MA542 Lecture

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Consider $G = Gal(\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q})$ which is the splitting field of $x^4 - 2 \in \mathbb{Q}[x]$ since the roots are $i^t \sqrt[4]{2}$ for $t = 0, 1, 2, 3$.

Claim: $G = \langle x, t \mid x^4 = 1, t^2 = 1, xt = tx^{-1} \rangle$ where

- $x(\sqrt[4]{2}) = i\sqrt[4]{2}$
- $x(i) = i$
- $t(\sqrt[4]{2}) = \sqrt[4]{2}$
- $t(i) = -i$

and $x(c) = c$ and $t(c) = c$ for all $c \in \mathbb{Q}$.

The basis for $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$ is

$$\mathcal{B} = \{1, \sqrt[4]{2}, \sqrt[4]{2}^2, \sqrt[4]{2}^3, i, i\sqrt[4]{2}, i\sqrt[4]{2}^2, i\sqrt[4]{2}^3\}$$

in keeping with fact that $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = 8$.

For an element $\sigma \in G$ the 'roots mapped to roots' requirement implies that

$$\sigma(\sqrt[4]{2}) = \pm\sqrt[4]{2} \text{ or } \pm i\sqrt[4]{2}$$
$$\sigma(i) = \pm i \text{ since } \pm i \text{ are the roots of } x^2 + 1$$

So for $x(\sqrt[4]{2}) = i\sqrt[4]{2}$ and $x(i) = i$ we have that
$x^2(\sqrt[4]{2}) = x(x(\sqrt[4]{2})) = x(i\sqrt[4]{2}) = x(i)x(\sqrt[4]{2}) = i(i\sqrt[4]{2}) = -\sqrt[4]{2}$.

Continuing this way $x^3(\sqrt[4]{2}) = x(-\sqrt[4]{2}) = -x(\sqrt[4]{2}) = -(i\sqrt[4]{2}) = -i\sqrt[4]{2}$ and similarly $x^4(\sqrt[4]{2}) = x(-i\sqrt[4]{2}) = -x(i)x(\sqrt[4]{2}) = -i(i\sqrt[4]{2}) = \sqrt[4]{2}$.

And $x(i) = i$ implies $x^k(i) = i$ for $k = 0, 1, 2, 3$.

Similarly $t(i) = -i$ and $t(\sqrt[4]{2}) = \sqrt[4]{2}$ implies that $t^2(i) = i$ and $t^2(\sqrt[4]{2}) = \sqrt[4]{2}$ of course.

As such, we deduce that $|x| = 4$ and $|t| = 2$.

What one also shows is that $xt = tx^{-1} = tx^3$.

Also, one can show that $|t| = 2$, $|tx| = 2$, $|tx^2| = 2$ and $|tx^3| = 2$. As such $G = \{1, x, x^2, x^3, t, tx, tx^2, tx^3\}$ and is isomorphic to D_4 where one can identify x with the 90° rotation and t (as well as tx, tx^2, and tx^3) are all the 'flips' one can perform on a square.

Note, $Z(G) = \langle x^2 \rangle$ which is non-trivial because for any even n, $|Z(D_n)| = 2$ while for n odd, $|Z(D_n)| = 1$. (i.e. the center is given by the 180° rotation)

Also, it is evident G acts transitively on the roots of $x^4 - 2$, namely $\{\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}\}$ since,

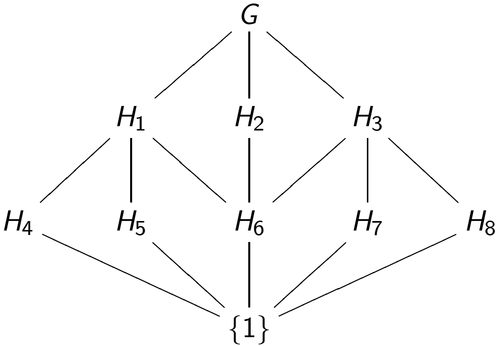
$x(\sqrt[4]{2}) = i\sqrt[4]{2}$	$x^2(\sqrt[4]{2}) = -\sqrt[4]{2}$	$x^3(\sqrt[4]{2}) = -i\sqrt[4]{2}$
$x(-\sqrt[4]{2}) = -i\sqrt[4]{2}$	$x^2(-\sqrt[4]{2}) = \sqrt[4]{2}$	$x^3(-\sqrt[4]{2}) = i\sqrt[4]{2}$
$x(i\sqrt[4]{2}) = -\sqrt[4]{2}$	$x^2(i\sqrt[4]{2}) = -i\sqrt[4]{2}$	$x^3(i\sqrt[4]{2}) = \sqrt[4]{2}$
$x(-i\sqrt[4]{2}) = \sqrt[4]{2}$	$x^2(-i\sqrt[4]{2}) = i\sqrt[4]{2}$	$x^3(-i\sqrt[4]{2}) = -\sqrt[4]{2}$

Indeed, this shows that the subgroup $\langle x\rangle$ itself acts transitively on the roots, let alone the whole group G.

Now, let's consider the subgroups of G and the corresponding fixed fields.

$$
\begin{aligned}
H_1 &= \langle x^2, t\rangle = \{1, x^2, t, tx^2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \\
H_2 &= \langle x\rangle = \{1, x, x^2, x^3\} \cong \mathbb{Z}_4 \\
H_3 &= \langle x^2, tx\rangle = \{1, x^2, tx, tx^3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \\
H_4 &= \langle t\rangle = \{1, t\} \cong \mathbb{Z}_2 \\
H_5 &= \langle tx^2\rangle = \{1, tx^2\} \cong \mathbb{Z}_2 \\
H_6 &= \langle x^2\rangle = \{1, x^2\} \cong \mathbb{Z}_2 \\
H_7 &= \langle tx^3\rangle = \{1, tx^3\} \cong \mathbb{Z}_2 \\
H_8 &= \langle tx\rangle = \{1, tx\} \cong \mathbb{Z}_2
\end{aligned}
$$

And, of course, we have G and $\{1\}$.


G
H1
H2
H3
H4
H5
H6
H7
H8
{1}

We can compute the corresponding fixed fields.

$$
\begin{aligned}
&H_1 = \langle x^2, t\rangle = \{1, x^2, t, tx^2\}\\
&\quad\downarrow\\
&x^2(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)\\
&\quad = \boxed{a} - b\sqrt[4]{2} + \boxed{c\sqrt[4]{2}^2} - d\sqrt[4]{2}^3 + \boxed{ei} - fi\sqrt[4]{2} + \boxed{gi\sqrt[4]{2}^2} - hi\sqrt[4]{2}^3\\
&\ t(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)\\
&\quad = \boxed{a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3} - ei - fi\sqrt[4]{2} - gi\sqrt[4]{2}^2 - hi\sqrt[4]{2}^3\\
&\quad\downarrow\\
&\ b = 0,\ d = 0,\ e = 0,\ f = 0\ g = 0,\ h = 0\\
&\quad\downarrow\\
&E_{H_1} = \mathbb{Q}(\sqrt[4]{2}^2) = \mathbb{Q}(\sqrt{2})
\end{aligned}
$$

Note that $[E_{H_1} : F] = 2 = [G : H_1]$

$$
\begin{aligned}
&H_2 = \langle x \rangle \\
&\quad\downarrow \\
&x(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3) \\
&\quad = \boxed{a} + bi\sqrt[4]{2} - c\sqrt[4]{2}^2 - di\sqrt[4]{2}^3 + \boxed{ei} - f\sqrt[4]{2} - gi\sqrt[4]{2}^2 + h\sqrt[4]{2}^3 \\
&\quad\downarrow \\
&b = 0,\ c = 0,\ d = 0,\ f = 0,\ g = 0,\ h = 0, \\
&\quad\downarrow \\
&E_{H_2} = \mathbb{Q}(i)
\end{aligned}
$$

Note that $[E_{H_2} : F] = 2 = [G : H_2]$

$$H_3 = \langle x^2, tx\rangle = \{1, x^2, tx, tx^3\}$$

$$\downarrow$$

$$x^2(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$

$$= \boxed{a} - b\sqrt[4]{2} + \boxed{c\sqrt[4]{2}^2} - d\sqrt[4]{2}^3 + \boxed{ei} - fi\sqrt[4]{2} + \boxed{gi\sqrt[4]{2}^2} - hi\sqrt[4]{2}^3$$

$$tx(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$

$$= \boxed{a} - bi\sqrt[4]{2} - c\sqrt[4]{2}^2 + di\sqrt[4]{2}^3 - ei - f\sqrt[4]{2} + \boxed{gi\sqrt[4]{2}^2} + h\sqrt[4]{2}^3$$

$$\downarrow$$

$$E_{H_3} = \mathbb{Q}(i\sqrt[4]{2}^2) = \mathbb{Q}(i\sqrt{2})$$

Note that $[E_{H_3} : F] = 2 = [G : H_3]$

$$H_4 = \langle t \rangle = \{1, t\}$$

$$\downarrow$$

$$t(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$

$$= \boxed{a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3} - ei - fi\sqrt[4]{2} - gi\sqrt[4]{2}^2 - hi\sqrt[4]{2}^3$$

$$\downarrow$$

$$E_{H_4} = \mathbb{Q}(\sqrt[4]{2})$$

Note that $[E_{H_4} : F] = 4 = [G : H_4]$

$$H_5 = \langle tx^2 \rangle = \{1, tx^2\}$$

$$\downarrow$$

$$tx^2(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$

$$= \boxed{a} - b\sqrt[4]{2} + \boxed{c\sqrt[4]{2}^2} - d\sqrt[4]{2}^3 - ei + \boxed{fi\sqrt[4]{2}} - gi\sqrt[4]{2}^2 + \boxed{hi\sqrt[4]{2}^3}$$

$$\downarrow$$

$$E_{H_5} = \mathbb{Q}(i\sqrt[4]{2})$$

Note that $[E_{H_5} : F] = 4 = [G : H_5]$

$$H_6 = \langle x^2 \rangle = \{1, x^2\}$$

$$\downarrow$$

$$x^2(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$

$$= \boxed{a} - b\sqrt[4]{2} + \boxed{c\sqrt[4]{2}^2} - d\sqrt[4]{2}^3 + \boxed{ei} - fi\sqrt[4]{2} + \boxed{gi\sqrt[4]{2}^2} 7 - hi\sqrt[4]{2}^3$$

$$\downarrow$$

$$E_{H_6} = \mathbb{Q}(\sqrt{2}, i)$$

Note that $[E_{H_6} : F] = 4 = [G : H_6]$

This fixed field calculation is a bit more subtle.

$$H_7 = \langle tx^3 \rangle = \{1, tx^3\}$$

$$\downarrow$$

$$tx^3(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$
$$= a + bi\sqrt[4]{2} - c\sqrt[4]{2}^2 - di\sqrt[4]{2}^3 - ei + f\sqrt[4]{2} + gi\sqrt[4]{2}^2 - h\sqrt[4]{2}^3$$

So the question is, what is fixed by tx^3?
If $v = a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3$ then $v - tx^3(v)$ can be written in terms of the basis:

$$(b-f)\sqrt[4]{2} + (2c)\sqrt[4]{2}^2 + (d+h)\sqrt[4]{2}^3 + (2e)i + (f-b)i\sqrt[4]{2} + (h+d)i\sqrt[4]{2}^3$$

where we note that 'a' and 'g' do not appear.

So we wish to solve

$$v - tx^3(v) = (b-f)\sqrt[4]{2} + (2c)\sqrt[4]{2}^2 + (d+h)\sqrt[4]{2}^3 + (2e)i + (f-b)i\sqrt[4]{2} + (h+d)i\sqrt[4]{2}^3$$

for b, c, d, e, f, h which make $v - tx^3(v) = 0$, which yields $b = f$, $c = 0$, $e = 0$, $d = -h$, that is a, g can be anything and so

$$\begin{aligned} v &= a + b\sqrt[4]{2} + d\sqrt[4]{2}^3 + bi\sqrt[4]{2} + gi\sqrt[4]{2}^2 - di\sqrt[4]{2}^3 \\ &= a + b(\sqrt[4]{2} + i\sqrt[4]{2}) + d(\sqrt[4]{2}^3 - i\sqrt[4]{2}^3) + gi\sqrt[4]{2}^2 \end{aligned}$$

and if we let $q = \sqrt[4]{2} + i\sqrt[4]{2}$ then $q^2 = 2i\sqrt[4]{2}^2$, $q^3 = 2(i\sqrt[4]{2}^3 - \sqrt[4]{2}^3)$, and $q^4 = -8$.
The point is that $E_{H_7} = \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2})$ and $[E_{H_7} : F] = 4 = [G : H_7]$.

$$H_8 = \langle tx \rangle = \{1, tx\} \cong \mathbb{Z}_2$$

$$\downarrow$$

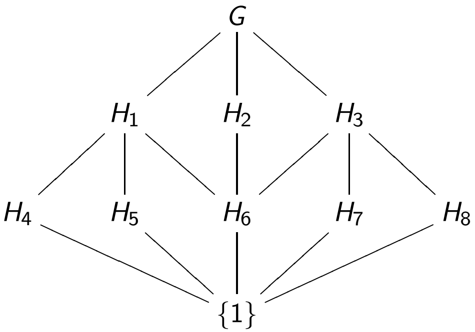
$$tx(a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3)$$
$$= a - bi\sqrt[4]{2} - c\sqrt[4]{2}^2 + di\sqrt[4]{2}^3 - ei - f\sqrt[4]{2} + gi\sqrt[4]{2}^2 + h\sqrt[4]{2}^3$$

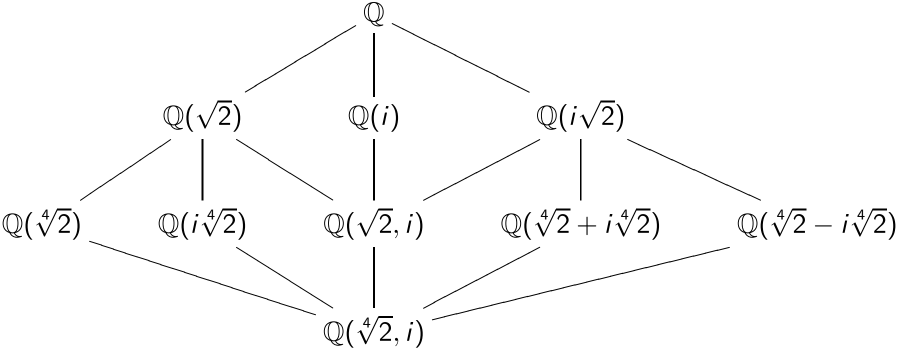
So for $v = a + b\sqrt[4]{2} + c\sqrt[4]{2}^2 + d\sqrt[4]{2}^3 + ei + fi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + hi\sqrt[4]{2}^3$, $tx(v) = v$ implies $d = h$, $b = -f$, $c = 0$, $e = 0$, and a, g are arbitrary which means v has the form

$$a + b\sqrt[4]{2} + d\sqrt[4]{2}^3 - bi\sqrt[4]{2} + gi\sqrt[4]{2}^2 + di\sqrt[4]{2}^3$$
$$= a + b(\sqrt[4]{2} - i\sqrt[4]{2}) + d(\sqrt[4]{2}^3 + i\sqrt[4]{2}^3) + gi\sqrt[4]{2}^2$$

so that $E_{H_8} = \mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$ and $[E_{H_8} : F] = 4 = [G : H_8]$.

Now, if we pass from subgroups H to fixed fields E_H we again get the 'inverted' subfield lattice for $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$.




$\mathbb{Q}$
$\mathbb{Q}(\sqrt{2})$
$\mathbb{Q}(i)$
$\mathbb{Q}(i\sqrt{2})$
$\mathbb{Q}(\sqrt[4]{2})$
$\mathbb{Q}(i\sqrt[4]{2})$
$\mathbb{Q}(\sqrt{2}, i)$
$\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2})$
$\mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$
$\mathbb{Q}(\sqrt[4]{2}, i)$

Observe that

- $H_1 = \langle x^2, t \rangle$
- $H_2 = \langle x \rangle$
- $H_3 = \langle x^2, tx \rangle$
- $H_6 = \langle x^2 \rangle$

are the only normal subgroups of G, and concordantly the fixed fields

- $E_{H_1} = \mathbb{Q}(\sqrt{2})$
- $E_{H_2} = \mathbb{Q}(i)$
- $E_{H_3} = \mathbb{Q}(i\sqrt{2})$
- $E_{H_6} = \mathbb{Q}(i, \sqrt{2})$

are the only intermediate fields which are splitting fields over $\mathbb{Q}$ (i.e. Galois extensions of $\mathbb{Q}$)

In contrast, a non-normal subgroup such as $H_4 = \langle t \rangle$ gives rise to the intermediate field $E_{H_4} = \mathbb{Q}(\sqrt[4]{2})$, which is not a splitting field over $\mathbb{Q}$.

i.e. It only contains the root $\sqrt[4]{2}$ of $x^4 - 2$ but none of the others.