_0^t w(x, s) ds \right), \quad t \in [0, t_1].$$

Now, define $L_1 = \inf_x L_0 \exp\left(\frac{1}{\epsilon} \int_0^{t_1} w(x, s) ds\right) = L_1(L_0)$, and set

$$L(x, t) = L_1 \exp\left(\frac{1}{\epsilon} \int_{t_1}^t w(x, s) ds\right), \quad t \in]t_1, t_2].$$

We continue in this manner until $L(x, t)$ is defined as a piecewise continuous function for all $t \in [0, t_{\max}]$ and all $x \in [0, 1]$, with

$$L(x, t) = L_i \exp\left(\frac{1}{\epsilon} \int_{t_i}^t w(x, s) ds\right), \quad t \in]t_i, t_{i+1}]$$

for $L_i = L_i(L_{i-1})$.

On each time interval $]t_i, t_{i+1}]$, the functions $L(x, t)$ and $U(x, t)$ clearly satisfy the hypotheses of Corollary 1, and hence the solution $u(x, t)$ satisfies

$$L(x, t) \leq u(x, t) \leq U(x, t), \quad \forall t \in [0, t_{\max}], \quad x \in [0, 1].$$

We remind the reader that the existence of a smooth solution has already been shown (see the discussion after the statement of Corollary 1); the piecewise application of Theorem 5 above only serves to improve the relevant bounds, and by uniqueness, the smoothness of the solution u is retained.

Finally, we derive the required lower bound on u , as claimed in the statement of Theorem 1, see (1.8). Let us first bound $\int w(x, t) dt$. We start with (2.5),

$$\begin{aligned} w(x, t) &= a(x_i, t, 0) + \eta(x - x_i) - C_i - D_i(t) - M\epsilon \\ &\geq \left[\max_x a(x, t, 0) - \delta \right] + \inf_x \eta(x - x_i) - (M + 2t_{\max})\delta - D_i(t) - M\epsilon \\ &\geq \left[\max_x a(x, t, 0) \right] - \underbrace{(1 + 1 + M + 2t_{\max})\delta}_{=: Y} - D_i(t) - M\epsilon, \end{aligned}$$

where we used (2.3) and (2.6) in the first inequality and $\eta \geq -\delta$ in the second. Hence, we find

$$\int_{t_i}^t w(x, s) ds \geq \left(\int_{t_i}^t \left[\max_x a(x, s, 0) \right] ds \right) - Y\delta(t - t_i) - M\frac{2}{\delta^2}\sqrt{M\epsilon} - M\epsilon(t - t_i),$$

where we used (2.7). For $i = 0$ and $t \in [0, t_1]$, we thus have

$$L(x, t) \geq L_0 \exp\left[\frac{1}{\epsilon} \left(\int_0^t \left[\max_{\tilde{x}} a(\tilde{x}, s, 0) \right] ds - Y\delta t_1 - M\frac{2}{\delta^2}\sqrt{M\epsilon} - M\epsilon t_1 \right)\right];$$

hence,

$$L_1 \geq L_0 \exp\left[\frac{1}{\epsilon} \left(\int_0^{t_1} \left[\max_{\tilde{x}} a(\tilde{x}, s, 0) \right] ds - Y\delta t_1 - M\frac{2}{\delta^2}\sqrt{M\epsilon} - M\epsilon t_1 \right)\right].$$

Using this bound on L_1 , we find a bound on L for $t \in]t_1, t_2]$, and we can continue in this manner up to $t = t_{\max}$. At the end, one finds that for all values of t ,

$$L(x, t) \geq L_0 \exp \left[\frac{1}{\epsilon} \left(\int_0^t \left[\max_{\tilde{x}} a(\tilde{x}, s, 0) \right] ds - Y \delta t_{\max} - M(N+1) \frac{2}{\delta^2} \sqrt{M\epsilon} - M\epsilon t_{\max} \right) \right],$$

where $N+1$ is the (δ -dependent) sequence length of the subdivision of the time interval $[0, t_{\max}]$ in (2.3). Observe that Y and t_{\max} are independent of δ ; hence, upon a linear rescaling of the parameter δ , we replace $Y \delta t_{\max}$ by $\frac{1}{2} \delta$. It now suffices to restrict ϵ to $]0, \epsilon_0]$, with

$$M(N+1) \frac{2}{\delta^2} \sqrt{M\epsilon_0} + M\epsilon t_{\max} \leq \frac{1}{2} \delta,$$

to prove Theorem 1.

Remark. By showing that $N = \mathcal{O}(\delta^{-1})$, we find that $\epsilon_0 = \mathcal{O}(\delta^8)$. Therefore, one can replace δ in the statement of Theorem 1 by any positive expression that is $o(\epsilon^{1/8})$ as $\epsilon \rightarrow 0$. Of course, this estimate is most probably not optimal.

3. PROOF OF THEOREM 2

As in the proof of Theorem 1, we make an Ansatz for the lower solution of the form $L(x, t) = L_0 \exp \frac{1}{\epsilon} \int_0^t w(x, s) ds$. Instead of (2.1), the function w now has to satisfy

$$\begin{cases} w(x, t) - a(x, t, \epsilon) - D\epsilon^{\alpha-1} \int_0^t w_{xx}(x, s) ds - D\epsilon^{\alpha-2} \left(\int_0^t w_x(x, s) ds \right)^2 \leq 0, \\ w_x(0, t) \geq 0 \geq w_x(1, t). \end{cases} \quad (3.1)$$

When $1 \leq \alpha < 2$, the dominant terms in (3.1) are the same as those appearing in (2.1) in the proof of Theorem 1. Therefore, the proof is completely analogous. When $0 < \alpha < 1$, the term multiplied by $\epsilon^{\alpha-1}$ becomes unbounded as well, making this case slightly different. However, the proof can easily be adapted to cover that situation (in fact, due to the larger diffusivity than in the case where $\alpha = 1$, the result is not unexpected).

When $\alpha > 2$, the dominant terms in (2.5) are $w(x, t) - a(x, t, 0)$. Therefore, it is easy to define a good choice of w :

$$w(x, t) = a(x, t, 0) - \delta - E_\delta(x),$$

where E_δ is some C^2 function that lies between 0 and $\frac{1}{2}\delta$ and that has steep enough slopes near $x = 0$ and $x = 1$ in order to satisfy the boundary conditions $w_x(0, t) \geq 0 \geq w_x(1, t)$. Choosing $\epsilon > 0$ small enough for given δ , (3.1) is then satisfied. With the same ease, one can define an upper bound. As a consequence, when $\alpha > 2$, we find that

$$L_0 \exp \frac{1}{\epsilon} \left(\int_0^t a(x, s, \epsilon) ds - \delta \right) \leq u(x, t; \epsilon) \leq U_0 \exp \frac{1}{\epsilon} \left(\int_0^t a(x, s, \epsilon) ds + \delta \right),$$

from which one can conclude directly that $t_{\text{PDEexit}}(x) = t_{\text{exit}}(x)$.

4. THE NONLINEAR SETTING

In this section, we prove Theorems 3 and 4. First, we observe that the notions of lower and upper solutions are applicable to the nonlinear problem in (1.12), as well. Also, Theorem 5, the result of Pao [15], applies in this nonlinear setting. Therefore, it suffices to find good lower and upper estimates.

4.1. Proof of Theorem 3. Let us first treat the case of large diffusivities, i.e., the context of Theorem 3. The proof is split up into two parts: given any small $\delta > 0$, we first apply Theorem 5 on the time interval $[0, \delta]$, which yields a solution $u = u(x, t)$ of (1.12) for $t \leq \delta$. In the second step, we consider the same equation, but this time with initial data given by $u(x, \delta)$ at $t = \delta$. We then apply Theorem 5 again, this time on $[\delta, t_{\text{max}}]$, to deduce that u can be extended for $t \geq \delta$. A priori, this argument leads to a solution that is only continuous at $t = \delta$. However, by varying δ slightly and by repeating the above construction, one shows that $u(x, t)$ is in fact differentiable on the entire time interval.

For the first part of our argument, we introduce two auxiliary functions $\bar{a}_\delta(x, t, \epsilon)$ and $\underline{a}_\delta(x, t, \epsilon)$, as follows: for δ small enough, it is certainly possible to find new initial data $\bar{u}_0(x)$ and $\underline{u}_0(x)$ that satisfy

$$0 < u_0(x) - \delta \leq \underline{u}_0(x) \leq u_0(x) \leq \bar{u}_0(x) \leq u_0(x) + \delta$$

as well as the boundary conditions $\bar{u}'_0(0) = \bar{u}'_0(1) = 0$ and $\underline{u}'_0(0) = \underline{u}'_0(1) = 0$, respectively. Now, we let

$$\bar{a}_\delta(x, t, \epsilon) = \max_{u \in [0, \bar{u}_0(x)]} f(u, x, t, \epsilon) < 0, \quad t \in [0, \delta]$$

and

$$\underline{a}_\delta(x, t, \epsilon) = \min_{u \in [0, \underline{u}_0(x)]} f(u, x, t, \epsilon) < 0, \quad t \in [0, \delta],$$

where the signs of \bar{a}_δ and \underline{a}_δ follow from condition (1.14) as long as $\delta > 0$ is small enough. The solution of the linear problem $\epsilon U_t - \epsilon^\alpha U_{xx} = \bar{a}_\delta(x, t, \epsilon)U$, with initial and boundary conditions $U(x, 0) = \bar{u}_0(x)$ and $U_x(0, t) = U_x(1, t) = 0$, is clearly an upper solution of the nonlinear problem (1.12), since

$$\epsilon U_t - \epsilon^\alpha U_{xx} - f(U, x, t, \epsilon)U = [\bar{a}_\delta(x, t, \epsilon) - f(U, x, t, \epsilon)]U \geq 0, \quad \forall t \in [0, \delta].$$

Similarly, the solution of the linear problem $\epsilon L_t - \epsilon^\alpha L_{xx} = \underline{a}_\delta(x, t, \epsilon)L$, with initial and boundary conditions $L(x, 0) = \underline{u}_0(x)$ and $L_x(0, t) = L_x(1, t) = 0$, is a lower solution of (1.12). As a consequence,

$$L(x, t, \epsilon) \leq u(x, t, \epsilon) \leq U(x, t, \epsilon)$$

for all $t \leq \delta$. By choosing $\epsilon > 0$ small enough, and by observing that U is exponentially small with respect to ϵ for $t > 0$, we may assume that $U \leq \delta$ at $t = \delta$.

For $t \geq \delta$, we define

$$\bar{a}_\delta(x, t, \epsilon) = \max_{u \in [0, \delta]} f(u, x, t, \epsilon) \quad \text{and} \quad \underline{a}_\delta(x, t, \epsilon) = \min_{u \in [0, \delta]} f(u, x, t, \epsilon).$$

Extending the solutions L and U to times $t \geq \delta$ will still provide lower and upper solutions for the nonlinear problem (1.12), at least as long as $U \leq \delta$, i.e., at least until the exit time of U . This exit time is a δ -dependent homogeneous exit time $t = t_{\text{PDEexit}}(\delta)$ that is determined by the equation

$$\int_0^{t_{\text{PDEexit}}(\delta)} \left[\max_x \bar{a}_\delta(x, t, 0) \right] dt = 0.$$

It follows that $t_{\text{PDEexit}}(\delta)$ is a lower bound for the actual exit time of the nonlinear equation.

Now, we observe that $t_{\text{PDEexit}}(\delta)$ is a decreasing function in δ : for $\delta > 0$, the exit takes place earlier than for $\delta = 0$, since the attraction before the turning point is weaker and the repulsion after the turning point is stronger. Since $t_{\text{PDEexit}}(\delta) = t_{\text{PDEexit}} + \mathcal{O}(\delta)$ and since $\delta > 0$ is arbitrary, it follows that the exit time of the nonlinear equation is bounded from below by the homogeneous bound $t = t_{\text{PDEexit}}$ given by

$$\int_0^{t_{\text{PDEexit}}} \left[\max_x f(0, x, t, 0) \right] dt = 0.$$

Similarly, by replacing L and \underline{a}_δ with U and \bar{a}_δ , respectively, one can show that that same time t_{PDEexit} is also an upper bound for the exit time, which establishes Theorem 3.

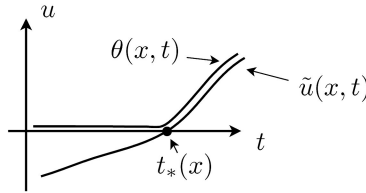


FIGURE 3. The function $\theta(x, t)$ used in the proof of Theorem 4, for x fixed.

4.2. Proof of Theorem 4. Let us finally prove Theorem 4. In the context of this theorem, we have a bistable regime where, after the passage past the turning point, two branches of singular points exist, $u = 0$ and $u = \tilde{u}(x, t)$, both of which are simple zeros of $f(u, x, t, 0)u = 0$. The simplicity of $\tilde{u}(x, t)$ ensures that the sign of $f(u, x, t, 0)$ is negative for $u > \tilde{u}(x, t)$. We first prove that a solution of the nonlinear problem (1.12) exists, and that it stays approximately in between the two branches $u = 0$ and $u = \tilde{u}(x, t)$ after passing the turning point. For time values $t \in [0, \delta]$, we proceed as in the proof of Theorem 3; from that proof, we know that the solution $u = u(x, t)$ is exponentially small with respect to ϵ at $t = \delta$.

Let us define the piecewise function $\theta(x, t)$ by 0 for $\delta \leq t \leq t_*(x)$ and by $\tilde{u}(x, t)$ for $t > t_*(x)$, see Figure 3. Ideally, we would like to be able to show that the solution of (1.12) stays between 0 and θ . While that is not necessarily the case, we can find an upper solution in a δ -neighborhood of θ , as follows. Using appropriately defined bump functions, we can define a smooth function θ_1 so that the graph $u = \theta_1(x, t)$ lies in a $\frac{\delta}{4}$ -neighborhood of the piecewise smooth graph $u = \theta(x, t)$. The function θ_1 does not necessarily satisfy the required boundary conditions. However, introducing bump functions near $x = 0$ and $x = 1$, we can find a $\frac{\delta}{4}$ -perturbation $\theta_2(x, t)$ of θ_1 so that the boundary conditions at $x = 0$ and at $x = 1$ are satisfied. Then, setting $\tilde{U} = \theta_2 + \frac{\delta}{2}$ gives a smooth graph in a δ -neighborhood of θ that lies above $u = \theta(x, t)$. Consequently, we have $f(\tilde{U}(x, t), x, t, 0) < 0$. It is now easy to see that for ϵ small enough,

$$\epsilon \tilde{U}_t - \epsilon^\alpha \tilde{U}_{xx} - f(\tilde{U}, x, t, \epsilon) \tilde{U} \geq 0, \quad \tilde{U}_x(0, t) = \tilde{U}_x(1, t) = 0,$$

and that at $t = \delta$, the function \tilde{U} is larger than the exponentially small solution $u(x, \delta)$. Hence, we conclude that \tilde{U} is an upper solution for $t \geq \delta$.

For the remainder of the proof, we define

$$\tilde{f}(u, x, t, \epsilon) = \begin{cases} f(u, x, t, \epsilon), & u \leq \tilde{U}(x, t), \\ f(\tilde{U}(x, t), x, t, \epsilon), & u \geq \tilde{U}(x, t) + \delta, \end{cases}$$

and we use bump functions to connect smoothly between the two cases. Since \tilde{U} is an upper solution for (1.12), we can safely replace f by \tilde{f} in the definition of the problem, as that will not change the solution u itself.

From this point on, we can continue the proof of Theorem 4 in the same manner as the proof of Theorem 3 above, this time defining

$$\bar{a}_\delta(x, t, \epsilon) = \max_{u \in [0, \infty[} \tilde{f}(u, x, t, \epsilon), \quad t > \delta.$$

Given \bar{a}_δ , we now proceed as before to obtain a uniformly valid upper solution for u . We note that the modified problem obtained from replacing f by \tilde{f} in (1.12) coincides with the original one as long as $u \leq \tilde{U}$; the modification ensures that \bar{a}_δ will provide a uniform bound even if $u > \tilde{U}$ and, hence, that Pao's result (Theorem 5) can again be applied, as in the proof of Theorem 3.

Specifically, due to condition (1.15) stated in the hypotheses of Theorem 4, we have

$$\bar{a}_\delta(x, t, \epsilon) = f(0, x, t, \epsilon) + \mathcal{O}(\delta)$$

(since $f(0, x, t, \epsilon) > f(u, x, t, \epsilon)$), and we can repeat the entire construction, which provides us with the upper solution $U(x, t)$ and the required $\mathcal{O}(\delta)$ -lower bound on the inhomogeneous exit time. The upper bound for the inhomogeneous exit time is easier to obtain, and can be found as in the proof of Theorem 3.

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