Generalized Helmholtz-Kirchhoff model for two dimensional distributed vortex motion

Raymond Nagem[1] Guido Sandri[1] David Uminsky[2,3]
C. Eugene Wayne[2,3]

February 1, 2008

Abstract

The two-dimensional Navier-Stokes equations are rewritten as a system of coupled nonlinear ordinary differential equations. These equations describe the evolution of the moments of an expansion of the vorticity with respect to Hermite functions and of the centers of vorticity concentrations. We prove the convergence of this expansion and show that in the zero viscosity and zero core size limit we formally recover the Helmholtz-Kirchhoff model for the evolution of point-vortices. The present expansion systematically incorporates the effects of both viscosity and finite vortex core size. We also show that a low-order truncation of our expansion leads to the representation of the flow as a system of interacting Gaussian (i.e. Oseen) vortices which previous experimental work has shown to be an accurate approximation to many important physical flows [9].

1 Introduction

In this paper we represent solutions of the two-dimensional Navier-Stokes equations as a system of interacting vortices. This expansion, which generalizes the Helmholtz-Kirchhoff model of interacting point vortices in an inviscid fluid, systematically incorporates the effects of both vorticity and finite vortex core size. Furthermore, we

[1]Department of Aerospace and Mechanical Engineering, Boston University
[2]Dept. of Mathematics and Statistics and Center for Biodynamics, Boston University
[3]Supported in part by NSF grant DMS-0405724

give conditions which guarantee the convergence of our expansion. Incompressible viscous flow has two standard analytic representations: a formulation in terms of the primitive velocity and pressure variables, and a formulation in terms of the velocity and vorticity variables [7]. The velocity-vorticity representation has particular advantages when boundaries are unimportant, since vorticity cannot be created or destroyed in the interior of a fluid. The vorticity field can also be directly related to physically observed flow structures such as line and ring vortices.

In two space dimensions, the vorticity field has the additional advantage of reducing to a scalar. An early representation of two-dimensional flow in terms of moving point vortices was developed by Helmholtz-Kirchhoff [4] and by Helmholtz [15] . The point vortex model has been studied extensively - a thorough review of the model and recent developments are described in the monograph by Newton [11]. While the Helmholtz-Kirchhoff point vortex model captures many of the basic physical phenomena observed in two-dimensional rotational flows, experiments with even relatively simple vortex configurations exhibit complications far beyond the point vortex predictions [9]. Additionally, the classical point vortex model neglects the effects of *viscosity*. However, these experiments also reinforce the idea that in many circumstances the fluid flow may be well approximated by a collection of interacting vortices - albeit vortices with finite core size, subject to the effects of viscosity. A few recent studies of the interaction of viscous vortices can be found in [1, 5, 6, 8, 9, 14]. The main focus of these papers is the merger of two like signed vortices. In [8] the authors use a spatial moment model for 2-D Euler equations which later incorporates weak Newtonian viscosity to derive equations of motions for two like signed vortices. A metastable state is found before merger which consists of two rotating, near-circular, vortices. More recently, it has been conjectured that in a two-vortex system the profiles relax to a pair of gaussian vortices before merging [6] . Thus, it is of interest to extend and generalize the Helmholtz-Kirchhoff point vortex model to a model that incorporates non-zero vortex core size and viscous effects while retaining its basic form. Such an extension is the goal of this paper.

The governing equations for the velocity ($\mathbf{u}$) and pressure (p) variables are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \frac{p}{\rho} + \nu \Delta \mathbf{u}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where ρ is the fluid density and ν is the kinematic viscosity. Taking the curl of (1) and (2) gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega}, \tag{3}$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}, \quad \nabla \cdot \boldsymbol{\omega} = 0, \tag{4}$$

which are the governing equations for the velocity-vorticity variables. For two-dimensional flows, the vorticity vector $\boldsymbol{\omega}$ is perpendicular to the plane of the flow, and the third term on the left-hand side of (3) vanishes. The condition $\nabla \cdot \boldsymbol{\omega} = 0$ is identically satisfied, and equation (3) then reduces to the single scalar equation:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega, \tag{5}$$

where ω is the single, non-zero component of the vorticity. A drawback of the formulation in (5) is that the velocity of the fluid is still present in the equation. However, assuming that the vorticity field is sufficiently localized, the velocity vector can be computed in terms of the vorticity ω by the Biot-Savart law

$$\mathbf{u}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy \ , \tag{6}$$

where for a two-vector $z = (z_1, z_2)$, $z^\perp = (-z_2, z_1)$.

In this paper we use equations (5) and (6) to develop a vorticity representation of two-dimensional viscous flow. Our representation is based on a decomposition of the vorticity field into a set of moving distributed vortices. Differential equations are derived for the motion of the vortex centers and for the time evolution of the vortex distributions. The evolution of each individual vortex is represented as an expansion with respect to a sequence of Hermite functions. Such expansions have proven useful in theoretical studies of two-dimensional fluid flows ([2], [3]) and the leading order term in this expansion is precisely the Gaussian vortex (i.e. Oseen vortex [13],[12]) whose utility as an approximation for vortex interaction was shown in [9]. We show that the coefficients in this expansion satisfy a system of ordinary differential equations whose coefficients can be explicitly represented in terms of a fixed, computable kernel function. We also prove the convergence of this expansion. It is shown that our representation reduces in the appropriate limit to the Helmholtz-Kirchhoff model, and allows at the same time arbitrarily complex evolution and interaction of the moving vortices.

In the present paper we concentrate on the mathematical formulation of the generalized Helmholtz-Kirchhoff model. In future work we will explore the predictions of this model both numerically and analytically in a number of different physical settings.

2 The "multi-vortex" expansion

In this section we separate the solution of the vorticity equation into N components and derive separate evolution equations for each component. From a physical point of view this decomposition will be most useful when each of the components corresponds to a localized region of vorticity (e.g. a vortex) well separated from the other lumps but the mathematical development described below is well defined without regard to these physical considerations. However, with this application in mind, we will often refer to each of the components as a "vortex".

Consider the initial value problem for the two-dimensional vorticity equation

$$\begin{aligned}
&\frac{\partial \omega}{\partial t} = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \\
&\omega = \omega(x,t), \;\; x \in \mathbb{R}^2, \;\; t > 0 \\
&\omega(x,0) = \omega_0(x)
\end{aligned} \tag{7}$$

where $\mathbf{u}$ is the velocity field associated to the vorticity field ω. We begin by decomposing the initial vorticity distribution by writing

$$\omega_0(x) = \sum_{j=1}^{N} \omega_0^j(x) \; . \tag{8}$$

Of course this decomposition is not unique - even the number of pieces, N, into which we decompose the vorticity is up to us to choose. In general the choice we make will be motivated by physical considerations, however, for the development below, all we require of the decomposition is that the total vorticity of each vortex is non-zero, i.e.

$$m_j = \int_{\mathbb{R}^2} \omega_0^j(x) dx \neq 0, \;\; j = 1, \ldots, N. \tag{9}$$

If (9) is satisfied we define x_0^j by

$$\int_{\mathbb{R}^2} (x - x_0^j) \omega_0^j(x) dx = 0 \; , \tag{10}$$

or equivalently

$$x_0^j = \frac{1}{m_j} \int_{\mathbb{R}^2} x \omega_0^j(x) dx \; . \tag{11}$$

We now write the vorticity for $t > 0$ as

$$\omega(x,t) = \sum_{j=1}^{N} \omega^j(x - x^j(t); t) \tag{12}$$

and the velocity field as

$$\mathbf{u}(x,t) = \sum_{j=1}^{N} \mathbf{u}^j(x - x^j(t);t) \tag{13}$$

where $\mathbf{u}^j(y,t)$ is the velocity field associated to $\omega^j(y,t)$ by the Biot-Savart Law. Of course we still have to define the equations of motion for $\omega^j(y,t)$ and $x^j(t)$.

The centers of the vorticity regions, $x^j(t)$, and the vorticity regions themselves evolve via a coupled system of ordinary-partial differential equations constructed so that in the limit of zero viscosity and when the different components of the vorticity happen to be point vortices (i.e. Dirac-delta functions) we recover the Helmholtz-Kirchhoff point vortex equations. If we take the partial derivative of (12) and use the equation satisfied by the vorticity, we find:

$$\begin{aligned}\partial_t \omega(x,t) &= \sum_{j=1}^{N} \partial_t \omega^j(s - s^j(t), t) - \sum_{j=1}^{N} \dot{x}^j(t) \cdot \nabla \omega^j(x - x^j(t), t) \qquad (14)\\ &= \sum_{j=1}^{N} \nu \Delta \omega^j(x - x^j(t), t) - \sum_{j=1}^{N} \left(\sum_{\ell=1}^{N} \mathbf{u}^\ell(x - x^\ell(t), t) \right) \cdot \nabla \omega^j(x - x^j(t), t) \ .\end{aligned}$$

Given this equation it is natural to define ω^j as the solution of the equation:

$$\begin{aligned}\frac{\partial \omega^j}{\partial t}(x - x^j(t), t) &= \nu \Delta \omega^j(x - x^j(t), t) - \left(\sum_{\ell=1}^{N} \mathbf{u}^\ell(x - x^\ell(t), t) \right) \cdot \nabla \omega^j(x - x^j(t), t)\\ &\quad + \dot{x}^j(t) \cdot \nabla \omega^j(x - x^j(t), t) \ , \ j = 1, \ldots, N. \qquad (15)\end{aligned}$$

To close this system of equations we must specify how the centers of vorticity $x^j(t)$ evolve. We impose the condition that the first moment of each vorticity region must vanish at every time $t > 0$, i.e. we require that

$$\int_{\mathbb{R}^2} (x - x^j(t)) \omega^j(x - x^j(t), t) dx = 0 \ \text{ for all } \ t > 0, \ j = 1, \ldots N \ . \tag{16}$$

(Note that this equation really contains two conditions - one for each component of $(x - x^\ell(t))$.) We impose this condition to fix the evolution of $x^j(t)$ because Gallay and Wayne have recently shown [2] that if one considers the evolution of general solutions of (7) the solution will approach an Oseen vortex, and the rate of the approach will be faster if the vorticity distribution has first moment equal to zero. Solutions of

(7) preserve the first moment, and hence if the initial conditions have first moment equal to zero the solution will have first moment zero for all time. The equations (15) no longer preserve the first moment and thus we impose this condition for all time, which then defines the motion of the center of vorticity.

Note that if we change variables in (16) to $z = x - x^j(t)$, we find

$$\int_{\mathbb{R}^2} z\omega^j(z,t)dz = 0 \ . \tag{17}$$

Since this equation holds for all $t > 0$ we can differentiate both sides with respect to t to obtain

$$\int_{\mathbb{R}^2} z\partial_t\omega^j(z,t)dz = 0 \ . \tag{18}$$

Using (15) we can insert the formula for $\partial_t\omega^j$ into this integral and we obtain:

$$\begin{aligned} \nu\int_{\mathbb{R}^2} z\Delta\omega^j(z,t)dz - \int_{\mathbb{R}^2} z\left(\sum_{\ell=1}^{N}\mathbf{u}^\ell(z+x^j(t)-x^\ell(t),t)\right)\cdot\nabla\omega^j(z,t)dz \\ + \int_{\mathbb{R}^2} z\left(\dot{x}^j(t)\cdot\nabla\omega^j(z,t)\right)dz = 0 \ . \end{aligned} \tag{19}$$

We first note that if we integrate twice by parts, we have

$$\int_{\mathbb{R}^2} z\Delta\omega^j(z,t)dz = 0 \ . \tag{20}$$

Next if we take the n^{th} component and integrate by parts we find

$$\int_{\mathbb{R}^2} z\left(\dot{x}^j_n(t)\cdot\nabla\omega^j(z,t)\right)dz = \dot{x}^j_n(t)\int_{\mathbb{R}^2} z\partial_{z_n}\omega^j(z,t)dz = -m_j\dot{x}^j_n(t) \ , \tag{21}$$

where $m_j = \int_{\mathbb{R}^2}\omega^j(z,t)dz$ and $n = 1, 2$.

Remark 2.1. *Note that the equations* (15) *do preserve the total integral ("mass") of the solution so this definition of* m_j *is consistent with* (9)*.*

Finally, recalling that the velocity field is incompressible we can rewrite

$$\left(\sum_{\ell=1}^{N}\mathbf{u}^\ell(z+x^j(t)-x^\ell(t),t)\right)\cdot\nabla\omega^j(z,t) = \nabla\cdot\left(\sum_{\ell=1}^{N}\mathbf{u}^\ell(z+x^j(t)-x^\ell(t),t)\omega^j(z,t)\right)$$

so, again considering the n^{th} component and integrating by parts we have

$$\int_{\mathbb{R}^2} z_n \left(\sum_{\ell=1}^{N} \mathbf{u}^\ell(z + x^j(t) - x^\ell(t), t) \right) \cdot \nabla \omega^j(z,t) dz \tag{22}$$
$$= - \int_{\mathbb{R}^2} \left(\sum_{\ell=1}^{N} \mathbf{u}^\ell_n(z + x^j(t) - x^\ell(t), t) \right) \omega^j(z,t) dz$$

Thus, if we combine (20) (21), and (22) we see that (19) reduces to the system of ordinary differential equations for the centers of the vorticity distributions:

$$\frac{dx^j_n}{dt}(t) = \frac{1}{m_j} \sum_{\ell=1}^{N} \int_{\mathbb{R}^2} \left(\mathbf{u}^\ell_n(z + x^j(t) - x^\ell(t), t) \omega^j(z,t) \right) dz \ , \tag{23}$$

supplemented by the initial conditions (10), while the N components of the vorticity evolve according to the partial differential equations (15) with initial conditions

$$\omega^j(z,0) = \omega^j_0(z + x^j_0) \tag{24}$$

obtained by combining (8) and (12).

Remark 2.2. *Consider* (23) *in the limit in which the components ω^j are all point vortices, i.e. $\omega^j(z,t) = m_j \delta(z)$, with $\delta(z)$ the Dirac-delta function. Recall that the velocity field associated with such a point vortex is*

$$U_n(z_1, z_2, t) = - \sum_{j=1}^{2} \epsilon_{n,j} z_j \frac{1}{(z_1^2 + z_2^2)}$$

where $\epsilon_{m,j}$ is the antisymmetric tensor with two indices. Then if we ignore the (singular) term with $\ell = j$ in the sum on the right hand side of (23) *we find*

$$\frac{dx^j_n}{dt}(t) = \sum_{\ell=1;\ell\neq j}^{N} m_\ell \frac{\left(\sum_{k=1}^{2} \epsilon_{n,k} (x^j(t) - x^\ell(t))_k \right)}{|x^j(t) - x^\ell(t)|^2} \tag{25}$$

These of course are just Helmholtz-Kirchhoff equations for the inviscid motion of a system of point vortices. Thus, our expansion can be regarded as a generalization of the this approximation which allows for both nonzero viscosity and vortices of finite size. To justify omitting the term with $\ell = j$ on the right hand side of (25) *we note that if we approximate the delta function with a narrow, Gaussian vorticity distribution, and $\mathbf{u}^j$ by the corresponding velocity field, this term will vanish by symmetry.*

3 The moment expansion; case of a single center

In this section we introduce another idea - an expansion of the vorticity in terms of Hermite functions. Then, in the next section we will combine the Hermite expansion with the multi-vortex expansion of the previous section.

The moment expansion is an expansion of the solution of the vorticity equation in terms of Hermite functions. Define

$$\phi_{00}(x,t;\lambda) = \frac{1}{\pi\lambda^2} e^{-|x|^2/\lambda^2} \tag{26}$$

where $\lambda^2 = \lambda_0^2 + 4\nu t$. Three simple facts that we will use repeatedly are

(i) $\partial_t \phi_{00} = \nu \Delta \phi_{00}$

(ii) $\int_{\mathbb{R}^2} \phi_{00}(x,t;\lambda) dx = 1$ for all $t \geq 0$.

(iii) Finally, and crucially for what follows, the vorticity function $\omega(x,t) = \alpha\phi_{00}(x,t)$ is an exact solution (called the Oseen, or Lamb, vortex) of the two dimensional vorticity equation for all values of α.

Note that we will often supress the dependence of ϕ_{00} on λ when there is no fear of confusion.

We now define the Hermite functions of order (k_1, k_2) by

$$\phi_{k_1,k_2}(x,t;\lambda) = D_{x_1}^{k_1} D_{x_2}^{k_2} \phi_{00}(x,t;\lambda) \tag{27}$$

and the corresponding moment expansion of a function by

$$\omega(x,t) = \sum_{k_1,k_2=1}^{\infty} M[k_1,k_2;t] \phi_{k_1,k_2}(x,t;\lambda) \tag{28}$$

Note that if the function $\omega(x,t)$ in (28) is the vorticity field of some fluid the linearity of the Biot-Savart law implies that we can expand the associated velocity field as:

$$\mathbf{V}(x,t) = \sum_{k_1,k_2=1}^{\infty} M[k_1,k_2;t] \mathbf{V}_{k_1,k_2}(x,t;\lambda) \tag{29}$$

where

$$\mathbf{V}_{k_1,k_2}(x,t;\lambda) = D_{x_1}^{k_1} D_{x_2}^{k_2} \mathbf{V}_{00}(x,t;\lambda) \tag{30}$$

and $\mathbf{V}_{00}(x,t;\lambda)$ is the velocity field associated with the Gaussian vorticity distribution ϕ_{00} - explicitly we have:

$$\mathbf{V}_{00}(x,t;\lambda) = \frac{1}{2\pi}\frac{(-x_2,x_1)}{|x|^2}\left(1-e^{-|x|^2/\lambda^2}\right) \; , \tag{31}$$

We define the Hermite polynomials via their generating function:

$$H_{n_1,n_2}(z;\lambda) = \left((D_{t_1}^{n_1}D_{t_2}^{n_2}e^{\left(\frac{2t\cdot z-t^2}{\lambda^2}\right)}\right)|_{t=0} \tag{32}$$

Note that the "standard" Hermite polynomials correspond to taking $\lambda = 1$.

Then using the standard orthogonality relationship for the Hermite polynomials:

$$\int_{\mathbb{R}^2} H_{n_1,n_2}(z;\lambda=1)H_{m_1,m_2}(z;\lambda=1)e^{-z^2}dz = \pi 2^{n_1+n_2}(n_1!)(n_2!)\delta_{n1,m1}\delta_{n_2,m_2} \tag{33}$$

we see that the coefficients in the expansion (28) are defined by the projection operators:

$$M[k_1,k_2;t] = (P_{k_1,k_2}\omega)(t) = \frac{(-1)^{k_1+k_2}\lambda^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)}\int_{\mathbb{R}^2} H_{k_1,k_2}(z;\lambda)\omega(z,t)dz \tag{34}$$

3.1 Convergence of the moment expansion

In this subsection we derive a criterion for the convergence of the moment expansion derived above and we show that if this criterion is satisfied for $t=0$ then it is satisfied for all subsequent times $t>0$.

Our convergence criterion is based on the observation that the Hermite functions $\phi_{k_1,k_2}(x,t;\lambda)$ are, for any value of t, the eigenfunctions of the linear operator

$$\mathcal{L}^\lambda\psi = \frac{1}{4}\lambda^2\Delta\psi + \frac{1}{2}\nabla\cdot(x\psi) \; . \tag{35}$$

This fact can be verified by direct computation and is related to the fact that $\mathcal{L}^\lambda$ can be transformed into the Hamiltonian quantum mechanical harmonic oscillator.

The Gaussian function ϕ_{00} plays a crucial role in the convergence proof, and its dependence on the parameter λ is particularly important in this discussion, so for this subsection only we will define

$$\phi_{00}(x,t;\lambda) = \Phi_\lambda(x,t)$$

to empasize this dependence.

If one now proceeds as in Lemma 4.7 of [3] one can prove

Proposition 3.1. *The operator $\mathcal{L}^\lambda$ is self-adjoint in the Hilbert space*

$$X^\lambda = \{f \in L^2(\mathbb{R}^2) \mid \Phi_\lambda^{-1/2} f \in L^2(\mathbb{R}^2)\}$$

with innerproduct $(f,g)_\lambda = \int_{\mathbb{R}^2} \Phi_\lambda^{-1} \overline{f} g dx$.

An immediate corollary of this proposition and the general theory of self-adjoint operators is

Corollary 3.2. *The eigenfunctions of $\mathcal{L}^\lambda$ form a complete orthogonal set in the Hilbert space X^λ.*

and as a corollary of this result and the observation that the eigenfunctions of $\mathcal{L}^\lambda$ are precisely our Hermite functions $\{\phi_{k_1,k_2}\}$, we have finally

Proposition 3.3. *Suppose that*

$$\|f\|_\lambda^2 = \int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)|f(x)|^2 dx < \infty \ ,$$

then the expansion

$$f(x) = \sum_{k_1,k_2} M[k_1,k_2]\phi_{k_1,k_2}(x)$$

converges with respect to the norm on the Hilbert space X^λ.

Thus, the following criterion guarantees that the expansion (28) for the vorticity converges:

$$\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)(\omega(x,t))^2 dx < \infty \ . \tag{36}$$

The main result of this subsection is the following theorem which proves that if our initial vorticity distribution satisfies (36) for some $\lambda = \lambda_0$, then the solution of the vorticity equation with that initial condition will satisfy (36) for all time t with $\lambda = \sqrt{4\nu t + \lambda_0^2}$ and hence as a corollary if the initial vorticity distribution satisfies (36) then our moment expansion converges for all times t.

Theorem 3.4. *Define the weighted enstrophy function*

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)(\omega(x,t))^2 dx \ .$$

If the initial vorticity distribution ω_0 is such that $\mathcal{E}(0) < \infty$ for some λ_0, and ω_0 is bounded (in the L^∞ norm) then $\mathcal{E}(t)$ is finite for all times $t > 0$.

Proof: The idea of the proof is to derive a differential inequality for $\mathcal{E}(t)$ which guarantees that if $\mathcal{E}(0)$ is finite then $\mathcal{E}(t)$ will be finite for all t. Differentiating $\mathcal{E}(t)$ we obtain

$$\begin{aligned}
\frac{d\mathcal{E}}{dt}(t) &= \frac{4\nu}{\lambda^2}\mathcal{E}(t) - \frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2} |x|^2 \Phi_\lambda^{-1}(x)(\omega(x,t))^2 dx \qquad (37)\\
&\quad +2\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)\omega(x,t)\partial_t\omega(x,t)dx \\
&= \frac{4\nu}{\lambda^2}\mathcal{E}(t) - \frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2} |x|^2 \Phi_\lambda^{-1}(x)(\omega(x,t))^2 dx \qquad (38)\\
&\quad +2\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)\omega(x,t)\left(\nu\Delta\omega - \mathbf{u}\cdot\nabla\omega\right)dx
\end{aligned}$$

We now consider the last term in (38). First note that upon integration by parts we have:

$$2\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)\omega(x,t)\left(\nu\Delta\omega(x,t)\right)dx = -2\nu\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)\left(|\nabla\omega|^2 + \frac{2}{\lambda^2}\omega x\cdot\nabla\omega\right)dx\ . \qquad (39)$$

The right hand side of the expression in (39) can again be broken up into two pieces and the second can be bounded by:

$$2\nu\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)\left(\frac{2}{\lambda^2}\omega x\cdot\nabla\omega\right)dx \le \nu\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)|\nabla\omega|^2 dx + \frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2} \Phi_\lambda^{-1}\left(x^2\omega^2\right)dx\ . \qquad (40)$$

Finally, we bound the last term in (37), which comes from the nonlinear term in the vorticity equation. In this estimate we use the fact (see [2], Lemma 2.1) that the L^∞ norm of the velocity field $\mathbf{u}$ can be bounded by a constant times the sum of the L^1 and L^∞ norms of the vorticity field - i.e by $C(\|\omega\|_{L^1(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)})$. This observation, combined with the fact that $\|\omega(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \|\omega_0\|_{L^p(\mathbb{R}^2)}$, which is a consequence of the maximum principle, implies that

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \le C(\|\omega_0\|_{L^1(\mathbb{R}^2)} + \|\omega_0\|_{L^\infty(\mathbb{R}^2)})\ .$$

and hence that

$$\begin{aligned}
2\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)\omega(x,t)\left(\mathbf{u}\cdot\nabla\omega\right)dx &\le 2C(\omega_0)\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)|\omega(x,t)||\nabla\omega|dx \qquad (41)\\
&\le \frac{4C(\omega_0)}{\nu}\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)(\omega(x,t))^2 dx + \nu\int_{\mathbb{R}^2} \Phi_\lambda^{-1}(x)|\nabla\omega|^2 dx
\end{aligned}$$

If we now combine the inequalities in (39), (40) and (41), with the expression for $\frac{d\mathcal{E}}{dt}$ in (38) we obtain:

$$\frac{d\mathcal{E}}{dt}(t) \leq \left(\frac{4C(\omega_0)}{\nu} + \frac{4\nu}{\lambda^2} \right) \mathcal{E}(t) \ , \tag{42}$$

from which we see immediately that if $\mathcal{E}(0)$ is bounded, $\mathcal{E}(t)$ remains bounded for all time. $\square$

3.2 Differential equations for the moments

Assuming that the function $\omega(z,t)$ is a solution of (7), we can derive differential equations satisfied by the moments $M[i_1, k_2, t]$ in (28). Surprisingly the expressions for the coefficients in these expansions are quite simple and explicit.

If we differentiate (28) and assume that ω is a solution of the two-dimensional vorticity equation we obtain:

$$\begin{aligned}\partial_t \omega = & \sum_{k_1,k_2=1}^{\infty} \frac{dM[k_1,k_2;t]}{dt} \phi_{k_1,k_2}(x,t;\lambda) + \sum_{k_1,k_2=1}^{\infty} M[k_1,k_2;t] \partial_t \phi_{k_1,k_2}(x,t;\lambda) \\ = & \sum_{k_1,k_2=1}^{\infty} M[k_1,k_2;t] \left(\nu \Delta \phi_{k_1,k_2}(x,t;\lambda) \right) \\ & - \left(\sum_{\ell_1,\ell_2=1}^{\infty} M[\ell_1,\ell_2;t] \mathbf{V}_{\ell_1,\ell_2}(x,t;\lambda) \right) \cdot \nabla \left(\sum_{k_1,k_2=1}^{\infty} M[k_1,k_2;t] \phi_{k_1,k_2}(x,t;\lambda) \right)\end{aligned} \tag{43}$$

From the first of the "simple facts" we stated about ϕ_{00}, we see that the last term on the first line cancels the middle line and hence if we apply the projection operators defined in (34), we are left with the system of ordinary differential equations for the moments

$$\begin{aligned}\frac{dM[k_1,k_2;t]}{dt} \ = \ -P_{k_1,k_2} \Bigg[& \left(\sum_{\ell_1,\ell_2=1}^{\infty} M[\ell_1,\ell_2;t] \mathbf{V}_{\ell_1,\ell_2}(x,t;\lambda) \right) \\ & \cdot \nabla \left(\sum_{m_1,m_2=1}^{\infty} M[m_1,m_2;t] \phi_{m_1,m_2}(x,t;\lambda) \right) \Bigg]\end{aligned} \tag{44}$$

The somewhat surprising fact which, in our opinion, makes the preceding straightforward calculations interesting is that the projection on the right hand side of (44) can be computed explicitly in terms of the derivatives of an relatively simple function.

We now explain how this is done. First recall that:

$$\phi_{m_1,m_2}(x,t;\lambda) = D_{x_1}^{m_1} D_{x_2}^{m_2} \phi_{00}(x,\lambda),\ \mathbf{V}_{\ell_1,\ell_2}(x,t;\lambda) = D_{x_1}^{\ell_1} D_{x_2}^{\ell_2} \mathbf{V}_{00}(x,\lambda) \tag{45}$$

In order to avoid confusing the two sets of derivatives we will rewrite these formulas as

$$\begin{aligned} \phi_{m_1,m_2}(x,t;\lambda) &= (D_{a_1}^{m_1} D_{a_2}^{m_2} \phi_{00}(x+a,\lambda))|_{a=0}, \\ \mathbf{V}_{\ell_1,\ell_2}(x,t;\lambda) &= (D_{b_1}^{\ell_1} D_{b_2}^{\ell_2} \mathbf{V}_{00}(x+b,\lambda))|_{b=0} \end{aligned} \tag{46}$$

Inserting these formulas into the right hand side of (44) and using the formula for the projection operator P_{k_1,k_2} in terms of the integration against a Hermite polynomial we obtain

$$\begin{aligned} \frac{dM}{dt}[k_1,k_2,t] &= -\frac{(-1)^{(k_1+k_2)}\lambda^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)} \sum_{\ell_1,\ell_2} \sum_{m_1,m_2} M[\ell_1,\ell_2,t] M[m_1,m_2,t] \\ &\quad \times \int_{\mathbb{R}^2} H_{k_1,k_2}(\mathbf{x}) (D_{x_1}^{m_1} D_{x_2}^{m_2} \mathbf{V}_{00}(\mathbf{x};\lambda)) \cdot \nabla_{\mathbf{x}} (D_{x_1}^{\ell_1} D_{x_2}^{\ell_2} \phi_{00}(\mathbf{x};\lambda)) d\mathbf{x} \\ &= -\frac{(-1)^{(k_1+k_2)}\lambda^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)} \sum_{\ell_1,\ell_2} \sum_{m_1,m_2} M[\ell_1,\ell_2,t] M[m_1,m_2,t] \\ &\quad \times D_{t_1}^{k_1} D_{t_2}^{k_2} D_{b_1}^{m_1} D_{b_2}^{m_2} D_{a_1}^{\ell_1} D_{a_2}^{\ell_2} \nabla_a \cdot \left(\int_{\mathbb{R}^2} e^{\frac{(-t_1^2 - t_2^2 + 2t_1 x_1 + 2t_2 x_2)}{\lambda^2}} \mathbf{V}_{00}(x+b;\lambda) \phi_{00}(x+a;\lambda) d\mathbf{x} \right) |_{t=0,a=0,b=0} . \end{aligned} \tag{47}$$

The last equality in this expression results from rewriting the Hermite polynomial H_{l_1,k_2} in terms of its generating function.

The last step in deriving the equations for the moments is to evaluate the integral in the last line of (47). The key step in this evaluation is to recall that for these incompressible flows the velocity field can be written in terms of the derivatives of the stream function and that the Laplacian of the stream function is minus the vorticity. Thus, we can write:

$$\mathbf{V}_{00}(x+b;\lambda) = -\nabla_b^* (\Delta_b)^{-1} \phi_{00}(x+b) \ , \tag{48}$$

where $\nabla_b^* f = (\partial_{x_2} f, -\partial_{x_1} f)$.

Inserting this into the integral in (47) we find

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{\frac{(-t_1^2 - t_2^2 + 2t_1 x_1 + 2t_2 x_2)}{\lambda^2}} \mathbf{V}_{00}(x+b;\lambda) \phi_{00}(x+a;\lambda) d\mathbf{x} \\ &\qquad = -\nabla_b^* (\Delta_b)^{-1} \int_{\mathbb{R}^2} e^{\frac{(-t_1^2 - t_2^2 + 2t_1 x_1 + 2t_2 x_2)}{\lambda^2}} \phi_{00}(x+b;\lambda) \phi_{00}(x+a;\lambda) d\mathbf{x} \end{aligned} \tag{49}$$

Now note that all three factors in the integrand are Gaussians and thus the integral can be evaluated explicitly, and we find

$$\int_{\mathbb{R}^2} e^{\frac{(-t_1^2-t_2^2+2t_1x_1+2t_2x_2)}{\lambda^2}}\phi_{00}(x+b;\lambda)\phi_{00}(x+a;\lambda)d\mathbf{x} \tag{50}$$
$$= \frac{1}{2\pi\lambda^2}e^{-\frac{1}{2\lambda^2}\left(a_1^2+a_2^2-2a_1b_1+b_1^2-2a_2b_2+b_2^2+2a_1t_1+2b_1t_1+t_1^2+2a_2t_2+2b_2t_2+t_2^2\right)}$$

We next compute the expression

$$-\nabla_b^*(\Delta_b)^{-1}\frac{1}{2\pi\lambda^2}e^{-\frac{1}{2\lambda^2}\left(a_1^2+a_2^2-2a_1b_1+b_1^2-2a_2b_2+b_2^2+2a_1t_1+2b_1t_1+t_1^2+2a_2t_2+2b_2t_2+t_2^2\right)} \tag{51}$$

Recall that given a vorticity field ω, $-(\Delta)^{-1}\omega$ is the associated stream function and $-\nabla(\Delta)^{-1}\omega$ the velocity field associated with ω. Since the inverse Laplacian and derivatives in (51) act only on the b-dependent parts of the expression we need to evaluate

$$-\nabla_b^*(\Delta_b)^{-1}\frac{1}{2\pi\lambda^2}e^{-\frac{1}{2\lambda^2}\left(-2a_1b_1+b_1^2-2a_2b_2+b_2^2+2b_1t_1+2b_2t_2\right)} \tag{52}$$
$$= -e^{\frac{1}{2\lambda^2}\left((t_1-a_1)^2+(t_2-a_2)^2\right)}\nabla_b^*(\Delta_b)^{-1}\frac{1}{2\pi\lambda^2}e^{-\frac{1}{2\lambda^2}\left((b_1+(t_1-a_1))^2+(b_2+(t_2-a_2))^2\right)}$$

But

$$-\nabla_b^*(\Delta_b)^{-1}\frac{1}{\pi\lambda^2}e^{-\frac{1}{2\lambda^2}\left((b_1+(t_1-a_1))^2+(b_2+(t_2-a_2))^2\right)}$$

is just the velocity field associated with a Gaussian vorticity distribution (i.e. an Oseen vortex) centered at the point $-((t_1-a_1),(t_2-a_2))$ which we know explicitly. Hence, the expression on the right hand side of (49) has the explicit representation:

$$\frac{1}{2\pi}e^{-\frac{1}{2\lambda^2}\left(a_1^2+a_2^2+t_1^2+t_2^2\right)}e^{\frac{1}{2\lambda^2}\left((t_1-a_1)^2+(t_2-a_2)^2\right)} \tag{53}$$
$$\times\frac{(-(b_2+(t_2-a_2)),(b_1+(t_1-a_1))}{((b_1+(t_1-a_1))^2+(b_2+(t_2-a_2))^2)}\left(1-e^{-\frac{1}{2\lambda^2}\left((b_1+(t_1-a_1))^2+(b_2+(t_2-a_2))^2\right)}\right)$$

If we now return to (47) we see that in order to compute the coefficients in the moment equations we need to evaluate the divergence of this last expression with respect to a which gives

$$K(a_1,a_2,b_1,b_2,t_1,t_2;\lambda) =$$
$$-\left(\frac{(a_2\,t_1-b_2\,t_1+(-a_1+b_1)\;t_2)}{\pi\;\left(a_1{}^2+a_2{}^2+b_1{}^2+b_2{}^2+2\,b_1\,t_1+t_1{}^2-2\,a_1\;(b_1+t_1)+2\,b_2\,t_2+t_2{}^2-2\,a_2\;(b_2+t_2)\right)\;\lambda^2}\right)$$
$$\left(-1+e^{\frac{(-a_1+b_1+t_1)^2+(-a_2+b_2+t_2)^2}{2\,\lambda^2}}\right)e^{\frac{a_1{}^2+a_2{}^2-2\,a_1\,b_1+b_1{}^2-2\,a_2\,b_2+b_2{}^2+2\,a_1\,t_1+2\,b_1\,t_1+t_1{}^2+2\,a_2\,t_2+2\,b_2\,t_2+t_2{}^2}{2\,\lambda^2}} \tag{54}$$

Returning to equation (47) we finally conclude that

$$\frac{dM}{dt}[k_1,k_2,t] = \tag{55}$$
$$= -\frac{(-1)^{(k_1+k_2)}\lambda^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)}\sum_{\ell_1,\ell_2}\sum_{m_1,m_2}\Gamma[k_1,k_2;\ell_1,\ell_2,m_1,m_2;\lambda]M[\ell_1,\ell_2,t]M[m_1,m_2,t]$$

where

$$\Gamma[k_1,k_2;\ell_1,\ell_2,m_1,m_2;\lambda] = \tag{56}$$
$$= D_{t_1}^{k_1}D_{t_2}^{k_2}D_{b_1}^{m_1}D_{b_2}^{m_2}D_{a_1}^{\ell_1}D_{a_2}^{\ell_2}K(a_1,a_2,b_1,b_2,t_1,t_2;\lambda)|_{t=0,a=0,b=0}$$

Thus, we have succeeded in rewriting the two-dimensional vorticity equation as a system of ordinary differential equations with simple, quadratic nonlinear terms whose coefficients can be evaluated in terms of derivatives of a single explicit function. Furthermore, we have given a sufficient condition on the initial vorticity distribution to guarantee that the expansion of the vorticity generated by the solution of these ordinary differential equations converges for all time.

4 The moment expansion for several vortex centers

In this section we extend the Hermite moment expansion of the previous section to the case in which there are two or more centers of vorticity by combining this expansion with the multi-vortex representation of Section 2. For simplicity of exposition we limit the discussion here to the case of two vortices but the expansion can be extended to any finite number of vortices.

The basic idea is just to consider the equations (15) for the evolution of each vortex and then expand each of the functions ω^j in Hermite moments as in the previous section. Thus, we define

$$\omega^j(z,t) = \sum_{k_1,k_2=1}^{\infty} M^j[k_1,k_2;t]\phi_{k_1,k_2}(z,t;\lambda) \tag{57}$$

for $j = 1,2$. We make a similar expansion for the velocity field in terms of the functions $\mathbf{V}_{\ell_1,\ell_2}$, and insert the expansions into (15). Letting $z = x - x^j(t)$, and recalling that $\partial_t \phi_{k_1,k_1} = \nu \Delta \phi_{k_1,k_2}$ we obtain:

$$
\begin{aligned}
\frac{dM^j[k_1,k_2;t]}{dt} &= \\
&= -P_{k_1,k_2}\left[\left(\sum_{j'=1}^{2}\sum_{\ell_1,\ell_2=1}^{\infty} M^{j'}[\ell_1,\ell_2;t]\mathbf{V}_{\ell_1,\ell_2}(z-s_{j,j'},t;\lambda)\right)\right.\\
&\qquad\left.\cdot\nabla\left(\sum_{k_1,k_2=1}^{\infty} M^j[m_1,m_2;t]\phi_{m_1,m_2}(z,t;\lambda)\right)\right]
\end{aligned} \tag{58}
$$

where $s_{j,j'} = x^{j'}(t) - x^j(t)$. Proceeding as in the previous section and using equation (47) we finally conclude that

$$
\begin{aligned}
\frac{dM^j}{dt}[k_1,k_2,t] &= \\
&= -\frac{(-1)^{(k_1+k_2)}\lambda^{2(k_1+k_2)}}{2^{k_1+k_2}(k_1!)(k_2!)}\sum_{j'=1}^{2}\sum_{\ell_1,\ell_2}\sum_{m_1,m_2}\Gamma^{j,j'}[k_1,k_2;\ell_1,\ell_2,m_1,m_2;s_{j,j'},\lambda]\\
&\qquad \times M^j[\ell_1,\ell_2,t]M^{j'}[m_1,m_2,t]
\end{aligned} \tag{59}
$$

where

$$
\begin{aligned}
&\Gamma^{j,j'}[k_1,k_2;\ell_1,\ell_2,m_1,m_2;s_{j,j'},\lambda] = \\
&\quad = D^{k_1}_{t_1}D^{k_2}_{t_2}D^{m_1}_{b_1}D^{m_2}_{b_2}D^{\ell_1}_{a_1}D^{\ell_2}_{a_2}K^{multi}(a_1,a_2,b_1,b_2,t_1,t_2;s_{j,j'},\lambda)|_{t=0,a=0,b=0}
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
&K^{multi}(a_1,a_2,b_1,b_2,t_1,t_2;s_1,s_2,\lambda) = \\
&\qquad \nabla_a\cdot\left(\int_{\mathbb{R}^2} e^{\frac{(-t_1^2-t_2^2+2t_1x_1+2t_2x_2)}{\lambda^2}}\mathbf{V}_{00}(x-s+b;\lambda)\phi_{00}(x+a;\lambda)d\mathbf{x}\right)
\end{aligned} \tag{61}
$$

Remark 4.1. *Note that comparing* (60) *with the calculation leading* (54) *we see that* K^{multi} *can be written in terms of the expression for* K *via the simple formula:*

$$
K^{multi}(a_1,a_2,b_1,b_2,t_1,t_2;s_1,s_2,\lambda) = K(a_1,a_2,b_1-s_1,b_2-s_2,t_1,t_2;\lambda)
$$

We now derive a similar expansion for the evolution of the centers of each vortex. We begin with (23)

$$
\frac{dx^j}{dt}(t) = \frac{1}{m_j}\sum_{j'=1}^{N}\int_{\mathbb{R}^2}\left(\mathbf{u}^{\ell}(z+x^j(t)-x^{j'}(t),t)\omega^j(z,t)\right)dz\ .
$$

(Recall that this is really a pair of equations, one for each component of x^j.) Now insert the moment expansion of $\mathbf{u}^{j'}$ and ω^j into this expression and we obtain:

$$\frac{dx^j}{dt}(t) = \sum_{j'=1}^{N} \sum_{\ell_1,\ell_2} \sum_{m_1,m_2} M^j[\ell_1,\ell_2,t] M^{j'}[m_1,m_2,t] \int_{\mathbb{R}} \mathbf{V}_{\ell_1,\ell_2}(z - s_{j,j'},t)\phi_{k_1,k_2}(z,t)dz \ , \tag{62}$$

where as before $s_{j,j'} = x^{j'}(t) - x^j(t)$. The integral of the velocity and vorticity can be evaluated just as in the preceding section and we find

$$\frac{dx^j}{dt}(t) = \sum_{j'=1}^{N} \sum_{\ell_1,\ell_2} \sum_{m_1,m_2} \Xi^{j,j'}[\ell_1,\ell_2,m_1,m_2;s_{j,j'},\lambda] M^j[\ell_1,\ell_2,t] M^{j'}[m_1,m_2,t] \ , \tag{63}$$

where in this case the coefficients $\Xi^{j,j'}[k_1,k_2;\ell_1,\ell_2,m_1,m_2;s_{j,j'},\lambda]$ is given by the expression

$$\begin{aligned} &\Xi^{j,j'}[\ell_1,\ell_2,m_1,m_2;s,\lambda] \\ &= \frac{1}{2\pi} D_{b_1}^{m_1} D_{b_2}^{m_2} D_{a_1}^{\ell_1} D_{a_2}^{\ell_2} \left(\left(\frac{(-(b_2 - s_2 - a_2),(b_1 - s_1 - a - 1))}{(b_2 - s_2 - a_2)^2 + (b_1 - s_1 - a - 1)^2} \right) \right. \\ &\quad \left. \left(1 - e^{-\frac{1}{2\lambda^2}((b_2 - s_2 - a_2)^2 + (b_1 - s_1 - a - 1)^2)} \right) \right) \Big|_{a=0,b=0} \ . \end{aligned} \tag{64}$$

4.1 Convergence of Multi-Vortex Expansion

We note that it is easy to extend our previous result on the convergence of the moment expansion to the multi-vortex expansion. To do so, we change variables to re-center each vortex at the origin. Thus if we let

$$\omega^j(x - x^j(t),t) = w^j(x,t), \quad \mathbf{u}^j(x - x^j(t),t) = \mathbf{v}^j(x,t)$$

then equation (15) becomes

$$\frac{\partial w^j}{\partial t}(x,t) \;=\; \nu \Delta w^j(x,t) - \left(\sum_{\ell=1}^{N} \mathbf{v}^\ell(x - x^\ell(t) + x^j(t),t) \right) \cdot \nabla w^j(x,t),$$

for $j = 1,\ldots,N$.

We are now ready to state our result.

Theorem 4.2. *Define*

$$\mathcal{E}^j(t) = \int_{\mathcal{R}^2} G_\lambda^{-1}(x) w^j(x,t)^2 dx.$$

If the initial vorticity distribution $\omega_0 \equiv \sum_{\ell=1}^N \omega^\ell(x - x^\ell(0), 0)$ *is such that* $\mathcal{E}^j(0) < \infty$ *for some* λ_0*, and for all* $j = 1, 2, \ldots, N$*, and if* ω_0 *is bounded (in the* L^∞ *norm) then each* $\mathcal{E}^j(t)$ *is finite for all times* $t > 0$*.*

Proof. We use the same idea as the single vortex case and differentiate $\mathcal{E}^j(t)$:

$$\begin{aligned}
\frac{d\mathcal{E}^i}{dt}(t) &= \frac{4\nu}{\lambda^2}\mathcal{E}^i(t) - \frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2} |x|^2 G_\lambda^{-1}(x) w^i(x)^2 dx \\
&\quad + 2\int_{\mathbb{R}^2} G_\lambda^{-1}(x) w^i(x,t)(\partial_t w^i(x,t) dx \\
&= \frac{4\nu}{\lambda^2}\mathcal{E}^i(t) - \frac{4\nu}{\lambda^4}\int_{\mathbb{R}^2} |x|^2 G_\lambda^{-1}(x)(w^i(x)^2 dx \\
&\quad + 2\int_{\mathbb{R}^2} G_\lambda^{-1} w^i (\nu \Delta w^i - \left(\sum_{\ell=1}^N \mathbf{v}^\ell(x - x^\ell(t) + x^j(t), t)\right) \cdot \nabla w^i) dx.
\end{aligned}$$

From here the proof proceeds identically to the proof of theorem 3.3 except we must examine the nonlinear term that comes from the vorticity equation a bit closer. In general, it is unknown whether $w^j(\cdot, t)$ satisfies a maximum principle, but since $\left(\sum_{\ell=1}^N \mathbf{v}^\ell(x - x^\ell(t) + x^j(t), t)\right) = \mathbf{u}$, the solution to (1), it does satisfy a maximum principle. Hence, as in Subsection 3.1 we can bound the L^∞ norm of $\sum_{\ell=1}^N \mathbf{v}^\ell(x - x^\ell(t) + x^j(t), t)$ by a constant depending only on the *initial* vorticity distribution. Hence we proceed to bound the integral

$$\int_{\mathbb{R}^2} G_\lambda^{-1} w^i (\left(\sum_{\ell=1}^N \mathbf{v}^\ell(x - x^\ell(t) + x^j(t), t)\right) \cdot \nabla w^i) dx \leq 2C(\omega_0) \int_{\mathbb{R}^2} G_\lambda^{-1} |w^i| |\nabla w^i| dx$$

and thus the rest of the bounds are the same. Again, putting everything together we arrive at:

$$\frac{d\mathcal{E}^i}{dt}(t) \leq \left(\frac{4\nu}{\lambda^2} + \frac{4C(\omega_0)}{\nu}\right)\mathcal{E}^i(t) \tag{65}$$

□

4.2 Interaction of Gaussian Vortices

The experimental and numerical work of [9] has shown that widely separated regions of vorticity can be well approximated by Gaussians for long periods of time. This corresponds to truncating our expansions for the vorticity so that they contain only a single term. In this subsection we analyze the equations that result from this truncation. This approximation can be viewed as a generalization of the Helmholtz-Kirchhoff approximation in which we include the effects of vorticity and finite core size to lowest order. We show that the total vorticity of each of the vortices is constant while the centers of vorticity evolve either along straight lines or circles.

Remark 4.3. *It should be noted here that the effect of truncating our expansion after one term allows for only viscosity as the driving force in vortex merger. Allowing more terms in the expansion introduces convective forces.*

If we start with two Gaussian distributions for our initial vorticity with the same value of λ_0 , and truncate the equations of motion for the moments so that all terms containing higher order moments are omitted then we can conclude that the two vortices travel along circular or straight line orbits around the "center of vorticity." Moreover, the leading coefficients $M^j[0,0:t]$ of the expansions are constant. To be precise, we let

$$\begin{aligned} \omega^1(x,t) &= M^1[0,0;t]\phi_{0,0}(x-x^1(t),t;\lambda) && (66)\\ \omega^2(x,t) &= M^2[0,0;t]\phi_{0,0}(x-x^2(t),t;\lambda). && (67) \end{aligned}$$

Let us also write $s_{i,j} = x^j(t) - x^i(t)$, then using (58)- (61) we first calculate the evolution of $M^1(0,0:t)$

$$\begin{aligned} \frac{dM^1}{dt}[0,0;t] &= -M^1[0,0;t]^2 K(0,0,0,0,0,0,\lambda) \\ &\quad -M^1[0,0;t]M^2[0,0;t]K^{multi}(0,0,0,0,0,0,s_{1,2},\lambda) \\ &= 0, \end{aligned}$$

since $K(0,0,0,0,0,0,\lambda) = 0$ and $K^{multi}(0,0,0,0,0,0,\lambda) = 0$, respectively. The calculation for $M^2[0,0;t]$ is the same. Thus the evolutions of the coefficients of leading order are constant.

Remark 4.4. *The fact that $M^1[0,0;t]$ is constant in time is not a consequence of the truncation of the moment equations to first order. One can show that the equations* (14) *conserve the zeroth moment of ω^j, independent of any truncation.*

More interestingly, though, is the calculation for the evolution of $x^j(t)$, the centers of these vortices. Again if we denote $x^j = (x_1^j, x_2^j)$ then the evolution for each component as defined by equation (23) can be written as:

$$\frac{dx_i^j}{dt} = \frac{1}{M^j[0,0,t]} \int V_i^k(y - s_{1,2}, t)\omega^j(y,t)dy. \tag{68}$$

Now using equation (48) to evaluate
$V^k(j - s_{1,2}, t)$, equation (68) yields the following equations for each x_i^j, $j = 1, 2$ and $i = 1, 2$:

$$\begin{aligned}
\dot{x}_1^1 &= -\frac{M^2}{2\pi}\frac{(e^{-\frac{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2}{2\lambda(t)^2}}-1)(x_2^1-x_2^2)}{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2} \\
\dot{x}_2^1 &= -\frac{M^2}{2\pi}\frac{(e^{-\frac{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2}{2\lambda(t)^2}}-1)(-x_1^1+x_1^2)}{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2} \\
\dot{x}_1^2 &= \frac{M^1}{2\pi}\frac{(e^{-\frac{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2}{2\lambda(t)^2}}-1)(x_2^1-x_2^2)}{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2} \\
\dot{x}_2^2 &= \frac{M^1}{2\pi}\frac{(e^{-\frac{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2}{2\lambda(t)^2}}-1)(-x_1^1+x_1^2)}{(x_1^1-x_1^2)^2+(x_2^1-x_2^2)^2}
\end{aligned} \tag{69}$$

where $M^j \equiv M^j[0,0,t]$ is constant and represents the total vorticity of the j^{th} vortex.

Remark 4.5. *If the vortices have different $\lambda(t)$ values say $\lambda_1(t)$ and $\lambda_2(t)$ then one just needs to replace the $2\lambda(t)^2$ with $\lambda_1(t)^2 + \lambda_2(t)^2$ in the exponential to arrive at the correct system.*

We now state the main result of this section.

Theorem 4.6. *System (69) admits only circular or straight line trajectories.*

Proof: Away from rest points our system (69) can be transformed to:

$$\begin{aligned}
\frac{\partial x_2^1}{\partial x_1^1} &= \frac{-x_1^1+x_1^2}{x_2^1-x_2^2} \\
\frac{\partial x_2^2}{\partial x_1^2} &= \frac{-x_1^1+x_1^2}{x_2^1-x_2^2} \\
\frac{\partial x_2^1}{\partial x_2^2} &= -\frac{M^2}{M^1} \\
\frac{\partial x_1^1}{\partial x_1^2} &= -\frac{M^2}{M^1}.
\end{aligned} \tag{70}$$

Integrating out the bottom two equations we get

$$\begin{array}{rcl} x_2^1 & = & -\frac{M^2}{M^1}(x_2^2 + k_2) \\ x_1^1 & = & -\frac{M^2}{M^1}(x_1^2 + k_1). \end{array} \tag{71}$$

First let us assume that $M^2 \neq -M^1$. Then plugging back into the first two equations of (70) we arrive at:

$$\begin{array}{rcccl} \frac{\partial x_2^1}{\partial x_1^1} & = & \frac{(-1-\frac{M^2}{M^1})x_1^1 - k_1}{(1+\frac{M^2}{M^1})x_2^1 - k_2} & = & -\frac{x_1^1 - \tilde{k}_1}{x_2^1 - \tilde{k}_2} \\ \frac{\partial x_2^2}{\partial x_1^2} & = & \frac{(1+\frac{M^2}{M^1})x_1^2 - \frac{M^2}{M^1}k_1}{(-1-\frac{M^2}{M^1})x_2^1 - \frac{M^2}{M^1}k_2} & = & -\frac{x_1^1 - \hat{k}_1}{x_2^1 - \hat{k}_2} \end{array} \tag{72}$$

for appropriate constants $\tilde{k}_1, \tilde{k}_2, \hat{k}_1, \hat{k}_2$. Thus we integrate again and get

$$\begin{array}{c} (x_2^1(t) - \tilde{k}_2)^2 + (x_1^1(t) - \tilde{k}_1)^2 = C_1 \\ (x_2^2(t) - \hat{k}_2)^2 + (x_1^2(t) - \hat{k}_1)^2 = C_2. \end{array} \tag{73}$$

If $M^2 = -M^1$ then we have equal but opposite size vortices and equations (69) become:

$$\begin{array}{rcl} \frac{\partial x_2^1}{\partial x_1^1} & = & -\frac{k2}{k1} \\ \frac{\partial x_2^2}{\partial x_1^2} & = & -\frac{k2}{k1} \end{array} \tag{74}$$

which gives us straight line solutions with slope $-\frac{k2}{k1}$, as desired.

$\square$

If we now consider the case of equal total vorticity for the two vortices ($M^2 = M^1 \equiv M$), the classical point vortex result is that the vortices will rotate around the center of vorticity at a constant frequency, $\frac{2M}{2\pi D^2}$, where D is the distance between the vortex centers. We will now compute the viscous and finite core size effects on the frequency of rotation, predicted by our model. For simplicity we will center the vorticess at the origin and place them on the circle of radius, r. We apply the polar change of variables,

$$x_1^i = r\cos(\theta_i) \quad x_2^i = r\sin(\theta_i) \quad i = 1, 2 \tag{75}$$

then using the fact that our vortices are out of phase by π we can compute that

$$(x_1^1 - x_1^2) = r\cos(\theta_1) - r\cos(\theta_2) = 2r\cos(\theta_1) \tag{76}$$

$$(x_2^1 - x_2^2) = r\sin(\theta_1) - r\sin(\theta_2) = 2r\sin(\theta_1), \tag{77}$$

and thus we arrive at the expression for the frequency of rotation

$$\Omega = (-1)^{i+1}\frac{M}{4\pi r^2}(e^{\frac{-2r^2}{\lambda(t)^2}} - 1). \tag{78}$$

Since $\lambda(t)^2 = \lambda_0^2 + 4\nu t$ we notice two things: the first is that viscosity, ν, slows the frequency of rotation down and, second, in the formal limit as $\nu \to 0$ and $\lambda_0^2 \to 0$ we recover the constant frequency, $\frac{2M}{2\pi D^2}$, which is what the Helmholtz-Kirchoff model for the rotation of two point vortices predicts.

With this calculation we may now compare the results of expanding a two gaussian initial distribution as a single vortex or as two independent vortices. In [10] a two Gaussian initial distribution a distance $2r$ apart with core size λ_0, each with mass M, is approximated by a single vortex expansion using (28). The authors truncate the expansion to quadrapole moment ($n = 2$) and calculated the frequency of rotation to be:

$$\Omega = \frac{M}{8\pi}\left[\frac{1}{2\nu t}\ln\left(1+\frac{4\nu t}{r^2}\right) - \frac{\lambda_0^2}{r^4}\frac{1}{1+4\nu t/r^2}\right]. \tag{79}$$

This equation is directly comparable to (78). Notice that both equations for frequency of rotation indicate slowing of rotation over time, albeit at different rates. In addition, both equations recover the Helmholtz-Kirchhoff approximation of $\frac{2M}{2\pi D^2}$ in the limit as $\nu \to 0$ and $\lambda_0 \to 0$. In fact, in the sufficiently localized regime, $r << \lambda_0$ both equations have similar initial frequencies and remain close asymptotically. A typical example is shown in figure 4.2

5 Conclusions:

In this paper we have derived a system of ordinary differential equations whose solutions give a representation of solutions of the two dimensional vorticity equation in terms of a system of interacting vortices. We have also derived a sufficient condition on the initial vorticity distribution which guarantees that this representation in terms of interacting vortices is equivalent to the original solution of the two-dimensional vorticity equation. This model generalizes the classical Helholtz-Kirchhoff model of interacting, inviscid, point vortices to include the effects of both finite core size and viscosity. We have also looked at the analytical predictions of our model for the interaction of two vortices which the expansion is truncated at leading order. We plan in future work to further explore the analytical and numerical predictions of this model.

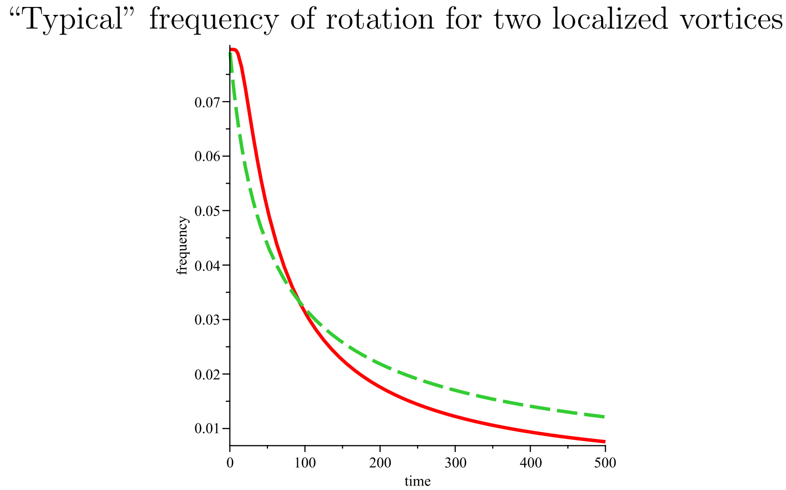

Figure 1: Here we plot both the frequency of rotation predicted by single vortex expansion up to quadrapole order (dashed) and the frequency predicted by two vortex expansions truncated to leading order (solid). The parameter values used are $\nu = .01$,$M = 1$,$r = 1$, and $\lambda_0 = .01$.

6 Acknowledgements:

CEW wishes to acknowledge many useful discussions about two dimensional fluid flows with Th. Gallay. DU and CEW wish also to acknowledge a very helpful discussion of Hermite expansions with G. Van Baalen.

References

[1] C. Cerretelli and C. H. K. Williamson. The physical mechanism for vortex merging. *Journal of Fluid Mechanics*, 475:41–77, 2003.

[2] Thierry Gallay and C. Eugene Wayne. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^2$. *Arch. Ration. Mech. Anal.*, 163(3):209–258, 2002.

[3] Thierry Gallay and C. Eugene Wayne. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.*, 255(1):97–129, 2005.

[4] G. R. Kirchhoff. *Vorlesungen über Mathematische Physik. Mechanik*. Teubner, Leipzig, 1876.

[5] Stéphane Le Dizès. Non-axisymmetric vortices in two-dimensional flows. *Journal of Fluid Mechanics*, 406:175–198, 2000.

[6] Stéphane Le Dizès and Alberto Verga. Viscous interactions of two co-rotating vortices before merging. *Journal of Fluid Mechanics*, 467:389–410, 2002.

[7] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[8] M. V. Melander, N. J. Zabusky, and J. C. Mcwilliams. Symmetric vortex merger in two dimensions: causes and conditions. *Journal of Fluid Mechanics Digital Archive*, 195:303–340, 2006.

[9] Patrice Meunier, Stéphane Le Dizès, and Thomas Leweke. Physics of vortex merging. *Comptes Rendus Physique*, 6:431–450, 2005.

[10] Raymond J. Nagem, Guido Sandri, and David Uminsky. Vorticity dynamics and sound generation in two-dimensional fluid flow. *The Journal of the Acoustical Society of America*, 122(1):128–134, 2007.

[11] Paul K. Newton. *The N-vortex problem*, volume 145 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Analytical techniques.

[12] C. W. Oseen. Über Wirbelbewegung in einer reibenden Flüssigkeit. *Ark. f. Mat. Astron. och Fys.*, 7:1–11, 1911.

[13] C. W. Oseen. *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Akademische Verlagsgesellschaft, Leipzig, 1927.

[14] L. Ting and R. Klein. *Viscous vortical flows*, volume 374 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1991.

[15] H. von Helmholtz. Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *Journal für die reine und angewandte Mathematik*, 55:25–55, 1858.