§4.5 HW Set 8 (11/3/09) p. 1/2

4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x \to a} [f(x)]^{p(x)} = \lim_{x \to a} y = \lim_{x \to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x \to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x \to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x \to a} [p(x)]^{q(x)} = \lim_{x \to a} y = \lim_{x \to a} e^{\ln y} = \infty$.

(f) $\lim_{x \to a} \sqrt[q(x)]{p(x)} = \lim_{x \to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

7. This limit has the form $\frac{0}{0}$. $\lim_{x \to (\pi/2)^+} \dfrac{\cos x}{1 - \sin x} \overset{\text{H}}{=} \lim_{x \to (\pi/2)^+} \dfrac{-\sin x}{-\cos x} = \lim_{x \to (\pi/2)^+} \tan x = -\infty$.

~~**8.** This limit has the form $\frac{0}{0}$.~~ $\cancel{\lim_{x \to 0} \dfrac{\sin 4x}{\tan 5x} \overset{\text{H}}{=} \lim_{x \to 0} \dfrac{4 \cos 4x}{5 \sec^2(5x)} = \dfrac{4(1)}{5(1)^2} = \dfrac{4}{5}}$

9. This limit has the form $\frac{0}{0}$. $\lim_{t \to 0} \dfrac{e^t - 1}{t^3} \overset{\text{H}}{=} \lim_{t \to 0} \dfrac{e^t}{3t^2} = \infty$ since $e^t \to 1$ and $3t^2 \to 0^+$ as $t \to 0$.

10. This limit has the form $\frac{0}{0}$. $\lim_{t \to 0} \dfrac{e^{3t} - 1}{t} \overset{\text{H}}{=} \lim_{t \to 0} \dfrac{3e^{3t}}{1} = 3$

11. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x \to \infty} \dfrac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim_{x \to \infty} \dfrac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x \to \infty} \dfrac{2}{\sqrt{x}} = 0$

18. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u \to \infty} \frac{e^{u/10}}{u^3} \overset{\text{H}}{=} \lim_{u \to \infty} \frac{e^{u/10} \cdot \frac{1}{10}}{3u^2} \overset{\text{H}}{=} \frac{1}{30} \lim_{u \to \infty} \frac{e^{u/10} \cdot \frac{1}{10}}{2u} \overset{\text{H}}{=} \frac{1}{600} \lim_{u \to \infty} \frac{e^{u/10} \cdot \frac{1}{10}}{1} = \frac{1}{6000} \lim_{u \to \infty} e^{u/10} = \infty$$

19. This limit has the form $\frac{0}{0}$. $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2} \overset{\text{H}}{=} \lim_{x \to 0} \dfrac{e^x - 1}{2x} \overset{\text{H}}{=} \lim_{x \to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

27. This limit has the form $\infty \cdot 0$.

$$\lim_{x \to \infty} x \sin(\pi/x) = \lim_{x \to \infty} \frac{\sin(\pi/x)}{1/x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2} = \pi \lim_{x \to \infty} \cos(\pi/x) = \pi(1) = \pi$$

28. This limit has the form $\infty \cdot 0$. $\lim_{x \to -\infty} x^2 e^x = \lim_{x \to -\infty} \dfrac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim_{x \to -\infty} \dfrac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim_{x \to -\infty} \dfrac{2}{e^{-x}} = \lim_{x \to -\infty} 2e^x = 0$

37. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x \to \infty} (x - \ln x) = \lim_{x \to \infty} x\left(1 - \frac{\ln x}{x}\right) = \infty \text{ since } \lim_{x \to \infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

HWset 8

§4.5 (cont'd)

41. $y = (1-2x)^{1/x} \Rightarrow \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\dfrac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{-2/(1-2x)}{1} = -2 \Rightarrow$
$\lim\limits_{x\to 0}(1-2x)^{1/x} = \lim\limits_{x\to 0} e^{\ln y} = e^{-2}$.

43. $y = x^{1/x} \Rightarrow \ln y = (1/x)\ln x \Rightarrow \lim\limits_{x\to\infty}\ln y = \lim\limits_{x\to\infty}\dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{1} = 0 \Rightarrow$
$\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$

63. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{n!} = \infty$

64. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^p} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{px^{p-1}} = \lim\limits_{x\to\infty}\dfrac{1}{px^p} = 0$ since $p > 0$.

65. First we will find $\lim\limits_{n\to\infty}\left(1+\dfrac{r}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1+\dfrac{r}{n}\right)^{nt} \Rightarrow \ln y = nt\ln\left(1+\dfrac{r}{n}\right)$, so
$\lim\limits_{n\to\infty}\ln y = \lim\limits_{n\to\infty} nt\ln\left(1+\dfrac{r}{n}\right) = t\lim\limits_{n\to\infty}\dfrac{\ln(1+r/n)}{1/n} \overset{\text{H}}{=} t\lim\limits_{n\to\infty}\dfrac{(-r/n^2)}{(1+r/n)(-1/n^2)} = t\lim\limits_{n\to\infty}\dfrac{r}{1+i/n} = tr \Rightarrow$
$\lim\limits_{n\to\infty} y = e^{rt}$. Thus, as $n\to\infty$, $A = A_0\left(1+\dfrac{r}{n}\right)^{nt} \to A_0e^{rt}$.

§4.6

3. The two numbers are x and $\dfrac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \dfrac{100}{x}$. $f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2-100}{x^2}$. The critical
number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$.
The numbers are 10 and 10.

4. Call the two numbers x and y. Then $x + y = 16$, so $y = 16 - x$. Call the sum of their squares S. Then
$S = x^2 + y^2 = x^2 + (16-x)^2 \Rightarrow S' = 2x + 2(16-x)(-1) = 2x - 32 + 2x = 4x - 32$. $S' = 0 \Rightarrow x = 8$.
Since $S'(x) < 0$ for $0 < x < 8$ and $S'(x) > 0$ for $x > 8$, there is an absolute minimum at $x = 8$. Thus, $y = 16 - 8 = 8$
and $S = 8^2 + 8^2 = 128$.

5. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is
$A = xy = x(50-x)$. We wish to maximize the function $A(x) = x(50-x) = 50x - x^2$, where $0 < x < 50$. Since
$A'(x) = 50 - 2x = -2(x-25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute
maximum at $x = 25$, and $A(25) = 25^2 = 625\ \text{m}^2$. The dimensions of the rectangle that maximize its area are $x = y = 25$ m.
(The rectangle is a square.)

7. We need to maximize Y for $N \geq 0$. $Y(N) = \dfrac{kN}{1+N^2} \Rightarrow$

$Y'(N) = \dfrac{(1+N^2)k - kN(2N)}{(1+N^2)^2} = \dfrac{k(1-N^2)}{(1+N^2)^2} = \dfrac{k(1+N)(1-N)}{(1+N^2)^2}$. $Y'(N) > 0$ for $0 < N < 1$ and $Y'(N) < 0$ for $N > 1$. Thus, Y has an absolute maximum of $Y(1) = \frac{1}{2}k$ at $N = 1$.

11. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \Rightarrow h = (1200 - b^2)/(4b)$. The volume is $V = b^2h = b^2(1200 - b^2)/4b = 300b - b^3/4 \Rightarrow V'(b) = 300 - \frac{3}{4}b^2$.
$V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 302). If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000\text{ cm}^3$.

44. (a) The total profit is $P(x) = R(x) - C(x)$. In order to maximize profit we look for the critical numbers of P, that is, the numbers where the marginal profit is 0. But if $P'(x) = R'(x) - C'(x) = 0$, then $R'(x) = C'(x)$. Therefore, if the profit is a maximum, then the marginal revenue equals the marginal cost.

(b) $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 1700 - 14x = 500 - 3.2x + 0.012x^2 \Leftrightarrow 0.012x^2 + 10.8x - 1200 = 0 \Leftrightarrow$ $x^2 + 900x - 100{,}000 = 0 \Leftrightarrow (x + 1000)(x - 100) = 0 \Leftrightarrow x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

48. Let x denote the number of \$10 increases in rent. Then the price is $p(x) = 800 + 10x$, and the number of units occupied is $100 - x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit}) \times (\text{number of units rented}) \\ &= (800 + 10x)(100 - x) = -10x^2 + 200x + 80{,}000 \text{ for } 0 \leq x \leq 100 \quad \Rightarrow \end{aligned}$$

$R'(x) = -20x + 200 = 0 \Leftrightarrow x = 10$. This is a maximum since $R''(x) = -20 < 0$ for all x. Now we must check the value of $R(x) = (800 + 10x)(100 - x)$ at $x = 10$ and at the endpoints of the domain to see which value of x gives the maximum value of R. $R(0) = 80{,}000$, $R(10) = (900)(90) = 81{,}000$, and $R(100) = (1800)(0) = 0$. Thus, the maximum revenue of \$81,000/week occurs when 90 units are occupied at a rent of \$900/week.