Dissipative Partial Differential Equations and Dynamical Systems

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Abstract

This article surveys some recent applications of ideas from dynamical systems theory to understand the qualitative behavior of solutions of dissipative partial differential equations with a particular emphasis on the two-dimensional Navier-Stokes equations.

1 Introduction

The focus of this article is the application of dynamical systems ideas to the study of dissipative partial differential equations. The notion of dissipativity arises in physics where it is generally thought of as a dissipation of some "energy" associated with the system and such systems are contrasted with energy conserving systems like Hamiltonian systems. In finite dimensional systems, the notion of dissipativity is relatively easy to quantify. If we have a system of ordinary differential equations (ODEs) defined on $\mathbb{R}^n$,

$$\dot{x}_j = f_j(\mathbf{x}) = f_j(x_1, \ldots, x_n) \ , \quad j = 1, \ldots n \ , \tag{1}$$

then a common definition of dissipativity is that there be some bounded set, $\mathcal{B}$, which is forward invariant under the flow defined by our differential equations and such that every solution of the system of ODEs eventually enters $\mathcal{B}$, [Hal88]. Such a set is referred to as an absorbing set. If an absorbing set exists, and if ϕ^t is the flow defined by this dynamical system, then we can define an attractor for the system by

$$\mathcal{A} = \cap_{t \geq 0} \phi^t(\mathcal{B}) \ , \tag{2}$$

which will be compact and invariant, and will have the property that any trajectory will approach this set as $t \to \infty$.

Another property often associated with dissipativity is that the determinant of the Jacobian of the vectorfield in (1) is negative, i.e.

$$\det\left(\{\frac{\partial f_j}{\partial x_k}\}_{j,k=1,\ldots,n}\right) < 0 \ . \tag{3}$$

If this determinant is zero, then the system conserves phase space volumes, and this is one of the properties associated with Hamiltonian systems. Condition (3) means that the systems "dissipates" phase space volume, but note that it is not, in itself, sufficient to insure that we have a bounded attractor for the system, as the two dimensional example

$$\dot{x}_1 = \frac{1}{2}x_1 \ , \quad \dot{x}_2 = -x_2 \ , \tag{4}$$

for which almost every solution tends to infinity shows. However, if the system satisfies (3), and is in addition dissipative in the sense described above, then we can immediately conclude that the attractor for the system has zero n-dimensional volume. Thus, the asymptotic behavior of the system is determined by what happens on a very "small" set.

One important direction of research in the study of dissipative systems is to focus on the properties of the attractor. In special cases, the attractor may consist of a small number of simple orbits, like stationary solutions and their connecting orbits, or periodic orbits. In other, more complicated cases, the attractor may contain chaotic trajectories but it may itself live in a manifold of much lower dimension than the number of degrees of freedom of our original system. In such circumstances, the long-time behavior of arbitrary solutions of the original system can be determined from a study of the possibly much smaller system obtained by restricting the original ODEs to this manifold containing the attractor. This dimensional reduction has been a powerful tool in the study of dissipative systems.

When one turns from ODEs to partial differential equations (PDEs), the discussion becomes more complicated due to the infinite dimensional nature of the problem. For one thing, the long-term behavior of the system may well depend on the norm we choose on our space of solutions. However, even after one has fixed the norm on the system the situation can be problematic due to the fact that closed and bounded sets need no longer be compact. Thus, even if we find some bounded absorbing set $\mathcal{B}$, as above, we have no guarantee that $\cap_{t\geq 0}\phi^t(\overline{\mathcal{B}})$ will be non-empty! Thus, in addition to proving that the PDE is well-posed, one typically needs to establish some smoothing properties to show that not only is there a set $\mathcal{B}$ that eventually "absorbs" all trajectories but also that this set is precompact in the function space on which we are working. Fortunately, this smoothing can often be proven for the sort of dissipative systems we discuss here and the systems described in the subsequent sections all have well defined attractors.

The reduction of dimension of the problem which results from focussing one's attention on the restriction of the system to the attractor is an even more powerful tool in the context of PDEs than ODEs. As we will see in subsequent sections, many physically interesting PDEs have attractors of finite (sometimes even small) dimension. Thus, the complex behavior of solutions of the PDE can be captured by the behavior of the solutions restricted to this finite dimensional set.

In one sense this simply transfers the problem from the study of the infinite dimensional behavior of solutions of the PDE to computing the possibly very complicated structure of the attractor and the latter question remains an active and open area of research for many of even the most natural physical systems, but it is at least a finite dimensional question and one that is well suited to attack with the methods of dynamical systems theory.

As mentioned above, in infinite dimensional systems, dissipativity requires more than simply the existence of a bounded absorbing set to insure that the system has an attractor. There are several additional hypotheses one can make to insure the necessary compactness - see [Hal88] or [Tem88], for example. However, we follow Robinson ([Rob01], p. 264) and define:

Definition 1. *A semigroup is* dissipative *if it possesses a* compact *absorbing set $\mathcal{B}$.*

The drawback of this definition is that one must always check the compactness of the absorbing set.

Given a dissipative semigroup ϕ^t on a Banach space X, the ω-limit set of a set $U \subset X$ is

$$\omega(U) = \{u \in X \mid \text{there exists } u_0 \in U, \text{ and } \{t_n\}, t_n \to \infty, \\ \text{such that } \lim_{n\to\infty} \|\phi^{t_n}(u_0) - u\| = 0 \ \} \ . \tag{5}$$

A global attractor is then defined as ([Tem88], p. 21)

Definition 2. *A set $\mathcal{A} \subset X$ is a global attractor for the semiflow ϕ^t, if*

1. *$\mathcal{A}$ is invariant.*
2. *Every bounded set is uniformly attracted to $\mathcal{A}$. That is to say, if $\mathcal{U} \subset X$ is bounded, then*

$$\lim_{t\to\infty} \operatorname{dist}(\phi^t(\mathcal{U}), \mathcal{A}) = 0 \ . \tag{6}$$

With these definitions it is relatively easy to show:

Theorem 1. *If the semiflow ϕ^t is dissipative it has a universal attractor. If $\mathcal{B}$ is a compact absorbing set for ϕ^t, then the attractor is $\omega(\mathcal{B})$.*

Proof. The proof is relatively straightforward. The sets $\mathcal{A}(t_0) = \overline{\cup_{t\geq t_0}\phi^t(\mathcal{B})}$ are non-empty, compact and decreasing, and their intersection, which gives $\omega(\mathcal{B})$ is clearly invariant under ϕ^t and one checks the attractivity property by contradiction. The details are in [Rob01]. $\square$

Remark 1. *For a semiflow defined by a well-posed PDE such that $\phi^t(u_0) = \phi^t(v_0)$ implies $u_0 = v_0$, one has the important corollary that when restricted to the attractor, ϕ^t actually defines a flow, not just a semi-flow. That is, if $u_0 \in \mathcal{A}$, then $\phi^t(u_0)$ is defined for all $t \in \mathbb{R}$, not just for $t \geq 0$. This is somewhat surprising since in general it is not possible to solve dissipative PDEs "backwards" in time.*

One may wonder how hard it is to establish that an infinite dimensional dynamical is dissipative in the sense of Definition 1. Often the compactness needed in the definition follows in a relatively straightforward way from the same sorts of estimates that yield existence and uniqueness of solutions. Consider, for example, the family of non-linear heat-equations

$$u_t = u_{xx} + g(u)\ , 0 < x < L \tag{7}$$

with zero boundary conditions - i.e. $u(0,t) = u(L,t) = 0$. If one places mild growth conditions on the nonlinear term g this is known as the Chafee-Infante equation and was one of the first PDE to be systematically studied with the methods of dynamical systems theory. If one assumes that the initial conditions $u_0 \in H_0^1(0,L)$, then the methods of semigroup theory readily show that the orbit $u(\cdot,t)$ with this initial condition is relatively compact in this Hilbert space [Mik98], and one has an absorbing set. Hence the equation defines a dissipative dynamical system in the sense of the above definition. In this case the attractor is quite simple - for a special class of nonlinear terms, Henry showed that it consisted of the stationary solutions of the equation, and their unstable manifolds, [Hen81].

Of course, the converse is of this fact is also true - there are equations like the three-dimensional Navier-Stokes equation which are expected to be dissipative on physical grounds, but the fact that there is no proof that smooth solutions exist for all time for general initial data precludes proving that they are dissipative in the sense of the previous definition.

The main example of an infinite dimensional dissipative system that we'll examine in the remainder of this review is the two-dimensional Navier-Stokes equation (2D NSE). These equations describe the evolution of the velocity of a two-dimensional fluid. While it may seem unrealistic to study two-dimensional fluid flows in a three-dimensional world there are a number of circumstances (e.g. the Earth's atmosphere) where this is a reasonable physical approximation. For further discussion of this point, see [Way11].

Physically, the NSE arise from an application of Newton's law for the fluid, namely

$$\frac{d}{dt}(\text{momentum}) = \text{applied forces}\ . \tag{8}$$

If $\mathbf{u}(\mathbf{x},t)$ is the fluid velocity, and if we assume that the fluid is incompressible so that we can take the density to be constant, the time rate of change of the momentum is given by the convective derivative

$$\frac{d}{dt}(\text{momentum}) = \rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u}\ . \tag{9}$$

The second term in this expression reflects the fact that the momentum of a small region of the fluid can change not only due to changes in its velocity, but also because it is being simultaneously swept along by the background flow.

The forced are are typically split into three parts:

- forces due to pressure: $f_{\text{pressure}} = -\nabla p(x,t)$, where p is the pressure in the fluid.
- viscous forces: These involve modeling internal properties of the fluid. We will take a standard model which says $f_{\text{visc}} = \alpha \Delta \mathbf{u}$, for some constant α.
- external forces, which we denote by $\mathbf{g}$.

If we insert these expressions into Newton's law, we obtain

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = (\alpha/\rho)\Delta \mathbf{u} = \frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{g}. \tag{10}$$

Note that if we consider a fluid moving in d dimensions, this expression is actually a system of d partial differential equations. However, we have $d+1$ unknown functions - the d components of the velocity, plus the pressure. To close our system we append the additional equation

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{11}$$

which reflects the assumption that the fluid is incompressible. For a further discussion of the physical origin of these equations, one can consult [DG95].

As remarked above for the three-dimensional case it is not known whether or not the NSE possess smooth solutions for all times, even if one assumes that the initial velocity field is very smooth and there are no external forces acting on the system. Indeed, this is one of the famous Millennium Prize Problems. Thus, we will discuss only two dimensional fluids, i.e. we will assume that

$$\mathbf{u} = \mathbf{u}(\mathbf{x},t) \in \mathbb{R}^2 \ , \text{ for } \mathbf{x} \in \Omega \subset \mathbb{R}^2 \ . \tag{12}$$

In order to complete the specification of the problem, we must supplement equations (10)-(11) with appropriate boundary conditions. We'll focus on two special cases which are especially amenable to mathematical analysis, namely either

1. $\Omega = \mathbb{R}^2$, with boundary conditions imposed by assuming appropriate decay conditions on $\mathbf{u}$ at infinity, or
2. $\Omega = [-\pi,\pi]\times[-(\pi\delta),(\pi\delta)] = \mathbb{T}^2_\delta$ with periodic boundary conditions, i.e. $\mathbf{u}(x_1,x_2,t) = \mathbf{u}(x_1+2\pi,x_2,t) = \mathbf{u}(x_1,x_2+2\pi\delta,t)$. Note that δ is a parameter that measures the asymmetry of our domain - it is assumed to be $\mathcal{O}(1)$ and is not necessarily small.

Very early in the study of fluid mechanics it was pointed out by Helmholtz that it was often more useful to study the evolution of the fluid's *vorticity* than its velocity. The vorticity is defined by the curl of the velocity field - i.e. $\overline{\omega}(\mathbf{x},t) = \nabla \times \mathbf{u}(\mathbf{x},t)$ and in general it is a vector, like the velocity. However, in two-dimensions an important simplification occurs:

$$\overline{\omega}(x_1,x_2,t) = (0,0,\partial_{x_1}u_2 - \partial_{x_2}u_1) = (0,0,\omega(x_1,x_2,t)) \tag{13}$$

so we see that only one component of the vorticity is non-zero and we can treat it as a scalar. If we take the curl of (10) we see that in two dimensions, we arrive at the scalar PDE

$$\partial_t\omega + \mathbf{u}\cdot\nabla\omega = \nu\Delta\omega + f \ , \tag{14}$$

here the parameter $\nu = \alpha/\rho$, and $f = \frac{1}{\rho}(\partial_{x_1}g_2 - \partial_{x_2}g_1)$.

One advantage of this formulation of the problem is that the pressure term has disappeared entirely from the equation. However, the price we pay is that it appears at first that the equation is no longer well defined - the velocity $\mathbf{u}$ still appears in the equation, although we have no equation for its evolution. However, we can eliminate $\mathbf{u}$ from the equation by recalling that

- $\mathbf{u}$ is divergence free, and
- ω is the curl of $\mathbf{u}$.

This means that we can reconstruct $\mathbf{u}$ from the vorticity with the aid of Biot-Savart law, which in two-dimensions takes the form

$$\mathbf{u}(\mathbf{x},t) = B[\omega](\mathbf{x},t) = \frac{1}{2\pi}\int \frac{\mathbf{y}^{\perp}}{|\mathbf{y}|^2}\omega(\mathbf{x}-\mathbf{y},t)d\mathbf{y} = \frac{1}{2\pi}\int \frac{(\mathbf{x}-\mathbf{y})^{\perp}}{|\mathbf{x}-\mathbf{y}|^2}\omega(\mathbf{y},t)d\mathbf{y} \ , \tag{15}$$

where, if $\mathbf{y} = (y_1,y_2)$, $\mathbf{y}^{\perp} = (-y_2,y_1)$.

If we insert this representation into (14), we see that we can regard the vorticity equation:

$$\partial_t\omega = \nu\Delta\omega - B[\omega]\cdot\nabla\omega + f \ , \tag{16}$$

as a nonlinear heat equation, with a quadratic, but non-local, nonlinear term. This relationship with the heat equation, and in particular the fact that this means that the vorticity of two-dimensional flows satisfies a maximum principle will be used in subsequent sections.

In remaining sections we will discuss how dynamical systems ideas can be applied to dissipative PDEs, taking (16) as our principal example. It is perhaps not surprising that this equation defines a dissipative dynamical system given the relationship with the heat equation, which models the dissipation of heat energy in physical systems. We will verify the dissipativity in the sense of Definition 1 in a later section and see how ideas like invariant manifold theory and Lyapunov functions can give many insights into details of the behavior of its solutions.

2 The ω-limit set of the two-dimensional Navier-Stokes equation

In this section we focus on the long-time asymptotic behavior of solutions of the *unforced* two-dimensional vorticity equation

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega \tag{17}$$

defined on the whole plane - i.e. $\omega = \omega(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^2$. At first glance, it may seem that this equation is unlikely to yield interesting dynamics – the dissipation in the equation might be expected to dampen out any nontrivial motions. However, we will see that in the case of both bounded and unbounded domains, characteristic structures emerge in the solutions which numerical investigations indicate are also important features of two-dimensional forced flows.

Clearly, $\omega \equiv 0$ is a fixed point for this equation and from a dynamical systems point of view it is natural in this circumstance to linearize about the fixed point and ask what the linearization tells us about the behavior nearby. If we linearize the 2D vorticity equation about the zero solution the resulting linearized equation is the 2D heat equation:

$$\partial_t \omega = \nu \Delta \omega \ . \tag{18}$$

Approaching this problem from a dynamical systems perspective a natural next step would be to construct center/stable/unstable manifolds for the nonlinear equation which correspond to the eigenspaces of the linear problem with eigenvalues having zero, negative or positive real parts respectively.

Unfortunately we immediately face a problem if we attempt to apply those ideas in the present context. One can easily compute the spectrum of the linear operator on the RHS of the heat equation using the Fourier transform and one finds that the spectrum consists of the negative real axis, up to and including the origin. Since there is no gap in the spectrum there is no way to split the phase space into "center" or "stable" parts as required in the center manifold theorem and no obvious way of identifying the modes associated with particular decay properties.

A way of circumventing this difficulty emerges if we recall the form of the fundamental solution of the heat equation:

$$G_\nu(\mathbf{x}, t) = \frac{1}{4\pi\nu t} e^{-|\mathbf{x}|^2/(4\nu t)} \ . \tag{19}$$

This suggests that it may be natural to consider (17) not in the variables $(\mathbf{x}, t)$, but in new variables in which $\mathbf{x}$ and t are related as $\sim \mathbf{x}/\sqrt{t}$. With this in mind, we introduce new independent and dependent variables:

$$\begin{aligned} \omega(\mathbf{x}, t) &= \frac{1}{(1+\nu t)} w\left(\frac{\mathbf{x}}{\sqrt{1+\nu t}}, \log(1+\nu t)\right) \\ \xi &= \frac{\mathbf{x}}{\sqrt{1+\nu t}} \ , \quad \tau = \log(1+\nu t) \end{aligned} \tag{20}$$

Remark 2. *These types of variables are often used in studying parabolic PDE where they are sometimes referred to as scaling variables.*

If we now rewrite (18) in terms of these new variables we obtain the new PDE

$$\begin{aligned} \partial_\tau w &= \mathbb{L}w \ , \quad w = w(\xi, t) \ , \quad \xi \in \mathbb{R}^2 \\ \mathbb{L}w &= \Delta_\xi w + \frac{1}{2}\xi \cdot \nabla_\xi w + w = \Delta_\xi w + \frac{1}{2}\nabla_\xi \cdot (\xi w) \end{aligned} \tag{21}$$

At first sight, it may not be apparent why (21) is an improvement over (18) as we have made the equation more, rather than less, complicated. However, as we show below, in contrast to the Laplace operator which appears on the right hand side of the heat equation, the operator $\mathbb{L}$ has a gap in its spectrum between the part of the spectrum with zero real part and the remainder and this will allow us to apply the center-manifold theorem to understand the asymptotic behavior of solutions near the fixed point at the origin.

To understand the spectrum of $\mathbb{L}$, consider the eigenvalue problem

$$\mathbb{L}\psi_\lambda = \lambda \psi_\lambda \ . \tag{22}$$

If we separate variables in this PDE, we get a pair of ordinary differential equations of the form

$$\frac{d}{d\xi_1^2}\phi_\lambda + \frac{1}{2}\frac{d}{d\xi_1}(\xi_1 \phi_\lambda) = \lambda \phi_\lambda \ , \tag{23}$$

with a similar equation for the ξ_2 part of the solution. Taking the Fourier transform of this equation yields

$$-k^2 \hat{\phi}_\lambda - \frac{1}{2}k\frac{d}{dk}\hat{\phi}_\lambda = \lambda \hat{\phi}_\lambda \ . \tag{24}$$

This first order equation can be solved with the aid of integrating factors and one finds that for any $\lambda \in \mathbb{C}$ one has a solution:

$$\hat{\phi}_\lambda(k) = \frac{C^+}{|k|^{2\lambda}} e^{-k^2} H(k) + \frac{C^-}{|k|^{2\lambda}} e^{-k^2} H(-k) \tag{25}$$

where $H(k)$ is the Heaviside function and the fact that we have two constants of integration for a first order differential equation reflects the singular point at $k = 0$.

At first sight, this seems as if every point λ is in the spectrum of $\mathbb{L}$. However, recall that the spectrum of an operator depends on the space on which it acts. In particular, if $\Re(\lambda) > 0$, the functions in (25) "blow-up" at $k = 0$ and hence won't be in any well behaved function space. In order to say exactly what the spectrum of $\mathbb{L}$ is, we must decide what function space it acts on - like many operators, its spectrum will change, according to the domain chosen. It has long been known that the time-decay of solutions of parabolic PDEs is linked to the spatial decay rate of their solutions. With this in mind, we define a family of weighted Sobolev spaces:

$$\begin{aligned} L^2(m) &= \{f \in L^2(\mathbb{R}^2) \mid \|f\|_m < \infty\} \\ \|f\|_m &= \left(\int_{\mathbb{R}^2} (1+|\xi|^2)^m |f(\xi)|^2 d\xi\right)^{1/2} \\ H^s(m) &= \{f \in L^2(m) \mid \partial^\alpha f \in L^2(m) \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_d) \text{ with } |\alpha| \le s\} \end{aligned} \tag{26}$$

One reason that these spaces are so convenient for our purposes is the fact that Fourier transformation turns differentiation into multiplication and vice-versa. For these spaces, that makes it easy to check that for any non-negative integers s and m, Fourier transformation is an isomorphism between $H^s(m)$ and $H^m(s)$, i.e. a function f is in $H^s(m)$, if and only if its Fourier transform $\hat{f} \in H^m(s)$.

Applying this observation to the expression for $\hat{\phi}_\lambda$ in (25), we see that due to the singularity at $k = 0$ and the rapid decay as $|k| \to \infty$, $\hat{\phi}_\lambda \in H^m(s)$ if the first m derivatives of $\hat{\phi}_\lambda$ are square integrable in some neighborhood of the origin.

Note that there are some "special" values of λ. If $\lambda = 0, -1/2, -1, -3/2, \ldots$, we can choose $A^\pm$ so that $\hat{\phi}_{-n/2}(k) = Ak^n \exp(-k^2)$. These are entire, rapidly decaying functions and thus are elements of $H^m(s)$ for any values of m and s, so that the non-negative half integers are eigenvalues of $\mathbb{L}$ for any s and m, with eigenfunctions given by the inverse Fourier transform of these expressions. In particular, we see that $\lambda = 0$ is always an eigenvalue and its eigenfunction is the Gaussian $\phi_0(\xi) = C_0 \exp(-\xi^2/4)$. For non-half integral values of λ, we cannot choose the constants $A^\pm$ to make the eigenfunctions smooth, and thus, at least for some values of s, they will not be in the spaces $H^m(s)$. In fact, one can easily verify that in one-dimension, for no value of $\lambda \in \mathbb{C}$ with $\Re(\lambda) \ge 0$ will $\hat{\phi}_\lambda$ be in $H^1(s)$, because the derivative will have a non-square-integrable singularity at the origin. In two-dimensions, one can tolerate slightly worse singularities, but one still finds that no ϕ_λ with $\Re(\lambda) \ge 0$ will lie in $H^2(s)$. A careful examination of this argument allows one to compute exactly what the spectrum of $\mathbb{L}$ is and one finds

Theorem 2. *([GW02], Theorem A.1)Fix $m > 1$ and let $\mathbb{L}$ be the operator in (21) acting on its maximal domain in $L^2(m)$. Then*

$$\sigma(\mathbb{L}) = \left\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < \frac{1}{2} - \frac{m}{2}\right\} \cup \left\{-\frac{n}{2} \mid n = 0, 1, 2, \ldots\right\} .$$

The key fact here, from the point of view of dynamical systems theory, it that we have now created a spectral gap. So long as we choose the decay rate parameter $m \ge 2$ in our function space, the eigenvalue $\lambda = 0$ is an isolated point in the spectrum of $\mathbb{L}$, with all the rest of the spectrum lying strictly in the left half plane. Thus, at least intuitively, we can now hope to define the center subspace of the linear problem to be the span of the eigenfunction(s) of the zero eigenvalue, the stable subspace to be the spectral subspace corresponding to the remainder of the spectrum, and attempt to construct a center-manifold for the full nonlinear equation that it is tangent to the zero-eigenspace at the origin.

We first note that the zero-eigenspace is very simple in this case - it is one-dimensional and consists just of the span of the Gaussian function

$$\phi_0(\xi) = \frac{1}{4\pi} e^{-\xi^2/4} \ . \tag{27}$$

For future reference we note that the projection onto this eigenspace is given by the zero eigenfunction to the adjoint operator to $\mathbb{L}$, namely

$$\mathbb{L}^\dagger v = \nu \Delta v - \frac{1}{2}\xi \cdot \nabla v \ . \tag{28}$$

The constant function is clearly a zero eigenfunction for $\mathbb{L}^\dagger$ and consequently, the projection of $f \in H^2(m)$ onto the zero eigenspace of $\mathbb{L}$ is given just by $P_0 f = (\int_{\mathbb{R}^2} f(\xi) d\xi)\phi_0$.

While the spectral gap in the spectrum of the linear part of the equation makes it reasonable to expect that there will be an center manifold for the semi-flow defined by the 2D NSE, there are technical difficulties associated with constructing invariant manifolds for PDE that still must be overcome. In contrast to the case of ordinary differential equations where it is more or less clear what the optimal assumptions on the vectorfield should be in order to obtain an invariant manifold theorem, the situation is far less clear in the case of PDE. For instance, depending on the circumstances, it may be preferable to assume that the semi-group associated with the linear part of the equation may be more or less smoothing. Depending on the choices made in this case, one may need to make either stronger hypotheses about the nonlinear part of the equation, or draw weaker conclusions about the manifold one constructs. There is far less of a "one size fits all" invariant manifold theorem in infinite dimensional systems - instead one typically tailors the hypotheses of the theorem to the circumstances of interest. For some of the choices that have been made, see [BJ89], [Mie91] or [VI92]. One general principle that seems to emerge from these different contexts is that it is often easier to work with the semi-flow defined by the PDE, rather than the equation itself, if this semi-flow exists. This is because it already incorporates any smoothing associated with the linear evolution, and we don't need to worry about precisely what smoothing assumptions to make on the flow. (For instance, one common assumption is that the linear part of the equation defines an analytic semi-group, but while such a hypothesis would apply to the vorticity equation (14), it would no longer hold once we rewrite the equation in terms of the scaling variables (20).)

One infinite dimensional version of the center-manifold theorem that does apply to our problem is the version due to Chen, Hale and Tan (CHT) [CHT97]. (CHT) assume that the PDE defines a semi-flow ϕ^t on some Banach space, X. They then make four natural hypotheses about this semi-flow. We refer the reader to the original paper for the details on these assumptions, but roughly they are as follows:

(H1) $\phi^t(u)$ is Lipshitz continuous in both t and u with uniformly bounded Lipshitz constant for t in some interval.

(H2) For some fixed, positive τ, ϕ^τ can be split into a bounded linear operator and a globally Lipshitz operator.

(H3) The Banach space can be split into a direct sum of two pieces (a "center" subspace, X_1 and a "stable" subspace X_2) and when the linear part of ϕ^τ acts on the stable subspace it produces decay at a faster rate than the growth produced when the inverse of the linear part of ϕ^τ acts on the center subspace. (This is a reflection and consequence of a spectral gap in the spectrum of the linear part of the PDE.)

(H4) The ratio of the Lipshitz constant of the nonlinear term divided by the spectral gap must be small.

Assuming that these hypotheses are satisfied, (CHT) then prove the following:

- There exists a Lipshitz function $g : X_1 \to X_2$ whose graph is invariant with respect to ϕ^τ. This is the center manifold for our system.
- Any orbit of ϕ^τ will approach an orbit on the center manifold as time goes to infinity.

We will apply the (CHT) theorem to the two-dimensional vorticity equation, (17), rewritten in terms of the scaling variables (20). In addition to redefining the dependent vorticity variable ω as in (20) we also rescale the velocity variable as

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{\sqrt{1+\nu t}}\mathbf{v}(\frac{\mathbf{x}}{\sqrt{1+\nu t}}, \log(1+\nu t)) \ . \tag{29}$$

Remark 3. *With this definition one can check that the rescaled vorticity w and rescaled velocity $\mathbf{v}$ are still related to one another through the Biot-Savart law.*

If we rewrite (17) in terms of these new variables we find

$$\partial_\tau w = \mathbb{L}w - \frac{1}{\nu}\mathbf{v}\cdot\nabla_\xi w \ . \tag{30}$$

If we take initial condition $w|_{\tau=0} = w_0$, we can define the semi-group associated with (30) with the aid of DuHamel's formula as

$$w(\tau) = e^{\tau\mathbb{L}}w_0 - \frac{1}{\nu}\int_0^\tau e^{(\tau-s)\mathbb{L}}(\mathbf{v}(s)\cdot\nabla_\xi w(s))dx \ . \tag{31}$$

The two terms in (31) define the splitting into linear and nonlinear pieces in hypothesis $(H2)$, and the decay hypotheses of $(H3)$ are intuitively satisfied due to the gap in the spectrum of $\mathbb{L}$. One can check them rigorously just by noting that the semigroup $e^{t\mathbb{L}}$ is just the heat semigroup written in terms of the scaling variables. Expressing the heat semigroup in terms of these variables then leads to the expected estimates, [GW02].

The remaining hypotheses $(H1)$ and $(H4)$ as well as the second part of $(H2)$ concern the smoothness of the nonlinear term. In order to treat this term, one needs estimates that allow us to transfer information about the vorticity field to information about the velocity field. The types of estimates we will need are analogous to the Hardy-Little-Sobolev (HLS) inequality, [LL97]. Note that from (15), we have

$$|v_j(\mathbf{x})| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|\mathbf{x}-\mathbf{y}|} |w(\mathbf{x}-\mathbf{y})| d\mathbf{y} \ . \tag{32}$$

Thus, if $w \in L^p(\mathbb{R}^2)$, the HLS inequality immediately implies that

$$\|u\|_{L^q} \leq C\|\omega\|_{L^p} \ , \tag{33}$$

for $q = 2p/(2-p)$. We need to refine these inequalities to account for the weights in our function spaces (i.e. to account for the fact that we work not in L^2, but in $L^2(m)$), but very similar estimates hold in that case. An extensive set of such inequalities is proved in ([GW02], Appendix B).

These inequalities allow one to establish the Lipshitz bounds used in $(H1)$ and $(H4)$. However, they would not immediately lead the *global* Lipshitz estimate required in the second part of $(H2)$. This is obtained as in other constructions of the center-manifold theorem by "cutting off" the nonlinear term outside of some ball centered at the origin. More precisely we multiply the nonlinear term by $\chi(\|w\|_{L^2(m)})$, where χ is a smooth function equal to one on a neighborhood of zero and vanishing outside of some slightly larger neighborhood. This results in an equation whose nonlinear term now has a global Lipshitz bound and whose solutions agree with those of our original equation whenever $\|w\|_{L^2(m)}$ is sufficiently small. Again, the details are provided in [GW02].

The estimates above allow us to apply the (CHT) theorem to construct a one-dimensional center manifold for the two-dimensional vorticity equation, written in terms of the scaling variables as in (30). Furthermore, the (CHT) theorem also guarantees that all solutions of the equation which remain in some neighborhood of the origin will approach an orbit on the center manifold. (We note that this restriction to solutions near the origin is not contained in the statement of the (CHT) theorem above. It results from the fact that we had to cut-off the nonlinear term in (30) in order to verify the hypotheses of the theorem, and hence the conclusions of the theorem only apply to solution that remain in the region where the cutoff function is one.) Thus, all the long-time asymptotics of such solutions will be determined by the orbits on the center manifold, so our next step is to compute the dynamics of (30) when restricted to this manifold. In the vorticity equation the center subspace, X_1

is just the span of the eigenfunction of the zero eigenvalue which we showed above is the Gaussian function ϕ_0 - i.e. any function $w_c \in X_1$ has the form $\alpha\phi_0$. Thus, any point in the center manifold can be written as $w = \alpha\phi_0 + g(\alpha)$, where g is the function whose graph defines the center manifold, and the restriction of (30) to the center manifold can be written as

$$\dot{\alpha}\phi_0 = \mathbb{P}_0(N(\alpha\phi_0 + g(\alpha))) \ . \tag{34}$$

Here, N is short-hand notation for the nonlinear term in (30), $\mathbb{P}_0$ is the projection onto the center subspace, and there is no linear term in the equation because the eigenvalue corresponding to ϕ_0 which would give rise to the linear term in the equation is zero. Recall that as we showed just after (28), the projection of a function f onto the center subspace is given by multiplying by ϕ_0 by the integral of f over $\mathbb{R}^2$. Thus, we need to integrate the nonlinear term in (30) over $\mathbb{R}^2$ - i.e.

$$-\frac{1}{\nu}\int_{\mathbb{R}^2} (\mathbf{v} \cdot \nabla w) d\xi \tag{35}$$

But if we recall that $\mathbf{v}$ is incompressible, i.e. has zero divergence, we can rewrite the integrand in this expression as $\nabla \cdot (\mathbf{v}w)$, and then the integral vanishes due to the divergence theorem! Thus, the restriction of (30) to the center manifold is simply

$$\dot{\alpha} = 0 \ . \tag{36}$$

This implies the surprising fact that the center manifold consists entirely of fixed points! Even more surprising, we can write them down explicitly - they are just multiples of the Gaussian. That is, if we take

$$w(\xi, \tau) = \mathcal{O}^A(\xi, \tau) = \frac{A}{4\pi} e^{-|\xi|^2/4} \tag{37}$$

this is a fixed point of (30) for all values of A. The reason is that if one computes the velocity field corresponding to $\mathcal{O}^A$ from the Biot-Savart law it is

$$\mathbf{v}^A(\xi, \tau) = \frac{A}{2\pi}\frac{\xi^\perp}{|\xi|^2}\left(1 - e^{-|\xi|^2/4}\right) . \tag{38}$$

Note that this velocity field is a purely tangential vector field, while $\mathcal{O}^A$ is purely radial and as a consequence, $\mathbf{v}^A \cdot \nabla\mathcal{O}^A \equiv 0$. Recalling that $\mathbb{L}\phi_0 = 0$, since ϕ_0 is the eigenfunction with eigenvalue zero, we see that the RHS of (30) vanishes when evaluated at $\mathcal{O}^A$ and as a consequence these are all fixed points of the equation.

Remark 4. *These solutions of the two-dimensional Navier-Stokes equation have been known for a long time and are called Lamb-Oseen vortices. If expressed in terms of the original variables instead of the scaling variables, they are self-similar solutions of the equation, rather than fixed points.*

Remark 5. *We can use the (CHT) theorem to construct other invariant manifolds, besides just the center manifold. Recall the computation of the spectrum in Theorem 2. If we choose the decay rate m of our function space $m \geq 3$, we see that there are now two isolated eigenvalues in addition to the essential spectrum - namely $\lambda_0 = 0$ and $\lambda_1 = -1/2$. If one examines the eigenfunctions computed above, one finds that $-1/2$ is a double eigenvalue and its eigenfunctions are Hermite functions, $C_1 \xi_1 e^{-|\xi|^2/4}$ and $C_1 \xi_2 e^{-|\xi|^2/4}$, where the constant C_1 is chosen to normalize the eigenfunctions. One can then reapply the (CHT) theorem, this time taking as the "center" subspace X_1, the three dimensional space spanned by the Gaussian and these two Hermite functions. This gives rise to a three dimensional invariant manifold, and just as above one can compute explicitly the system of ordinary differential equations that results when one restricts (30) to this manifold. This computation uses the fact that the eigenfunctions of the adjoint operator $\mathbb{L}^\dagger$ with eigenvalues $-1/2$ are just the coordinate functions ξ_1 and ξ_2 and hence the projection onto these directions corresponds just to taking first moments of the solution. By choosing larger and larger values of m, one can expose more and more isolated eigenvalues, and one finds that their eigenfunctions are also given by Hermite functions of higher and higher order, and the projections onto these eigendirections are given by combinations of higher order moments of the solution.*

Remark 6. *There has been a fair amount of work using more traditional PDE techniques to identify special families of solutions of the Navier-Stokes equations which have specific temporal decay properties. For instance, Mayakawa and Schonbek [MS01] gave necessary and sufficient conditions for solutions to satisfy specific temporal decay rates in terms of integrals of various moments of the solutions. From the dynamical systems point of view, we see that solutions decaying with a particular rate can be identified as lying in a particular invariant manifold. For example, any solution approaching a non-zero Oseen vortex in the center manifold will decay in time with the same rate as the Oseen vortex, i.e. $\sim t^{-1/2}$ in the L^∞ norm in the original, unscaled variables. The only way a solution in a neighborhood of the origin can avoid approaching one of the Oseen vortices is if it lies in the invariant manifold of solutions asymptotic to the origin. (These manifolds are sometimes called Fenichel fibers, and they consist of all solutions sharing the same long-time asymptotics.) As we noted in the preceding remark, the projections onto the various eigenspaces are expressed as moments of the solution. Gallay and I were able to show that the condition that a solution lay on the Fenichel fiber through the origin was exactly the same condition on the moments that had earlier been found by analytic means in [MS01].*

Remark 7. *In Remark 5 we observed that the Hermite functions are eigenfunctions corresponding to the isolated eigenvalues of $\mathbb{L}$ and these can be used to construct invariant manifolds that govern the long-time asymptotics of solutions. They can also be used as the basis of a numerical method which expresses the solution of (14) as a sum of finitely many vortex blobs and then replaces the PDE (14) with a system of ordinary differential equations that track how the centers and the moments of each* *of these blobs evolve, [NSUW09]. The coefficients in these ordinary differential equations are expressed in terms of integrals over Hermite functions and thanks to the integration formulas for products of Hermite functions, one can derive compact combinatorial formulas for these coefficients, allowing efficient numerical implementation of these equations [UEWB12].*

The invariant manifold approach above has allowed us to identify the family of Oseen vortices as the only candidates for the long time asymptotic behavior of small solutions of the 2D NSE. However, there is still the possibility that if one chooses large initial data, some other type of behavior might emerge. In order to investigate that possibility we turn to a more global tool from dynamical systems, namely Lyapunov functions. Recall that roughly speaking Lyapunov functions are functions defined on the phase space of a dynamical system which are monotonic non-increasing along orbits of the system. Because of this monotonicity, we see that if an orbit approaches some long-time limit, the Lyapunov function, evaluated along that orbit, must also approach a limit and hence the orbit must approach a region of the phase space in which the Lyapunov function is constant. This last observation is the heart of the LaSalle Invariance Principle, and it is extremely useful in pinning down the possible locations of the ω-limit set of a dynamical system.

We now make these observations more precise.

Definition 3. *If X is a Banach space, a Lyapunov function for the semi-flow ϕ^t is a continuous, real-valued function Ψ, such that*

$$\limsup_{t\to 0^+} \frac{\Psi(\phi^t(u_0)) - \Psi(u_0)}{t} \le 0 \text{ for all } u_0 \in X \ . \tag{39}$$

The LaSalle Invariance Principle can then be stated as:

Proposition 1. *Let Ψ be a Lyapunov function for the semi-flow ϕ^t. Define $\mathbb{E} = \{u \in X \mid \frac{d}{dt}\Psi \cdot \phi^t(u)|_{t=0} = 0\}$. If the forward orbit of u_0 is contained in a compact subset of X, then the ω-limit set of u_0 lies in $\mathbb{E}$.*

Proof. The proof makes precise the idea sketched above. The compactness of the forward orbit of u_0, plus the continuity of Ψ means that $\Psi(\phi^t(u_0))$ is bounded below as a function of t. The monotonicity of Ψ along orbits then implies that there exists Ψ^∞ such that $\lim_{t\to\infty} \Psi(\phi^t(u_0)) = \Psi^\infty$. If we choose any point w in the ω-limit set, then there exists a sequence of times t_n tending toward infinity such that $\phi^{t_n}(u_0) \to w$, and this, combined again with the continuity of Ψ, means that $\Psi(w) = \Psi^\infty$. Since the ω-limit set is invariant under ϕ^t, and since w was an arbitrary point in the ω-limit set, we find that $\Psi(\phi^t(w)) = \Psi^\infty$ for all t and thus $w \in \mathbb{E}$. □

We now apply to the method of Lyapunov functions to the 2D NSE. We'll continue to work with the vorticity form of the equation, and since we're particularly interested in the Oseen vortices as possible ω-limit sets for solutions of this equation, we'll also continue to use the rescaled form of the equation (30).

The first thing we address is whether or not the forward orbit of a general initial condition w_0 is compact, since this plays an important role, both in the existence of the ω-limit set and in the LaSalle Invariance Principle. In fact, the first question to address is whether or not solutions even exist for general initial data. This turns out to be relatively easy to establish if we work in the weighted L^2 spaces we introduced earlier due to the relationship of (14) to the heat equation and the well understood smoothing and decay properties of the heat kernel. In fact, one can establish decay in much larger spaces of initial data. Work by Giga, Miyakawa and Osada, Ben-Artzi, Gallagher and Gallay, and other over the past twenty years or so have proven that the equation is globally wellposed for an initial vorticity distribution in $L^1(\mathbb{R}^2)$, or even is one takes measures as initial data, [GMO88], [BA94], [GG05]. We won't discuss the the proofs of these results because they don't have a particularly dynamical systems "flavor" which is the focus of this review, but instead refer the reader to the original articles for details.

Suppose, given the well-posedness results of the previous paragraph that we consider an arbitrary initial vorticity distribution $w_0 \in L^1(\mathbb{R}^2)$. The forward orbit of this point exists, and we would like to know if it has an ω-limit. This will follow if the orbit is relatively compact in L^1, and by the Rellich compactness criterion, this will in turn follow if we can demonstrate that the solutions have some smoothness and decay at infinity. In the case of (30):

- the smoothness of the solution comes from the smoothing properties of the heat kernel, which are preserved by the nonlinear term in the equation, and
- the decay at infinity come from estimates on solutions of the vorticity equation due to Carlen and Loss [CL95].

The details of this argument are presented in [GW05], and establish that the ω-limit set exists for any solution of (30) with initial vorticity in L^1.

We now compute what the ω-limit set actually is with the aid of two Lyapunov functions.

(A) The first Lyapunov function is motivated by relative entropy functions of kinetic theory. In kinetic theory, one often seeks to prove that the probability distribution for the velocities of a gas of particles converge to the Maxwellian distribution, i.e. a Gaussian distribution of the velocities. In our case, if the Oseen vortex is indeed the ω-limit set, then we also are looking for convergence to a Gaussian distribution - in this case of vorticity, rather than velocity.

(B) A significant problem with the analogy between kinetic theory and the vorticity equation is that while it is very natural to assume that the solutions of kinetic equations are non-negative (since they represent probability distributions) it is quite unnatural to assume that the vorticity is always of one sign. Since the relative entropy functional is only defined for functions that are everywhere positive (or negative), our second Lyapunov functional will ensure that even for solutions of (30) which change sign, the ω-limit set will still lie in the space of solution that are everywhere positive or everywhere negative.

We first focus on the relative entropy function from kinetic theory. The entropy functional, which originated in the study of statistical physics, is given by $\int w(x)\ln(w(x))dx$. The relative entropy function modifies this to look at the entropy relative to some fixed state - in our case the Oseen vortex. Thus, we define

$$H(w(\tau)) = \int_{\mathbb{R}^2} w(\xi,\tau)\ln\left(\frac{w(\xi,\tau)}{\phi_0(\xi)}\right)d\xi \ , \tag{40}$$

where ϕ_0 is the Gaussian function defined in (27). A straightforward computation shows that H is defined, continuous, and bounded below for any function $w \in L^2(m)$ which is everywhere positive, if $m > 3$. A similar definition can be constructed for everywhere negative functions, but it is not obvious how this functional can be modified to accommodate solutions that change sign. Differentiating $H(w(\tau))$ with respect to τ, we find

$$\frac{d}{d\tau}H(w(\tau)) = \int_{\mathbb{R}^2} w_\tau\left(1+\ln\left(\frac{w(\tau)}{\phi_0}\right)\right)d\xi \ . \tag{41}$$

If one now inserts the expression for w_τ from the RHS of (30) and integrates by parts (repeatedly!) one finds that

$$\frac{d}{d\tau}H(w(\tau)) = -\int_{\mathbb{R}^2} w\left|\nabla\left(\ln\frac{w}{\phi_0}\right)\right|d\xi \ . \tag{42}$$

Since $w(\xi,\tau) > 0$, this calculation implies that H is strictly decreasing unless the $\left|\nabla\left(\ln\frac{w}{\phi_0}\right)\right| = 0$, that is, unless $w = A\phi_0$ for some constant A. But then, by the LaSalle Invariance Principle, the ω-limit set of the orbit $w(\tau)$ must lie in the set of functions proportional to ϕ_0 - i.e. the ω-limit set must be one of the Oseen vortices. Thus, we have established that for solutions of (30) which do not change sign, the ω-limit set, must be one of the Oseen vortices, regardless of the size of the initial data, and we now turn to a consideration of what to do when the solution changes sign.

Remark 8. *In the calculation above, we used that if* $\left|\nabla\left(\ln\frac{w}{\phi_0}\right)\right| = 0$*, then* $w = A\phi_0$*. In principle, the constant* A *could depend on* τ*. This cannot occur in our context because of the fact that* $\int_{\mathbb{R}^2} w(\xi,\tau)d\xi$ *is constant. Hence the total "mass" of the solution is conserved and* A *cannot change with time. Note that we do assume in this calculation that the initial conditions are chosen so that* $\int_{\mathbb{R}^2} w_0(\xi)d\xi \neq 0$*.*

In order to treat solutions of (30) that change sign we exploit the similarity of the vorticity equation to the heat equation and in particular, we use the fact that its solutions satisfy a maximum principle. Given an initial condition ω_0 for (14) (or w_0 for (30)), split it into its positive and negative pieces - i.e. define

$$\begin{aligned} \omega_0^+(\mathbf{x}) &= \max(\omega_0(\mathbf{x}), 0) \\ \omega_0^-(\mathbf{x}) &= -\min(\omega_0(\mathbf{x}), 0) \ . \end{aligned}$$

Then define the evolution of the positive and negative parts of the data by

$$\partial_t^{\pm}\omega = \nu\Delta\omega^{\pm} - \mathbf{u}\cdot\nabla\omega^{\pm} \ . \tag{43}$$

Then if $\omega(\xi, t)$ is the solution of (14), with initial condition ω_0, we have

- $\omega(\mathbf{x}, t) = \omega^+(\mathbf{x}, t) - \omega^-(\mathbf{x}, t)$, and

- Both ω^+ and ω^- satisfy a maximum principle. In particular, since $\omega_0^{\pm}(\mathbf{x}) \geq 0$, we have $\omega^{\pm}(\mathbf{x}, t) > 0$ for all $\mathbf{x}$ and $t > 0$.

With these observations, it is easy to show that the L^1 norm of ω is a Lyapunov functional (for the details of this calculation, see [GW05].) Namely, we have

Lemma 1. *Define* $\Phi(\omega(t)) = \int_{\mathbb{R}^2} |\omega(\mathbf{x}, t)|\, d\mathbf{x}$. *Then* $\Phi(\omega(t))$ *is non-increasing in time, and is strictly decreasing unless* $\omega(\mathbf{x}, t)$ *is everywhere positive or everywhere negative.*

Putting together our two Lyapunov functionals we can now show that for any solution of (30), the ω-limit set must be an Oseen vortex. Let Ω be the ω-limit set of a solution of (30) with initial condition $w_0 \in L^1(\mathbb{R}^2)$. Assume that $\int_{\mathbb{R}^2} w_0(\xi) d\xi \neq 0$. Applying the LaSalle Invariance Principle to the Lyapunov functional Φ, we see that any point Ω must lie in the set of functions which are everywhere positive or everywhere negative. But then, pick a point $\bar{\omega} \in \Omega$ and apply the LaSalle Invariance Principle again, this time with the relative entropy functional. From this we conclude that ω-limit set must be of the form $A\phi_0$, with $A = \int_{\mathbb{R}^2} w_0(\xi) d\xi$, and so the ω-limit set of every solution in L^1 is just an Oseen vortex.

Remark 9. *Note that there's one additional step that we have swept under the rug here. We only know that the relative entropy functional is continuous and bounded on the weighted Hilbert spaces,* $L^2(m)$*, not on all of* L^1*, so we can't directly apply the above argument to solutions in* L^1*. However, using the decay estimates of Carlen and Loss mentioned above, one can prove that the* ω*-limit set of any* L^1 *solution must lie in the spaces* $L^2(m)$ *for any* $m > 1$*, and then one can repeat the above argument.*

To conclude this section note that we have now shown that any solution of the two-dimension NSE (with integrable, nonzero total vorticity) will eventually approach an Oseen vortex. If we start with small initial data, the invariant manifold theorem gives

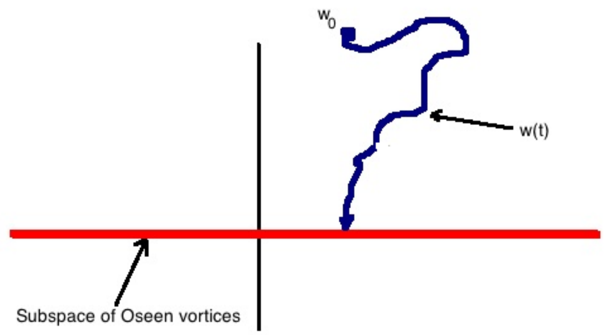

Figure 1: The phase space of the 2D NSE contains a line of fixed points (when expressed in terms of scaling variables) and almost every solution approaches some point on this line asymptotically, but we have little information about the rate of approach.

us very precise information about the asymptotic rate of approach of the solution to the Oseen vortex, but for general initial data, it may take a very long time for the solution to approach this limiting state. In the next section of this review, we examine some possible behaviors that may occur on intermediate time scales, before the solution finally converges to its asymptotic state.

3 Metastable states, pseudo-spectrum and intermediate time scales

In this section we look at another application of dynamical systems ideas to the two-dimensional NSE, namely the emergence of metastable states in the system. That part of this section which is original work, is all joint work with Margaret Beck, and the details of the proofs appear in [BW11a]. In contrast to the previous section we now consider the equations on a rectangular domain with periodic boundary conditions. We are specifically interested in this section in the appearance of structures *before* the long-time asymptotic state appears, and most of the numerical studies of these phenomena have been done on such periodic domains. As in the previous section, it is convenient to study the evolution of the vorticity

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega \ , \tag{44}$$

but this time we require that

$$\omega(x_1, x_2, t) = \omega(x_1 + 2\pi, x_2, t) = \omega(x_1, x_2 + 2\pi\delta, t) \ , \tag{45}$$

where $\delta \sim \mathcal{O}(1)$ is the asymmetry parameter of the domain (and will equal one for a square domain.) As in the previous section we can recover the velocity field in (44) from the vorticity via the Biot-Savart law, which in this case is most conveniently expressed in terms of the Fourier coefficients of the solution:

$$\hat{\omega}(k, \ell) = \frac{1}{4\pi^2 \delta} \int_{\mathbb{T}^2_\delta} \omega(x_1, x_2) e^{-i(kx_1 + \ell x_2/\delta)} dx_1 dx_2 \ , \tag{46}$$

with analogous definitions of $\hat{\mathbf{u}}_{1,2}(k, \ell)$. (To save space, we suppress the time dependence of the functions when it will not cause confusion.) The Biot-Savart law then takes the form

$$\hat{\mathbf{u}}(k, \ell) = i \frac{(-\ell/\delta, k)}{k^2 + (\ell/\delta)^2} \hat{\omega}(k, \ell) \ . \tag{47}$$

Remark 10. *We leave it as an easy exercise to show that because of the periodic boundary conditions, $\hat{\omega}(0,0) = 0$, so that (47) is well defined. One could choose $\hat{\mathbf{u}}(0,0)$ to be an arbitrary constant, but we will set it equal to zero. Given (47), one can derive estimates on the norm of the velocity in terms of those of the vorticity, analogous to those in Section 2.*

Note that from (47), we see immediately that if $\omega \in L^2(\mathbb{T}^2_\delta)$, then so are both components of the velocity field. If we apply the energy inequality derived below in (74), and take advantage of the fact that the external forcing is zero here, we see that all solutions will tend asymptotically to zero. We note here an important distinction between the Navier-Stokes equation on the torus and in the plane. In both cases, the smoothing of the evolution implies that if the initial vorticity, $\omega_0 \in L^1$, then $\omega(t) \in L^2$ for any $t > 0$. However, in the present case, as noted just above, this implies that the system has finite energy and hence will decay to zero as $t \to \infty$, rather than to an Oseen vortex, as in the previous section.

From the energy inequality, we see that the rate of convergence toward the asymptotic states occurs on the viscous time scale $t \sim \mathcal{O}(1/\nu)$, which in the weakly viscous regime in which turbulent fluids are typically studied is enormously long. However, in numerical experiments on two-dimensional turbulent flows, one sees that on time scales much shorter than the viscous time scale characteristic structures emerge which then come to dominate the flow for very long periods of time, until one finally reaches the asymptotic state. The goal in the present section is to describe some recent work which proposes an explanation of the emergence of these intermediate time scales based on dynamical systems theory.

In order to gain some insight into what the metastable states and their associated time scales are in the NSE, we first review some of the numerical results on this system. One of the starting points of Beck's and my work were the investigations of Yin, Mongomery and Clercx [YMC03]. While their numerical experiments are consistent with an eventual convergence of solutions to zero, much more striking is that characteristic structures like vortex dipole pairs or "bar states" (shear flows, in which the vorticity contours are constant in one direction) emerge quickly from an initially very disordered state, and then dominate the subsequent evolution of the flow for very long times.

`Insert figures from [YMC03] here if permission is granted.`

In the numerics of [YMC03], the most common metastable states that are observed in the system are the dipole states - only with rather carefully prepared initial conditions are the bar states observed. However, if instead of considering the equation on a square domain as in [YMC03], one considers the equation on a rectangular domain, the numerical experiments of Bouchet and Simonnet [BS09], indicate that the bar states can become the dominant metastable states. Furthermore, these states are sufficiently stable that they continue to dominate the evolution even if the equation is subjected to a random force. While the random force may cause an apparently random switching between the bar and dipole states, for the great majority of the time, the system is in one or the other of these two states.

The goal, in the remainder of this section is to propose an explanation for the rapid appearance and long persistence of these families of solutions of the two-dimensional NSE. In [BW11b], Beck and I proposed a dynamical systems explanation for similar families of metastable states in Burgers equation. These states, and their importance for the dynamics of the system, were first systematically investigated by Kim and Tzavaras in [KT01], where they are called "diffusive N-waves". Our explanation of the metastable behavior in Burgers equation began by showing that (in scaling variables, similar to those used in the previous section) there was a one-dimensional invariant manifold in the infinite dimensional phase space of the equation which is completely filled with fixed points and these fixed points represent the only possible long-time asymptotic states of the system. These are analogous to the family of Oseen vortices in the 2D NSE. If one linearizes about one of these fixed points one finds the spectrum of the linearized operator has a zero eigenvalue corresponding to motion along this manifold. The remainder of the spectrum lies strictly in the left half-plane. The next smallest eigenvalue is a simple eigenvalue $\lambda = -1/2$, with the rest of the spectrum having more negative real parts. Locally, invariant manifold theory allows one to construct a one-dimensional manifold tangent at the fixed point to the eigenfunction corresponding to this eigenvalue. Using the Cole-Hopf transformation, Beck and I extended this manifold globally and proved that these manifolds are normally stable - that is, if one enters a neighborhood of the manifold, one will remain in a neighborhood of the manifold for all subsequent times. We also showed that this manifold consists of exactly the diffusive N-waves previously identified as the important metastable states in Burgers equations by [KT01]. Thus, we referred to these manifolds as the "metastable manifolds". The final step in our construction was to show that "almost every" (in a sense made precise in [BW11b]) initial condition gives rise to a solution of Burgers equation which approaches one of these metastable manifolds on a short time scale. They evolve (due to the stability properties of the manifolds) slowly along the manifold until they eventually approach the long-time asymptotic state on the center manifold. If we represent this scenario graphically, we obtain the following picture of the phase space of the weakly viscous Burgers equation:

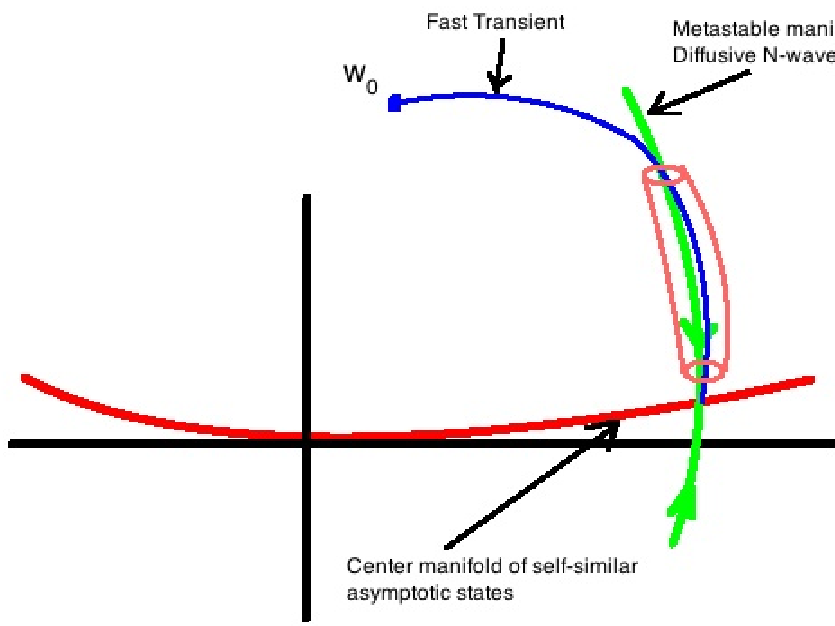

Figure 2: The phase space of the weakly viscous Burgers equation, showing the metastable manifolds of diffusive N-waves which govern the intermediate asymptotics.

Comparing this with Figure 1, we see that in this case we have a much more complete picture of the phase space structures which organize both the long-term and intermediate asymptotics, and we would now like to extend as much as possible of this model to understand the appearance of metastable states in the 2D NSE.

For the NSE equation on the torus we have already remarked that the only long-term asymptotic state is the zero solution. If we linearize the vorticity equation (44) around the zero solution we find, just as before, the heat equation,

$$\partial_t w = \nu \Delta w \ , \tag{48}$$

but this time with periodic boundary conditions. Note that unlike the case in $\mathbb{R}^2$ studied in the previous section, on the torus the heat equation has discrete spectrum (and a spectral gap) so there is no need to introduce scaling variables as we did there. Since we are only considering solutions of zero mean (see Remark 10), the eigenvalues of the right hand side of (48) are

$$\lambda(m,\ell) = -\nu(m^2 + \delta^{-2}\ell^2) \ , \quad (m,n) \neq (0,0) \ , \tag{49}$$

with the corresponding eigenfunctions given by simple combinations of sines and cosines. We recall that the parameter δ measures the asymmetry of our domain, and if $\delta < 1$, then the smallest eigenvalue is given by $\lambda(1,0) = -\nu$, with eigenfunction $w_{1,0}(x_1,x_2) = A \sin x_1$. More generally, we could choose the eigenfunction to be $A\sin x_1 + B\cos x_1$, but by a translation of the origin we can choose it to be proportional to $\sin x_1$. One expects on the basis of the general theory of dynamical systems that, modulo the technical difficulties that come from working in an infinite dimensional phase space, one should be able to construct an invariant manifold for the semiflow generated by the NSE that is tangent at the origin to the eigenspace of the eigenvalue $\lambda(1,0) = -\nu$. However, in general, we will only be able to approximate this manifold in a small neighborhood of the fixed point $w \equiv 0$. In the case of Burgers equation we used the Cole-Hopf transformation to extend this local manifold globally, but that tool is not available here. Remarkably though, one can write down an invariant family of solutions of the full vorticity equation that is tangent at the origin to $A\sin x_1$. It is simply,

$$\omega^b(x_1,y_x) = Ae^{-\nu t}\sin(x_1) \ , \quad \mathbf{u}^b(x_1,x_2) = -Ae^{-\nu t}\begin{pmatrix} 0 \\ \cos x_1 \end{pmatrix} \ . \tag{50}$$

We'll refer to this family of states as bar states following the terminology of [YMC03], though these states are also know as Kolmogorov flows, and physically they represent a simple shear flow.

Remark 11. *There are a number of related explicit solutions of the 2D NSE. One can of course replace the sine functions in (50) with cosines, or take a linear combination of sine and cosine. There are also analogous states associated with the eigenvalues* $-\nu m^2$ *which are proportional to* $\exp(-\nu m^2 t)\sin m x_1$*, as well as solutions associated with the eigenvalues* $-\nu(\ell/\delta)^2$ *corresponding to shear flows oriented along the* x_2 *coordinate direction and proportional to* $\exp(-\nu(\ell/\delta)^2 t)\sin \ell x_2/\delta$*. More generally, if one takes any solutions of the heat equation with periodic boundary conditions which only depends on the variable* x_1*, this will give a solution of the two-dimensional vorticity equation, because if one checks the form of the velocity field given by the Biot-Savart law (47), one finds that the nonlinear term in the equation vanishes identically.*

Remark 12. *There are also explicit solutions analogous to the dipole solutions observed in [YMC03]. These appear in square domains (i.e. when the parameter $\delta = 1$, are often known as Taylor-Green vortices and they are solutions of the vorticity equation with*

$$\omega(x_1, x_2.t) = Ae^{-\nu t}(\cos(x_1) + \cos(x_2)) \ , \quad \mathbf{u}(x_1, x_2, t) = Ae^{-2\nu t}(-\sin(x_2), \sin(x_1)) \tag{51}$$

While we believe that the framework we use to discuss metastability of the bar states below is probably also applicable to the dipole states, mathematically the analysis is significantly harder, so we focus here just on the bar states.

Remark 13. *If one plots the constant vorticity contours of the bar states and the Taylor-Green vortices, one sees that they are very similar to those of the bar and dipole states observed numerically.*

However, there are some discrepancies. If, for example, one computes the stream function associated with the bar states by solving Poisson's equation,

$$-\Delta\psi^b = \omega^b \ , \tag{52}$$

One sees that $\psi^b(x_1, x_2, t) = Ae^{-\nu t}\sin(x_1) = \omega^b(x_1, x_2, t)$ - i.e. the stream function is a linear function of the vorticity. Plots of ψ vs. ω for numerical solutions of the 2D NSE (see Fig. 9 of [YMC03], for example) show that while for small values of the vorticity the dependence is nearly linear, there is some departure from this linear behavior at large values of the vorticity. Nonetheless we believe that the bar states are good candidates for the metastable states in these systems because once the system gets close to such a state (as it appears to do in the numerics) the stability results described below show that it will remain nearby and actually converge toward these states at a rate much faster than expected from viscous effects alone.

We now examine the stability of the family of bar states. We hope both to show that they attract nearby trajectories, and also to understand why they appear on a time scale so much shorter than the viscous time scale. In the case of Burgers equation we proved that the metastable manifold was normally stable with the aid of the Cole-Hopf transformation. That tool is no longer available to us, so we resort to a more direct, dynamical systems, approach, namely we linearize the NSE about the bar states and study the evolution of this linearized equation. Linearizing (44) about ω^b leads to the linear PDE

$$\partial_t w = \nu\Delta w - \mathbf{u}^b \cdot w - \mathbf{v} \cdot \nabla\omega^b \ , \tag{53}$$

where $\mathbf{v}$ is the velocity field associated with w. Because of the form of the velocity field $\mathbf{u}^b$ (see (50)), $\mathbf{u}^b \cdot w = Ae^{-\nu t}\sin(x_1)\partial_{x_2} w$. Likewise, since ω^b is independent of x_2, the last term in (53) also simplifies to $\mathbf{v}_1 \partial_{x_1}\omega^b = Ae^{-\nu t}\cos(x_1)\mathbf{v}_1$. From the Biot-Savart law we see that we can write $\mathbf{v}_1 = \partial_{x_2}(\Delta^{-1}w)$, where Δ^{-1} can be computed via its action on the Fourier series of w, i.e. $\widehat{\Delta^{-1}w}(m, \ell) = -\hat{w}(m, \ell)/(m^2 + (\ell/\delta)^2)$.

Thus, the last two terms on the RHS of (53) simplify and we are left with the linear equation

$$\partial_t w = \nu \Delta w - A e^{-\nu t} (\cos(x_1)) \partial_{x_2} (\mathbb{1} + \Delta^{-1}) w \ . \tag{54}$$

Analyzing the stability of the family of bar states is more complicated than analyzing the stability of a fixed point of the equation because the linear equation (54) is non-autonomous. Nonetheless, computing the spectrum of the RHS of (54) for some fixed time t may give insight into the behavior of solutions. If we fix the time t and set $\tilde{A} = A e^{-nut}$, we can compute the eigenvalues of

$$\mathcal{L}_{\nu,\tilde{A}} w = \nu \Delta w - \tilde{A} (\cos(x_1)) \partial_{x_2} (\mathbb{1} + \Delta^{-1}) w \ , \tag{55}$$

on the space of functions satisfying periodic boundary conditions. This is a relatively easy computation (numerically) if we express w as a Fourier series and consider the way $\mathcal{L}_{\nu,\tilde{A}}$ acts on these series. If $\hat{w}(m,\ell)$ are the Fourier coefficients of w, defined as in (46) we see that $\mathcal{L}_{\nu,\tilde{A}}$ doesn't "mix" different values of ℓ because there is no non-trivial x_2 dependence in the operator. Thus, we can consider separately the action of $\mathcal{L}_{\nu,\tilde{A}}$ on spaces of functions with different, fixed values of ℓ. Denoting this operator by $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ we have

$$\begin{aligned} (\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}} \hat{w})(m,\ell) &= -\nu (m^2 + (\ell/\delta)^2) + \\ -i \frac{\tilde{A}\ell}{2\delta} &\left[(1 - \frac{1}{(m-1)^2 + (\ell/\delta)^2}) \hat{w}(m-1,\ell) + (1 - \frac{1}{(m+1)^2 + (\ell/\delta)^2}) \hat{w}(m+1,\ell) \right] \ . \end{aligned} \tag{56}$$

Note that the operator $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ has a special form. It has a real diagonal (and hence symmetric) piece with negative eigenvalues and a small coefficient in front of it, and a large off-diagonal piece which is "almost" skew-symmetric (due to the "i" in front of that term). In fact, as explained in [BW11a], a simple change of variables allows one to rewrite $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ as the sum of a diagonal piece and an exactly skew-symmetric off-diagonal piece. As we'll see in the remainder of this section, such operators which arise frequently in fluid mechanics, often have very special spectral properties.

As a first, simple remark about the properties of the operator $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$, note that if we are given any matrix of the form

$$L = D + A \tag{57}$$

where D is a real diagonal operator with eigenvalues lying in the set $\sigma_0 = \{\lambda \in \mathbb{R} \mid \lambda \leq -\nu\}$, and with A a skew-symmetric matrix, then no matter how large A is (i.e. no matter how large its norm), the eigenvalues of L remain to the left (in the complex plane) of the line $\Re(\lambda) = -\nu$. This follows from the following simple calculation. Suppose that λ is an eigenvalue of L with normalized eigenvector v. Then we have

$$\begin{aligned} \lambda &= \langle v, Lv \rangle = \langle v, (D+A)v \rangle = \langle (D-A)v, v \rangle \\ \overline{\lambda} &= \langle Lv, v \rangle = \langle (D+A)v, v \rangle \ . \end{aligned}$$

Adding these two expressions together and dividing by 2 we find

$$\Re(\lambda) = \langle v, Dv \rangle \le -\nu \ , \tag{58}$$

where the last inequality comes from the bound on the eigenvalues of D and the fact that $\|v\| = 1$.

In physical terms this means somewhat surprisingly that if we have a small, stable, symmetric dissipative linear operator, and add to it a skew-symmetric piece, a situation which comes up frequently in fluid mechanics when we linearize about some non-trivial solution of the NSE, we cannot destabilize the system, no matter how large the skew-symmetric piece is. In fact, what may happen is that the skew-symmetric piece, even though its own eigenvalues all lie on the imaginary axis, serves to further stabilize the system.

One example where this seems to occur is in the linearization of the 2D NSE in the whole plane about the Oseen vortex solutions that were discussed in the previous section. In that the case, the dissipative diagonal part again just comes from the Laplacian term in the vorticity equation, while the skew-symmetric piece comes from the linearization of the nonlinear terms about the vortex. The stability properties of this linearization were first investigated numerically by Prochazka and Pullin [PP95] who found that as the Reynold's number increased, (which means that the skew-symmetric piece of the operator became larger and larger with respect to the symmetric piece) the real part of almost all the eigenvalues of the linearization became increasingly negative – i.e. the vortex becomes more and more stable. The few eigenvalues whose real parts don't become more negative are fixed, independent of the Reynold's number and correspond to special, exact, symmetric solutions of the NSE.

A proposal to theoretically explain this stability phenomenon was first proposed by Gallagher, Gallay and Nier [GGN09] who linked this behavior to the hypercoercivity method developed by Villani [Vil09] and were able to prove rigorously that this "enhanced stability" occurs in a model problem. More recently, W. Deng [Den12], [Den11], has extended this method to the actual linearization about an Oseen vortex, at least for modes with a sufficiently strong angular dependence (i.e. if one expands the perturbation in a Fourier series in polar coordinates of the form $w(r,\theta) = \sum_n \hat{w}_n(r) \exp(in\theta)$, then n is required to be sufficiently large.)

Beck and I propose that a similar mechanism is responsible for the metastable properties of the bar states. We begin with numerical evidence that this is the case, by computing the eigenvalues of the operator $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ as $\nu \to 0$. We find that in this limit, in which the skew-symmetric operator is much larger than the symmetric part, the eigenvalues not only all have negative real part, but also the real parts are proportional not to ν, as the eigenvalues of the symmetric part are, but rather to $\sqrt{\nu}$. This indicates the presence of a new time scale in the problem

$$\tau_{meta} \sim \frac{1}{\sqrt{\nu}} << \tau_{viscous} \sim \frac{1}{\nu} \tag{59}$$

which is much shorter than the viscous time scale when ν is small.

To illustrate this effect, we first start with a simple example in which we take a 41×41 matrix approximation to the operator $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$. We choose the parameters $A = 1$, $\nu = 0.01$, $\delta = 0.9$ and $\ell = 1$, though we would obtain very similar results for essentially any other choices of parameters. The three figures below show first of all the eigenvalues of the diagonal part of the operator, then the off-diagonal, nearly anti-symmetric part of the operator, and finally the eigenvalues of the full operator, illustrating how the interplay between these two pieces actually results in eigenvalues whose real part is more negative, than the operator without the anti-symmetric part.

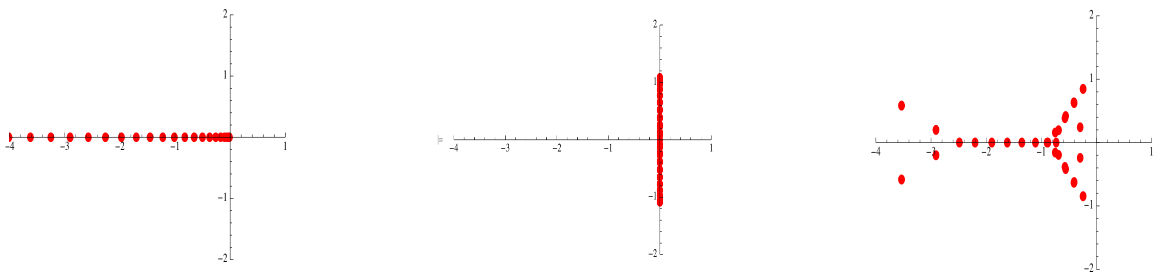

Figure 3: A numerical simulation showing how the interplay between the symmetric and anti-symmetric parts of an operator can lead to more negative eigenvalues (i.e. enhanced stability) than either part alone.

To study more quantitatively the effects of this interplay, and in particular, to examine the dependence of the eigenvalues on the viscosity, ν, the next figure considers the least negative eigenvalue of a 201×201 matrix approximation to $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$. The parameters A, δ, and ℓ are as above, but now we choose four different values of $\nu = 0.0005, 0.0001, 0.00005, 0.00001$. We then plot the logarithm of ν against the logarithm of the absolute value of the real part of the least negative eigenvalue. Intuitively, this eigenvalue should determine the least stable mode when we perturb the bar states. Superimposed on the log-log plot is a line of slope $1/2$, which would correspond to the real part of the eigenvalue being proportional to $\sqrt{\nu}$. As one can see, the agreement is very good.

Remark 14. *There is one special case in which the eigenvalues of $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ do scale as ν and that is when $\ell = 0$. These correspond to family of x_2 independent solutions of the vorticity equation discussed in Remark 11. There are also a small number of eigenvalues which scale anomalously when $\ell = \pm 1$. They are discussed in detail in [BW11a]. While the origin of these "anamolous" states is not clear, it may be that they are somehow connected with the dipole states.*

The eigenvalue calculations above are suggestive solutions of (55) will decay with rate $\sim \sqrt{\nu}$, but for highly non-symmetric, non-autonomous operators like $\mathcal{L}$, it is

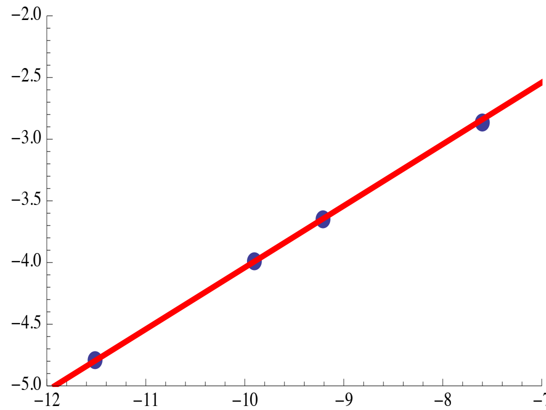

Figure 4: A log-log plot of ν against the absolute value of the real part of the least negative eigenvalue, superimposed on a line of slope 1/2.

well known that one cannot automatically assume that the spectrum of the linearized operator at a particular time determines the decay rate of solutions. Often, a more relevant quantity is the pseudo-spectrum, [TE05]. The behavior of the pseudo-spectrum may be qualitatively different than that of the eigenvalues and in the model of the linearization of the NSE about the Oseen vortex investigated by Gallagher, et al [GGN09], this turns out to be the case. However, for our problem numerical calculations suggest that the pseudospectral bounds also scale like $\sqrt{\nu}$, so these computations lead to conclusions about the stability of the bar states similar to those obtained from the eigenvalue calculations above.

Because of the difficulty in obtaining information about the decay rates of the evolution generated by $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ from either the spectrum or pseudo-spectrum of the operator at any fixed time, we will instead study the evolution with the aid of the hypercoercivity method developed by Villani [Vil09]. This method was developed to deal specifically with evolutions generated by operators of the form $L = A^*A + B$, where B is skew-symmetric. The method can account for the enhanced stability of the sort we observe by exploiting the lack of commutativity between A and B. This point is crucial – if $[A, B] = 0$, we could simultaneously diagonalize the two operators and wouldn't observe an qualitative difference in the behavior of the real parts of the eigenvalues of A^*A and $A^*A + B$. Villani systematized this by considering the time decay of a special functional which incorporates this non-commutativity and then relating this functional to more standard norms of solutions.

At the moment we're unable to apply Villani's method to the full linearized equation (55) due to the presence of the non-local term $\cos(x_1)(\Delta^{-1}w)$. We have computed the spectrum of the operator $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ both with this term present (those were the computations presented above) and without this term (see Figure 5, just below) and the spectrum is very similar in both cases. In particular, the real parts of the eigenvalues display the same $\sqrt{\nu}$ scaling. In addition the non-local term is a compact, lower order pertrubation so we believe it will have a small influence on the evolution of the system. Thus, instead of studying the evolution generated by $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$, we consider the approximate equation

$$\partial_t w = \mathcal{L}^{\ell}_{\text{approx}} w = \nu \Delta w - A e^{-\nu t}(\cos(x_1))\partial_{x_2} w \ , \tag{60}$$

or, if we again expand w in terms of its Fourier series and consider terms with the same fixed value of ℓ, as we did in (56), we have

$$\partial_t w = \mathcal{L}^{\ell}_{approx} w = -\nu(\partial_{x_1}^2 - \ell^2) w + i\frac{\ell}{\delta} A e^{-\nu t}(\cos(x_1)) w \ . \tag{61}$$

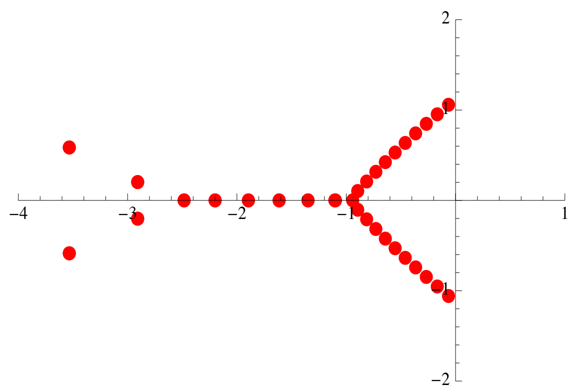

Figure 5: A plot of the eigenvalues of the approximate linear operator in (60). The parameter values are the same as in Figure 3

Remark 15. *Note that one other advantage of (61) as compared to (54) is that the operator on the RHS of (61) is already a sum of a symmetric piece plus a skew-symmetric piece, without resorting the initial change of variables needed to bring* $\hat{\mathcal{L}}^{\ell}_{\nu,\tilde{A}}$ *into this form.*

Remark 16. *In the study of the linearization of the 2D NSE about the Oseen vortex a similar nonlocal term appeared in linearization. In [Den12], Deng studied the pseudo-spectrum of the full linearization of the equation and the approximation to the linearization obtained by dropping the nonlocal term. She found similar results in both cases.*

Remark 17. *Note that (61) no longer contains any explicit dependence in x_2. Thus, for the remainder of this section we will replace x_1 by x to simplify the notation.*

As mentioned above, Villani's method exploits the non-commutativity of the symmetric and skew-symmetric parts of the operator and with that in mind we define the following operators

$$B^\ell w = i\frac{\ell}{\delta}Ae^{-\nu t}(\cos(x))w \ , \quad C^\ell w = [\partial_{x_1}, B^\ell]w = i\frac{\ell}{\delta}Ae^{-\nu t}(\sin(x))w \tag{62}$$

Note that like B^ℓ, C^ℓ is also skew-symmetric, B^ℓ and C^ℓ commute with each other, and both are bounded operators.

Following Villani, we now define a functional that incorporates the effects of C^ℓ on the evolution, namely,

$$\Phi^\ell(t) = \|w\|^2 + \alpha\|\partial_x w\|^2 - 2\beta\Re(\partial_x w, C^\ell w) + \gamma(C^\ell wl, C^\ell w). \tag{63}$$

where the constants α, β, and γ will be chosen in the course of the proof. The first restriction we place on their values comes from the fact that we want Φ^ℓ to control some norm of the solution of (61). If we require that

$$\beta^2 < \alpha\gamma/4 \ , \tag{64}$$

then we have

$$\|w\|^2 + \frac{\alpha}{2}\|w_x\|^2 + \frac{\gamma}{2}\|C^\ell w\|^2 < \Phi^\ell(t) < \|w\|^2 + \frac{3\alpha}{2}\|w_x\|^2 + \frac{3\gamma}{2}\|C^\ell w\|^2.$$

Now consider the time rate of change of the Φ^ℓ along a solution of (61). This is almost identical to the analogous computation in [Vil09], except for minor modifications due to the fact that the operators B^ℓ and C^ℓ are non-autonomous in our case. Thus, we obtain:

$$\begin{aligned}\frac{d}{dt}\Phi^\ell(t) = {} & ((w_t, w) + (w, w_t)) + \alpha\left((\partial_x w_t, \partial_x w) + (\partial_x w, \partial_x w_t)\right) \\ & -2\beta\Re\left((\partial_x w_t, C^\ell w) + (\partial_x w, C^\ell w_t)\right) + \gamma\left((C^\ell w_t, C^\ell w) + (C^\ell w, C^\ell w_t)\right) \\ & -2\beta\Re(\partial_x w, \frac{dC^\ell}{dt}w) + \gamma\left((\frac{dC^\ell}{dt}w, C^\ell w) + (C^\ell w, \frac{dC^\ell}{dt}w)\right).\end{aligned} \tag{65}$$

We now use (61) and the properties of B^ℓ and C^ℓ (specifically their anti-symmetry) to control each of the terms in (65). The details of that procedure are set out in [BW11a] – in this review we just focus on a few representative terms, including those that give the enhanced stability, to explain the ideas involved.

For instance, consider the terms

$$\begin{aligned}(w_t, w) + (w, w_t) &= -\nu\left[(\partial_x^2 w, w) + (w, \partial_x^2 w)\right] - 2\nu\ell^2(w, w) \\ &\quad +(B^\ell w, w) + (w, B^\ell w) \\ &= -2\nu\left[\|w\|_{L^2}^2 + \|w_x\|_{L^2}^2\right] \ ,\end{aligned} \tag{66}$$

where the terms involving B^ℓ vanish due to anti-symmetry. Note that the remaining terms contribute to the decay of Φ^ℓ, but only with a rate proportional to ν - so this would represent decay on the viscous time scale.

To see where the accelerated decay comes from, we consider those terms proportional to β, since they are the ones which exploit the lack of commutativity between the symmetric and anti-symmetric parts of $\mathcal{L}^\ell_{approx}$. Thus,

$$
\begin{aligned}
(\partial_x w_t, C^\ell w) + (\partial_x w, C^\ell w_t) &= -2\ell^2\nu\Re(w_x.C^\ell w) + \nu\left[(w_{xxx}, C^\ell w) + (w_x, C^\ell w_{xx})\right] \\
&\quad +(\partial_x(B^\ell w), C^\ell w) + (w_x, C^\ell(B^\ell w)) \ . \qquad (67)
\end{aligned}
$$

The terms involving derivatives of w can be handled with the aid of integration by parts and then absorbed in other terms because they are all proportional to ν. The two remaining terms can be combined if we recall from (62) that $\partial_x B^\ell = C^\ell + B^\ell\partial_x$, while $C^\ell B^\ell = B^\ell C^\ell$. Thus, we find

$$
\begin{aligned}
(\partial_x(B^\ell w), C^\ell w) + (w_x, C^\ell(B^\ell w)) &= (C^\ell w, C^\ell w) + (B^\ell w_x, C^\ell w) + (w_x, B^\ell(C^\ell w)) \\
&= \|C^\ell w\|^2 \ , \qquad (68)
\end{aligned}
$$

again, using the anti-symmetry of B^ℓ. Taking into account the fact that this term has a negative coefficient in front of it, we see that it gives a large (i.e. $\mathcal{O}(1)$, rather than $\mathcal{O}(\nu)$) negative contribution to $d\Phi^\ell/dt$ and is responsible for the accelerated convergence rate.

There are still a number of problems which must be overcome. The remaining terms must all be carefully bounded, and in particular, while the term discussed in the preceding inequality will yield an accelerated, convergence rate for the part of Φ^ℓ proportional to $\|C^\ell w\|^2$, one must show that it can also yield a bound for the other terms in Φ^ℓ. The details of these estimates are provided in [BW11a], where one finds that there exist positive constants M and K, independent of ν, such that

$$
\Phi^\ell(t) \leq K e^{-M\sqrt{\nu}t}\Phi^\ell(0) \ . \qquad (69)
$$

It is also explained there how one can combine the estimates for different values of ℓ to show that the norm of the full solution of (60) also decay with a rate proportional to $\sqrt{\nu}$.

Thus we see that the interplay between the symmetric and anti-symmetric parts of the linearized operator leads to a decay rate much faster than the $\mathcal{O}(\nu)$ viscous decay rate, and consequently a time scale associated with the convergence to the bar states that is much faster than suggested by viscous effects alone. For instance, for a Reynold's number of $\sim 10^4$, (in the non-dimensionalized variables in which we are working, this would be $\nu \sim 10^{-4}$), these estimates suggest an approach toward the metastable solutions like bar states on a time scale of the order of $\tau_{meta} \sim 100$ time units, rather that the $10,000$ time units expected if the convergence were governed by the viscous time scale. We note that these convergence rates and Reynold's numbers are qualitatively similar to those observed in numerical simulations of these systems.

Of course the picture in the case of metastable behavior for the 2D NSE is still far less complete than for Burgers equation. The most important missing piece is the fact that so far, the rigorous estimates only apply to the approximation of the true linear evolution obtained by dropping the nonlocal term in the equation. However, as we have seen from the numerical computations presented above, the spectral properties of the operator with or without the nonlocal term are very similar, and we hope that the sort of techniques Deng developed in [Den11] will also apply here and allow us to treat the full linearized operator.

A second important difference between Burgers equation and the 2D NSE is that in Burgers equation the viscous N-waves are the only metastable states for almost all solutions, but for the NSE there are other candidates, such as the dipole states observed in numerics (see Remark 12), or the states discussed in Remark 11 which like the bar states only depend on one of the two spatial variables. This means that the final picture of metastability for the NSE will undoubtedly be more complicated than the simple picture in Figure 2 for Burgers equation. Nonetheless, we feel that these dynamical systems ideas can yield important simplifying insights into the intermediate asymptotics of this important physical system and can explain how time scales much shorter than the viscous time scales can arise from the operators which have a small, symmetric dissipative piece, and a large, anti-symmetric component, which arise frequently in the study of fluid motions.

4 Finite Dimensional Attractors for the Navier-Stokes equations

In the two previous sections we focussed on the behavior of the two-dimensional Navier-Stokes equations in the absence of forcing and studied the types of structures that emerged on intermediate and long time scales. In the present section we consider solutions of the equation when it is subjected to external forcing. In practice, such forcing might arise through shearing at the boundaries of the domain, as is the case in the Taylor-Couette experiment, but as is often done in theoretical studies we will allow for a general body force, i.e. we will consider the form of the equations in (10) and (11).

$$
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= \nu \Delta \mathbf{u} - \nabla p + \mathbf{g} \qquad (70)\\
\nabla \cdot \mathbf{u} &= 0\,. \qquad (71)
\end{aligned}
$$

Here the forcing term $\mathbf{g}$ is some general vector valued function whose smoothness we specify below.

Numerical studies show that even in the presence of certain types of forcing, the structures we studied in previous sections may still play a dominant role in the dynamical behavior of the system [BS09]. However, in this section we focus on more general properties of the system such as the existence and dimension of the attractor for the system. We will present fewer details of the calculations in this section than in the prior ones since the work described here is somewhat older than that in the previous sections and several excellent monograph length reviews of these results already exist [Rob01], [DG95].

While most investigations of these questions work directly with the fluid velocity, in order to keep the discussion as close as possible to prior sections we will continue to work with the vorticity formulation of the problem. This has the disadvantage that we are restricted essentially to domains without boundary and thus we focus here as in section three on a rectangular domain with periodic boundary conditions, so we consider functions in

$$L^2(\mathbb{T}^2_\delta) = \{u_j \in L^2 \mid u_j(x_1,x_2) = u_j(x_1+2\pi,x_2) = u_j(x_1,x_2+2\pi\delta) \ , \ j=1,2\} \ . \tag{72}$$

Assuming that the forcing $\mathbf{g}$ is differentiable, we can then take the curl of (70) to obtain the scalar equation

$$\partial_t \omega = \nu\Delta\omega - \mathbf{u}\cdot\nabla\omega + f \ , \tag{73}$$

where $\mathbf{u}$ is again the velocity field recovered from the vorticity via the Biot-Savart law (47), and $f = \partial_{x_1}\mathbf{g}_2 - \partial_{x_2}\mathbf{g}_1$.

Proving the existence and uniqueness of solutions for this equation is straightforward so we will assume that has been done and focus instead on more detailed aspects of its dynamics. Recall from the introduction that an infinite dimensional dynamical system is dissipative if the phase space contains a precompact absorbing ball. That follows in the present circumstances from straightforward energy estimates. If we multiply (73) by ω and integrate over the torus, we find that

$$\partial_t \|\omega(t)\|_{L^2} = -\nu\|\nabla\omega\|^2_{L^2} - \int_{\mathbb{T}^2_\delta} \omega(\mathbf{u}\cdot\nabla\omega)dx + \int_{\mathbb{T}^2_\delta} \omega f dx \ . \tag{74}$$

Consider each of the terms on the RHS of this equation in turn. Since $\int \omega dx = 0$, we can apply Poincaré's inequality to the first term and bound it by $-c_p\nu\|\omega\|^2_{L^2}$. Turning to the second term, the incompressibility of the velocity field $\mathbf{u}$ allows us to rewrite it as:

$$\int_{\mathbb{T}^2_\delta} \omega(\mathbf{u}\cdot\nabla\omega)dx = \frac{1}{2}\int_{\mathbb{T}^2_\delta} \mathbf{u}\cdot\nabla\omega^2 dx = \frac{1}{2}\int_{\mathbb{T}^2_\delta} \nabla\cdot(\mathbf{u}\omega^2)dx = 0 \ , \tag{75}$$

where the last inequality follows from the periodic boundary conditions. Finally the Cauchy-Schwartz inequality implies that $|\int \omega f dx| \leq \|\omega\|_{L^2}\|f\|_{L^2} \leq \frac{1}{2}c_p\nu\|\omega\|^2_{L^2} + \frac{1}{2c_p\nu}\|f\|^2_{L^2}$. Inserting these estimates into (74) we find

$$\partial_t \|\omega(t)\|_{L^2} = -\frac{1}{2}c_p\nu\|\omega\|^2_{L^2} + \frac{1}{2c_p\nu}\|f\|^2_{L^2} \ . \tag{76}$$

If we now assume that the forcing function is an element of $L^\infty([0,\infty); L^2(\mathbb{T}^2_\delta))$, we can apply Gronwall's inequality to conclude that

$$\|\omega(t)\|^2_{L^2} \leq e^{-c_p \nu t/2}\|\omega_0\|^2_{L^2} + \frac{2}{c_p \nu}(1 - e^{-c_p\nu t/2})\|f\|_{L^\infty([0,\infty);L^2)} \ . \tag{77}$$

and so we see that all solutions of (73) will eventually enter the ball of radius $R = \frac{2}{c_p\nu}\|f\|_{L^\infty([0,\infty);L^2)}$.

As noted in the introduction, in infinite dimensional systems, the existence of an absorbing ball is not enough in itself to guarantee an attractor. We much show that the absorbing set is also (pre-)compact in L^2. One does this by showing the boundedness of the absorbing ball in H^s, with $s \geq 1$, and using the fact that H^s is compactly embedded in L^2. The boundedness of the H^s norm follows by energy estimates similar to those in (74)-(76). They are somewhat more complicated in this case though because the nonlinear term no longer vanishes as in (75). One still obtains a bound on the H^s norm though and as a consequence the existence of a bounded, compact attractor for the forced two-dimensional NSE. In fact, in two-dimensions, there is a whole sequence of energy estimates linking and bounding the various Sobelev norms of the solution - for an explication of these estimates, sometimes known as the ladder theorems, see [DG95].

As discussed in the introduction, one advantage of the existence of an attractor is that it often allows one to focus attention on a much smaller dimensional system than the original, infinite dimensional PDE. In fact, the attractor for the two-dimensional NSE is finite dimensional and one can estimate its dimension in terms of the forcing function f. This fact was first established for a class of reaction-diffusion equations by Mallet-Paret [MP76]. In the context of the two-dimensional NSE is first studied by Constantin and Foias, [CF85]. It is well known that the attractor of a chaotic dynamical system is often a fractal set and so to measure it, we must use a *fractal* dimension. There are several common fractal dimensions including the Hausdorff dimensional and the capacity dimension. Estimates of the both these quantities exist in the case of the attractor in the two-dimensional NSE, but we'll focus on the capacity here because it is the simplest to define. Given a set $\mathcal{A}$ in the phase-space of our PDE, define $N(r) =$ min number of balls of radius r needed to cover $\mathcal{A}$. Then the capacity dimension of $\mathcal{A}$ is defined to be:

$$d_C(\mathcal{A}) = \limsup_{r\to 0} \frac{N(r)}{\log(1/r)} \ . \tag{78}$$

It is easy to check that if one takes a segment of a curve or surface in $\mathbb{R}^d$, the capacity dimension gives the usual and expected values of 1 and 2 respectively. Suppose now that we consider the attractor for the two-dimensional NSE. The basic idea relating the dimensional estimates to the dynamics of the system is the following. Suppose one covers the attractor with d-dimensional balls of radius r. On can show that if the attractor is finite dimensional it suffices to use finite (although perhaps large) dimensional balls to compute the dimension. If one considers how these balls will be distorted by the flow, say by the time-one map of the system, they will typically be elongated in some directions, corresponding to the chaotic stretching that occurs in the attractor. However, the high-frequency modes in the Navier-Stokes equation are very strongly damped, and if we choose d, the dimension of the covering balls, sufficiently large, the number of contracting directions will overwhelm the effect of the expanding directions and result in the total volume of the ball shrinking. The capacity is defined in terms of the limit as the radius of the balls tends to zero so we are most interested in the action of the flow on balls of very small radius. In this limit, the action of the flow is well approximated by the linearization of the flow, which will distort our original ball into a d-dimensional ellipsoid. The lengths of the axes of this ellipsoid can then be related to the radius of the original ball by the singular values of the linearized system, which are in turn related to the Lyapunov exponents of the dynamical system. If the Lyapunov exponents of the system are $\mu_1 \geq \mu_2 \geq \dots$, (where we have listed the exponents along with their multiplicity), then the shortest axis of the deformed ellipsoid should have length $\sim re^{\mu_d}$, and the volume on the ellipsoid should be approximately $\sim r^d e^{\mu_1 + \cdots \mu_d}$. This in turn gives a recursion relation between $N(r)$ and $N(e^{\mu_d})$, which can then be used to evaluate the $\limsup$ in (78). The details of this calculation are very clearly explained in([DG95], p. 67) and lead to an estimate of the dimension in terms of the attractor. [1]

There are a number of points that must be addressed in order to make this argument rigorous including:

- The Lyapunov exponents concern the asymptotic growth rates of the linearized flow, not the amount of growth or shrinking that occurs in the time-one map.
- Typically, the Lyapunov exponents will depend on the initial point we choose, so it's not clear why this argument leads to a uniform estimate on the dimension of the attractor.
- Finally, we would like to have an estimate on the attractor dimension in terms of accessible quantities in the equation, like the viscosity, ν, or the forcing f, rather than generally unknown Lyapunov exponents.

Constantin and Foias address the first two points by introducing global Lyapunov exponents. They consider the usual definition of the Lyapunov exponents in terms of the growth rates of n-dimensional volumes induced by the linearized flow, but before taking the infinite time limit of these growth rates, they take the supremum of the growth rates over the attractor. This gives an estimate of the growth rate independent of the starting point, but might lead to an overestimate of the dimension. They also give rigorous estimates relating the size of a ball of radius r, transported for a finite

[1] The link between the attractor dimension and the Lyapunov exponents was first proposed by Kaplan and Yorke, [KY79], and is often known as the Kaplan-Yorke formula.

time t, to these global Lyapunov exponents, as well of making rigorous the details of the Kaplan-Yorke formula for infinite dimensional systems like the NSE.

To relate the Lyapunov exponents of the NSE to the quantities that appear in the equation, note that if we linearize (73) about a solution $\omega(t)$, with associated velocity field $\mathbf{u}(t)$, the linearized equation takes the form:

$$\partial_t W = \nu \Delta W - \mathbf{u} \cdot \nabla W - \mathbf{U} \cdot \nabla \omega \ , \tag{79}$$

where $\mathbf{U}$ is the velocity field associated with W.

Recall that the estimates of dimension were based on how the linearized flow causes d-dimensional spheres to expand or contract. Thus, instead of studying (79) directly, we consider the evolution of $(P_d W)$, the projection of W onto an d-dimensional subspace of the infinite dimensional phase space, and consider the maximum growth rate over all such projections. The growth rate of such projections can be controlled by energy estimates of the type that lead to (77), and one finds

$$d_C \leq C \left(\frac{\|\mathbf{g}\|_{L^2}}{\nu^2} \right)^{2/3} \left(1 + \log \left(\frac{4\pi^2 \delta}{\nu^2} \|\mathbf{g}\|_{L^2} \right) \right) \ . \tag{80}$$

In this estimate, the factor of $4\pi^2\delta$ is just the area of the domain on which the equation evolves, and $\mathbf{g}$ is the forcing term in the equation for the velocity field, not the vorticity, and this force is assumed to be constant in time. Note that although the forcing function for the fluid velocity appears in the estimate of dimension, the actual estimates are done in the vorticity formulation which seems to yield sharper bounds on the attractor dimension than working directly with the vorticity [CFT88]. For more details on the derivation of these estimates, one can consult either the original paper, [CF85], which explains the ideas behind the proof very clearly, or the monograph, [DG95].

Remark 18. *There is a non-rigorous estimate of the dimension of the attractor of the two-dimensional Navier-Stoke equation based on a scaling theory of turbulence. Remarkably, the rigorous estimate (80) agrees with the prediction of the scaling theory, up to the factor of* $\left(1 + \log\left(\frac{4\pi^2\delta}{\nu^2}\|\mathbf{g}\|_{L^2}\right)\right)$.

Remark 19. *Bounding the dimension of the attractor is one way to quantify the essentially finite dimensional nature of the two-dimensional NSE semi-flow. Another approach focuses on showing that there is a finite number of* determining modes *or* determining nodes*. In this case, one asks whether or not the knowledge of the behavior of the projection of the solution on a finite number of modes, or its value at a finite number of points, is sufficient to determine its long time behavior. More precisely, let* P_d *be the projection onto an* d*-dimensional subspace discussed above. One says that the system has* d *determining modes if for some choice of* P_d*, any two solutions* $\omega(t)$ *and* $\tilde{\omega}(t)$ *for which*

$$\lim_{t \to \infty} \|P_d(\omega(t)) - P_d(\tilde{\omega}(t))\|_{L^2} = 0 \ , \tag{81}$$

must also have

$$\lim_{t\to\infty} \|\omega(t) - \tilde{\omega}(t)\|_{L^2} = 0 \ , \tag{82}$$

with a similar definition for determining nodes. Note that this definition does not imply that given a knowledge of the first d modes of the system we can reconstruct the remaining modes, but merely that those (infinitely many) other modes cannot change the long-term asymptotics of the solution.

The first estimates establishing that the solutions of the two-dimensional NSE had a finite number of determining modes were by Foias and Prodi [FP67]. More recently, Jones and Titi, [JT93] have shown that the for periodic boundary conditions, the number of determining modes is bounded by

$$d \leq C \frac{\|\mathbf{g}\|_{L^2}}{\nu^2} \ , \tag{83}$$

which is larger than estimate for the dimension of the attractor established above.

While the finite dimensionality of the attractor for the two-dimensional NSE implies that the asymptotic behavior is determined by the motions on some "small" piece of the phase space, it doesn't immediately imply that there is a smooth finite dimensional dynamical system whose motion reproduces the long-time behavior of the NSE. The reason is that one knows very little about the structure of the attractor and in fact, for chaotic systems, it is expected to be a complicated, fractal set. Thus, simply restricting the NSE to the attractor will not generally lead to a smooth, finite dimensional, dynamical system. An alternate approach which has been pursued is to attempt to construct a finite dimensional manifold which is:

- smooth, or at least Lipshitz,
- invariant with respect to the semi-flow defined by the NSE, and
- attracts any solution of the equation at an exponential rate.

Such manifolds, if they exist, are called *inertial manifolds*.

If a PDE possesses an inertial manifold, then by its invariance and attractivity, the attractor must lie inside it. Thus, by restricting the PDE to the inertial manifold one obtains a smooth (or at least Lipshitz), finite dimensional dynamical system which captures all the asymptotic behavior of the original, infinite dimensional system.

Inertial manifolds bear much resemblance to center manifolds in finite dimensional systems of ordinary differential equations. Recall that the existence theory for center manifolds requires that the "spectral gap", that is, the difference between the real parts of the stable and unstable eigenvalues and the center eigenvalues, must be large in comparison with the Lipschitz constant of the nonlinear term in the equation.

In the inertial manifold case, this becomes the requirement that there be a large gap between the eigenvalues corresponding to modes related to the inertial manifold and those "off" the manifold. Typically the distinction between these two sets of modes is expressed by writing the inertial manifold is written as a graph of a function whose domain is the span of the first set of these modes and whose range is the span of the modes off the manifold. The size of the gap is again related to the properties of the nonlinear term, but are more complicated than those in the ODE case because the nonlinear term in the PDE is often not Lipshitz on the whole phase space of the system. For example, consider the nonlinear term in the vorticity equation (73). The presence of the derivative in the nonlinearity means it is not even defined on the whole phase space, $L^2(\mathbb{T}^2_\delta)$, let alone Lipschitz. This lack of smoothness results in much more stringent spectral conditions than in the center manifold case.

For nonlinear terms of the sort that appear in the NSE, the best estimates to date for the spectral gap require that if the eigenvalues of the linear part of the equation are ordered as $0 \geq -\lambda_1 \geq -\lambda_2 \geq \ldots$, then proof of existence for the inertial manifold requires that for a manifold on dimension N, one must have:

$$\lambda_{N+1} - \lambda_N \geq K(\lambda_{N+1}^{1/2} + \lambda_N^{1/2}) \ . \tag{84}$$

For (73), the eigenvalues of the linear part can be explicitly computed as $\nu(m^2+(\ell/\delta)^2$, and one sees that the large gaps required by (84) do not exist. As a consequence, there is no proof to date, that the two-dimensional NSE possesses an inertial manifold.

There are, however, large classes of dissipative PDEs for which the existence of an inertial manifold has been proven. Most of these fall into one of two categories:

1. 1-dimensional equations: In one dimension, if one considers a PDE with linear part
$$\partial_t u = \nu \partial_x^r u \ , \tag{85}$$
with periodic boundaries on the interval $[-\pi, \pi]$, the eigenvalues are $-\lambda_n = -\nu n^{2r}$. Thus, for $r > 1/2$, one obtains arbitrarily large gaps between eigenvalues as n grows. Using this growth, the existence of inertial manifolds for equations like the Kuramoto-Sivishinsky and Cahn-Hilliard, [FST85], [FST88], [CFNT89].

2. Equations with "nicer" nonlinear terms: If the nonlinear term in the equation is a smooth function on the phase space of the equation, as is true for many reaction-diffusion equations, the spectral gap condition can be greatly weakened and the existence of inertial manifolds for many two-dimensional domains, and even some special three-dimensional domains has been established, [MPS88].

In conclusion, we have seen that the tools of dynamical systems theory provide a number of insights into the qualitative behavior of solutions of dissipative PDEs, especially the two-dimensional Navier-Stokes equations. The identification of finite dimensional invariant structures in the infinite dimensional phase space of these systems, be they attractors, or invariant manifolds, allow one to better understand the long-time asymptotic behavior of both the forced and unforced Navier-Stokes equations, as well as providing a way of understanding the emergence of intermediate time scales in these systems.

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