# MA542 Lecture

Timothy Kohl

Boston University

February 3, 2025

# Field of Fractions

Observe that $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0\}$.

For domains other than $\mathbb{Z}$ we can construct other sets of 'fractions'.

**Theorem**

*Let $D$ be a domain, then there exists a field $F$ (called the field of fractions, denoted $Frac(D)$) that contains $D$ as a subring.*

Before we get into the proof, we should quantify what it means for $D$ to be contained as a subring of $Frac(D)$. And the best way is to consider the canonical example above, $\mathbb{Q} = Frac(\mathbb{Z})$.

In this situation, the fractions of the form $\{\frac{a}{1} \mid a \in \mathbb{Z}\}$ are a subring which is 'isomorphic' to $\mathbb{Z}$ (in the same way one views isomorphisms of groups) in that $a \mapsto \frac{a}{1}$ and $a + b \mapsto \frac{a}{1} + \frac{b}{1}$ and $ab \mapsto \frac{a}{1}\frac{b}{1} = \frac{ab}{1}$.

i.e. $\{\frac{a}{1} \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$ even though we haven't (yet) defined what an isomorphism of rings means.

PROOF:
We construct $S = \{(a,b) \mid a,b \in D, b \neq 0\}$ and define an equivalence relation on $S$ as follows:

$$(a,b) \equiv (c,d) \text{ if } ad = bc$$

and let $F$ be the set of equivalence classes under $\equiv$.

If we define $\frac{a}{b} = [(a,b)]$ (the equivalence class of $(a,b)$) then we let

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$
$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

where these make sense since $bd \neq 0$ unless $b = 0$ or $d = 0$ which doesn't happen since the elements of $S$ consist of ordered pairs where the second coordinate is not zero.

PROOF (continued)
One can verify that 0 in $F$ is $\frac{0}{1}$ and 1 in $F$ is $\frac{1}{1}$

The most important verification to make is that the operations are not sensitive to the choice of equivalence class representative.

We have $\frac{a}{b} = \frac{a'}{b'}$ if $ab' = a'b$ and $\frac{c}{d} = \frac{c'}{d'}$ if $cd' = c'd$ and so

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \; *$$
$$\frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd' + b'c'}{b'd'} \; **$$

The question is whether the right hand sides of (*) and (**) are the same.

Now

$$(ad+bc)(b'd') = db'd' + bcb'd' = (ab')dd' + b(cd')b' = a'bdd' + bc'd'b'$$
$$(a'd'+b'c')(bd) = a'd'bd + b'c'bd = a'bdd' + bc'd'b'$$

and so * and ** are the same.

In a similar way one may verify that $\frac{a}{b}\cdot\frac{c}{d} = \frac{a'}{b'}\cdot\frac{c'}{d'}$.

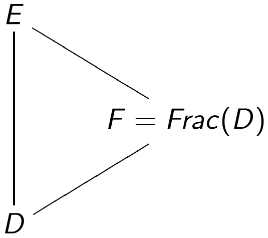
We verify that $F$ is a field by realizing that, for $b \neq 0$, $\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}$ if $\frac{ab}{ab} = \frac{1}{1}$ which is certainly true since (by cross multiplying) obviously $ab \cdot 1 = 1 \cdot ab$ in $D$.

Lastly, the set $\{\frac{a}{1} \ |a \in D\}$ is a sub-ring of $F$ that (under the addition and multiplication of fractions defined above) is isomorphic to $D$.

The overall point of this construction is to construct a ring which naturally contains the inverses of every element of $D$.

In this way, the field $F = Frac(D)$ we've constructed is unique in that if $D$ is the subring of some field $E$ then certainly $E$ contains all the inverses of every element of $D$ so it contains a subring isomorphic to $Frac(D)$ which we can diagram as follows:

i.e.



As mentioned above, the construction of the fraction field is kind of natural in that it is basically the field defined by 'allowing' every non-zero element of $D$ to be invertible.

Examples: $Frac(\mathbb{Z}) = \mathbb{Q}$ the prototype example
$Frac(\mathbb{Z}[i]) = \mathbb{Q}(i)$
Most elements $a + bi \in \mathbb{Z}[i]$ do not have inverses since $U(\mathbb{Z}[i]) = \{\pm 1, \pm i\}$ (exercise) and if we take the 'ratio' of two Gaussian integers $\frac{a+bi}{c+di}$ we get

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di}\frac{c-di}{c-di} \\ &= \frac{(ac-bd)+(ad+bc)i}{c^2+d^2} \\ &= \frac{(ac-bd)}{c^2+d^2} + \frac{(ad+bc)}{c^2+d^2}i \\ &\in \mathbb{Q}(i)\end{aligned}$$

We end this topic with two interesting exercises.

1) If $D$ is a field, show that $Frac(D) \cong D$. (i.e. $Frac(D)$ is no 'larger').

2) What 'fails' if we try to create $Frac(D)$ when $D$ isn't a domain?

# Polynomial Rings

## Definition

Let $R$ be a commutative ring, for an indeterminate '$x$' the set

$$R[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in R\}$$

where $n \geq 0$ and $a_n x^n + \cdots + a_0 = b_m x^m + \cdots + b_0$ if and only if $m = n$ and $a_i = b_i$ for each $i$ from 0 to $n$ is the polynomial ring over $R$ (in the variable '$x$') if we define the addition and multiplication as follows:

If $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ then

$$f(x) + g(x) = (a_s + b_s)x^s + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

where $s = max(m, n)$ and $a_i = 0$ for $i > n$ and $b_j = 0$ for $j > m$ and where

$$f(x)g(x) = c_{n+m}x^{n+m} + \cdots + c_1 x + c_0$$

where $c_k = a_k b_0 + a_{k-1} b_1 + \cdots + a_0 b_k$

Now we're familiar with how polynomial addition and multiplication work so this definition just formalizes things.

Here is another basic concept we're all familiar with.

**Definition**

For $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$ (not the zero polynomial) the degree $deg(f(x)) = n$ if $a_n \neq 0$.

What about $deg(0)$ where 0 is the (constant) zero polynomial? (We'll get to this in a bit.)

What we're examining is the role of the 'ring of coefficients' $R$ in understanding the 'arithmetic' of the polynomial ring $R[x]$.

In particular we have a basic fact about $R$ versus $R[x]$.

## Proposition

*If $R$ is an integral domain then so is $R[x]$.*

## Proof.

This isn't too difficult. If $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ are two non-zero polynomials then, in particular $a_n \neq 0$ and $b_m \neq 0$ so $f(x)g(x) = (a_n b_m)x^{n+m} + (\textit{lower degree terms})$ we have $a_n b_m \neq 0$ since $R$ is a domain. □

Also, we observe that if $R$ has unity 1 then 1 (viewed as a constant polynomial) is the unity element of $R[x]$.

Also, since $R$ is commutative, then $R[x]$ is commutative. (easy exercise)

So what about $deg(0)$? Is it 0? No.

In the definition of $R[x]$ we have that if $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ then $deg(f(x)) = n$ and $deg(g(x)) = m$ and

$$f(x)g(x) = (a_n b_m)x^{n+m} + \cdots + a_0 b_0$$

and so we ask, what is $deg(f(x)g(x))$?

Well, if $R$ is a domain then
$deg(f(x)g(x)) = n + m = deg(f(x)) + deg(g(x))$ since $a_n \neq 0$ and $b_m \neq 0$ implies $a_n b_m \neq 0$.
Even if $f(x)$ and $g(x)$ are (non-zero) constant polynomials (whence $deg(f(x)) = 0$ and $deg(g(x)) = 0$) we still have
$deg(f(x)g(x)) = deg(f(x)) + deg(g(x))$

So what if $g(x) = 0$?

In this case $f(x)g(x) = 0$ so $deg(f(x)g(x)) = deg(0)$ but we want $deg(f(x)g(x)) = def(f(x)) + deg(g(x))$ which means

$$deg(f(x)) + deg(0) = deg(0)$$

so... this would imply that $deg(f(x)) = 0$ but $f(x)$ need not have degree 0.

The way to resolve this is to define $deg(0) = -\infty$.

With this definition the $deg(f(x)g(x)) = deg(f(x)) + deg(g(x))$ holds for all polynomials in $R[x]$ when $R$ is a domain.