On the validity of 2D-surface water wave models

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November 20, 2002

Abstract

The prediction of the dynamics of surface water waves is a difficult task. In order to describe the dynamics of this nonlinear free boundary value problem a number of models, such as the Korteweg-de Vries or the Nonlinear Schrödinger equation have been formally derived to describe the solutions in various parameter regimes. Here for $2D$ surface water waves we give an overview of the possible models, their derivation, their properties and their history. The central issue of this paper is the question of the validity of these models. We point out a number of possible applications such as the linear and nonlinear interaction of solitary waves, speed limits for high speed ferries, and the description of tsunamis.

1 Introduction

More than 70 per cent of earth's surface is covered by water. A journey on a boat on a windy day immediately makes clear that water is not a static structure. The waves induced by earthquakes under the sea or by a meteorite falling into the ocean can cause serious damage at coastlines more than 5.000 kilometers away. A large ship, traveling at the wrong speed, can generate waves of nearly permanent form, which are very destructive when they come ashore. All these examples show that the prediction of the behavior of surface water waves is of great practical, as well as theoretical, interest.

Mathematically, the problem is described by the Navier-Stokes or Euler's equations with some free unknown top surface. The solution of this nonlinear free boundary value problem is a difficult task. Even an accurate numerical solution of the full problem over the length and time scales necessary for practical applications remains out of reach for the foreseeable future.

Thus, a mathematical analysis of the underlying partial differential equations is needed to explain the experimental data, to construct suitable models which are tractable to numerical analysis and to gain some intuition into the nature of solutions of the problem.

The simplest possible system of partial differential equations which still gives a realistic description of the problem is known as the $2D$ water wave problem. In this case the water is contained in a canal of fixed finite depth and infinite length subject to gravitational force. Dissipation of energy is neglected, so the fluid is described by Euler's equations rather than the Navier-Stokes equations. For Euler's equation the total vorticity of the flow is preserved and so we assume that the motion of the fluid is irrotational. This system has been widely accepted as relevant to the physical problem since 1871 when Boussinesq (cf. [5]) formally showed that it predicted the existence of permanent solitary waves which confirmed the experimental observation of Russell [30] in 1834. Boussinesq's prediction was rigorously proven in 1954 by Friedrichs and Hyers [14].

The present paper is concerned with this $2D$ water wave problem and with models that describe the phenomena present in it.
The simplest and most used model is given by the linear wave equation

$$\partial_\tau^2 A = \partial_X^2 A \qquad \text{or} \qquad \partial_\tau^2 A = \Delta A \quad \text{in the 3D case,}$$

with $A = A(X, \tau)$ proportional to the elevation of the free top surface at a point X and at a time τ. This model is used for the description of long waves with small amplitude. It is used in the numerical simulation of the spreading of tsunamis and in the construction of non-resonant harbor basins. A version with X-dependent coefficients has been used in the computation of the eigenmodes of Lake Constance in order to explain a documented time-periodic flooding in 1546 [39].

The most famous nonlinear model for $2D$ surface water waves is the Korteweg-de Vries (KdV) equation

$$\partial_T A = \partial_X^3 A + A\partial_X A,$$

with $T \in \mathbb{R}$, $X \in \mathbb{R}$, and $A(X,T) \in \mathbb{R}$. It has been derived in [5] and [22] for the description of unidirectional long waves. Again the quantity $A(X,T)$ is proportional to the elevation of the free surface. It is of interest by its own rights. Since the middle 1960's, (see [44]) the KdV equation has been known to be a completely integrable Hamiltonian system which can be solved explicitly with the help of the inverse scattering transform. Although the $2D$ water wave problem is not completely integrable many of the model equations which arise from it share this property.

For the KdV equation there exists a family of permanent solitary waves called the 1-solitons. Solitons turn out to be very robust with respect to collisions with other solitons. If two solitons collide, then after a relatively short "collision time" they regain the shape that they had prior to the collision and the only reminder of the fully nonlinear interaction is a finite phase shift of the solitons relative to each other. Thus, in this way, solitons share properties of both waves and particles. Indeed the impact of solitons on the understanding of the behavior of nonlinear systems has been so great that aside from the discovery of chaos, they have been called the greatest discovery in mathematics and nonlinear physics in recent decades [27].

A second nonlinear model for $2D$ water waves is the Nonlinear Schrödinger (NLS) equation

$$\partial_T A = i\partial_X^2 A + iA|A|^2, \qquad (\text{or} - iA|A|^2)$$

with $T \in \mathbb{R}$, $X \in \mathbb{R}$, and $A(X,T) \in \mathbb{C}$. It has been derived in 1968 by Zakharov [45] in order to describe slow modulations in space and time of a spatially and temporarily oscillating wave packet. See Figure 6. It possesses time-periodic solitary waves and shares with the KdV equation the property of being a completely integrable Hamiltonian system which can be solved by an inverse scattering scheme.

The central issue of this paper is the question of the validity of these models (see Theorem 4.1, Claim 6.1, Theorem 7.1). We point out a number of possible applications of these models, such as the linear and nonlinear interaction of solitary waves in the 2D water wave problem (see Section 4.3), speed limits for high speed ferries (see Section 4.4), and the description of some aspects of tsunamis (see Section 5.4). We also explain how, in the face of the plethora of formal models that have been proposed to describe the long-wavelength limit of the water wave problem one can show that at least one of them (the KdV model) can be placed on a rigorous footing and we describe why we feel it is the best suited to both mathematical or numerical analysis. Furthermore, the results we have proven give precise estimates of the amount by which the true solution of the problem differs from the predictions of the KdV model (see Section 5.3). We begin the paper (Section 2) by introducing the equations of the 2D-water wave problem and close the paper (Section 7) with a discussion about possible extensions as the inclusion of surface tension and the consideration of models for the 3D water wave problem. In order to gain a deeper understanding of how all these models arise, in Section 3.2 we discuss their relation to the linearized water wave problem in Fourier space.

Acknowledgement The work is partially supported by the Deutsche Forschungsgemeinschaft DFG under the grant Schn520/3-1 and by the NSF under grant DMS-0103915.

2 The 2D water wave problem

The water wave problem [37] consists in finding the irrotational flow of an incompressible fluid in an infinitely long canal of fixed depth with a free surface under the influence of gravity. The coordinates are denoted with $x_1 \in \mathbb{R}$ and x_2 in the bounded direction. The fluid is contained in the unbounded domain $\Omega(t)$ between the bottom $\{(x_1, -1)|x_1 \in \mathbb{R}\}$ and the free unknown top surface $\Gamma(t) = \{(x_1, \eta(x_1, t))|\, x_1 \in \mathbb{R}\}$. Under these assumptions the problem is completely determined by the evolution of the free surface $\Gamma(t)$.

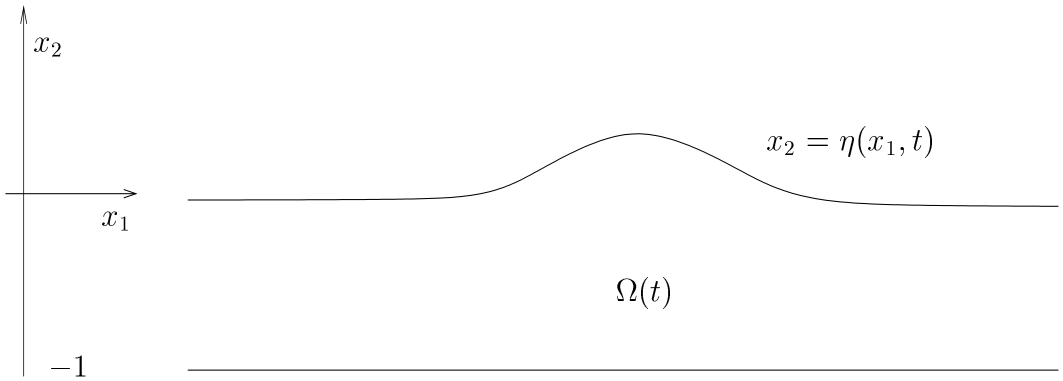

Figure 1: The water wave problem.

In detail, the velocity field $u = (u_1, u_2)$ satisfies Euler's equations

$$\partial_t u + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p - g\vec{e}_2, \tag{1}$$

$$\text{div } u = 0 \tag{2}$$

in $\Omega(t)$, with g being the gravitational constant, ρ the constant density of the fluid, and $\vec{e}_2$ the unit vector into the x_2-direction. Without loss of generality we will set $g = 1$ and the depth of the fluid at rest, i.e. $\eta = 0$, to one in the following. For the derivation of Euler's equations see [13, chapter 6]. In the derivation, equation (1) comes from the balance of forces and the second equation from the conservation of mass which simplifies for incompressible fluids to (2). In the derivation an inviscid fluid is assumed, i.e. a constitutive law $\sigma_{ij}(u) = -p\delta_{ij}$ for the stress tensor σ is chosen, where p is the pressure, and δ the Kronecker symbol.

The vorticity $\omega = \text{rot } u$ of the velocity field u satisfies

$$\partial_t \omega = -(u \cdot \nabla)\omega + (\omega \cdot \nabla)u. \tag{3}$$

From this equation we see that if the fluid's motion is initially irrotational – i.e. if the initial vorticity is zero, it will remain so for all time.

Henceforth, we will assume that the fluid's motion is irrotational, i.e. that $\text{rot } u = 0$ and from this it follows that the velocity field can be written as a gradient of a potential $\phi : \Omega(t) \to \mathbb{R}$, i.e. $u = \nabla\phi$, which satisfies

$$\partial_t \nabla\phi + \frac{1}{2}\nabla|\nabla\phi|^2 = -\frac{1}{\rho}\nabla p - g\nabla x_2.$$

Integration gives Bernoulli's law

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gx_2 = C(t). \tag{4}$$

Since the potential ϕ is only unique up to a constant we may take $C(t) \equiv 0$. Eqn. (4) will be used as one boundary condition on the top surface.

Due to the incompressibility of the fluid, i.e. $\operatorname{div} u = 0$, the potential ϕ satisfies

$$\Delta\phi = 0, \qquad \text{in} \quad \Omega(t). \tag{5}$$

The impermeability of the bottom gives the lower boundary condition

$$u_2|_{x_2=-1} = \partial_{x_2}\phi|_{x_2=-1} = 0. \tag{6}$$

On the free surface $\Gamma(t)$ it is assumed that surface particles remain surface particles, i.e. the vector $(1, \dot{x}_1(t), \dot{x}_2(t)) = (1, u_1(t), u_2(t))$ must be tangent to the surface $(t, \Gamma(t))$. Differentiating $x_2 = \eta(x_1, t)$ with respect to time using $u = \nabla\phi$ gives the kinematic boundary condition

$$\partial_t\eta = \partial_{x_2}\phi - (\partial_{x_1}\eta)\,(\partial_{x_1}\phi), \qquad \text{for } (x_1, x_2) \in \Gamma(t). \tag{7}$$

The balance of forces gives the second boundary condition on the top surface. Note that the pressure is continuous on the top surface and we assume that the pressure is constant as a function of (x_1, x_2) in the air. Thus at the top boundary without loss of generality $p(x, t) \equiv 0$ which leads to

$$\partial_t\phi = -\frac{1}{2}((\partial_{x_1}\phi)^2 + (\partial_{x_2}\phi)^2) - g\eta \qquad \text{for } (x_1, x_2) \in \Gamma(t). \tag{8}$$

remembering that we have also assumed w.l.o.g. $g = 1$. Equations (5)-(8) are called the Eulerian formulation of the water wave problem.

Laplace's equation (5) can be solved in the domain $\Omega(t)$ given the boundary conditions condition (6) on the bottom of the canal and the value of $u_1|_{\Gamma(t)} = \partial_{x_1}\phi|_{\Gamma(t)}$ given on the top surface. Furthermore, this solution is unique up to a constant. Thus, all terms on the right hand side of (7) and (8) can be computed if η and $w = u_1|_{\Gamma(t)}$ are known and the behavior of the entire system (5)-(8) will be determined by the evolution of the two variables $\eta = \eta(x_1, t)$ and $w = w(x_1, t)$. Therefore, in what follows, we restrict our consideration to these two variables. Moreover, we write x for x_1.

3 The initial value problem

3.1 Existence and uniqueness of solutions

For the Eulerian formulation of the 2D water wave problem in a canal of finite depth there is a direct local existence and uniqueness proof [20] for analytic initial conditions which is based on a Cauchy-Kowalevskaya method. See also [36].

All local existence and uniqueness theorems in Sobolev spaces work with the Lagrangian formulation [42] of the 2D water wave problem, in which the free surface possesses a fixed parameterization. The history of this local existence and uniqueness theory for the 2D water wave problem starts with a canal of infinite depth, i.e. when in Fourier space the operator $\hat{\mathcal{K}}_0$ which is introduced in (9) has to be replaced by the Heaviside function $\hat{H}$, where $\hat{H}(k) = 1$ for $k \geq 0$ and $\hat{H}(k) = -1$ for $k < 0$. In this case Nalimov [26] has established a local existence and uniqueness result for small data and Wu [40] extended this result to prove local well-posedness for an arbitrary non-self-intersecting initial surface profile.

In the case of a canal of finite depth, Yoshihara [42] and Craig [7] proved local existence and uniqueness results in the absence of surface tension, and Yoshihara included the effects of surface tension in [43]. These local existence and uniqueness theorems have been improved in [33] and [34].

Recently, Wu also proved the local existence and uniqueness of solutions of the 3D water wave problem in infinite depth without surface tension [41], again also for large data.

3.2 Behavior of the linearized problem

In order to gain a deeper understanding of where the KdV or the NLS equation come from, in this section we consider the linearized water wave problem given by

$$\partial_t \eta = \mathcal{K}_0 w, \qquad \partial_t w = -\partial_x \eta \tag{9}$$

where $\mathcal{K}_0$ is a multiplication operator in Fourier space defined by its symbol $\hat{\mathcal{K}}_0(k) = -i\tanh(k)$. System (9) comes from the solution of (5) in $\mathbb{R} \times (0,1)$ under the boundary conditions (6) and $\partial_{x_1}\phi|_{x_2=0} = w$. In Fourier space system (9) can be diagonalized and we obtain

$$\partial_t \hat{c}_j = \hat{\lambda}_j \hat{c}_j \tag{10}$$

which possesses the solutions

$$\hat{c}_j(k,t) = \hat{c}_j|_{t=0}(k)e^{\hat{\lambda}_j(k)t}$$

where the eigenvalues $\hat{\lambda}_j$ are given by

$$\hat{\lambda}_{1,2}(k) = i\hat{\omega}_{1,2}(k) = \pm i\sqrt{k\tanh k}.$$

plotted in Figure 2. All models which we consider in this paper are derived by multiple scaling analysis. An *Ansatz* $x \mapsto A(\varepsilon x)e^{ik_0x}$ in physical space, with $0 < \varepsilon \ll 1$ a small perturbation parameter, corresponds in Fourier space to a concentration of the Fourier modes at the wave number k_0, i.e $k \mapsto \frac{1}{\varepsilon}\hat{A}(\frac{k-k_0}{\varepsilon})$ in Fourier space. The initial conditions are chosen in such a way that they are non-zero only for one of the curves $\hat{\omega}_j$ which we denote in the following with $\hat{\omega}$.

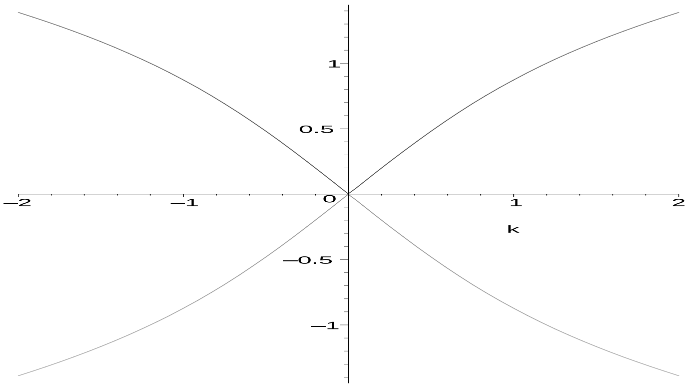

Figure 2: The curves of eigenvalues $k \mapsto \hat{\omega}_{1,2}(k)$.

Due to the concentration of the Fourier modes at the wave number k_0 for the dynamics of

$$\hat{u}(k,t) = \varepsilon^{\alpha-1} e^{i\hat{\omega}(k)t} \hat{A}_0(\frac{k-k_0}{\varepsilon}),$$

with $\alpha \in \mathbb{R}$ determined by the nonlinear analysis, the form of $\hat{\omega}$ at k_0 plays the major role. For $k_0 \neq 0$ we approximate $\hat{\omega}$ at k_0 by its second order Taylor polynomial

$$\tilde{\omega}(k) = \hat{\omega}(k_0) + \hat{\omega}'(k_0)(k-k_0) + \hat{\omega}''(k_0)(k-k_0)^2/2$$

and $\hat{u}(k,t)$ by

$$\hat{u}_{\text{app}}(k,t) = \varepsilon^{\alpha-1} e^{i\tilde{\omega}(k)t} \hat{A}_0(\frac{k-k_0}{\varepsilon}).$$

In order to see where the NLS equation comes from we introduce $k = k_0 + \varepsilon K$ and write

$$\hat{u}_{\text{app}}(k,t) = \varepsilon^{\alpha-1} e^{i\hat{\omega}(k_0)t} e^{i\hat{\omega}'(k_0)\varepsilon K t} \hat{A}(K,T).$$

with

$$\hat{A}(K,T) = e^{i\hat{\omega}''(k_0)K^2 T} \hat{A}_0(K)$$

and $T = \varepsilon^2 t$. The inverse Fourier transform gives in physical space

$$u(x,t) = \varepsilon^{\alpha} A(\varepsilon(x + \hat{\omega}'(k_0)t), \varepsilon^2 t) e^{i(k_0 x + \omega(k_0)t)}$$

where A solves the linear Schrödinger equation

$$\partial_T A = -i\hat{\omega}''(k_0)\partial_X^2 A/2$$

with $X = \varepsilon(x + \hat{\omega}'(k_0)t)$. We have

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}} |u(x,t) - u_{\text{app}}(x,t)| \\
\leq\ & \int |\hat{u}(k,t) - \hat{u}_{\text{app}}(k,t)| dk \\
\leq\ & \varepsilon^{\alpha-1} \int |(e^{i\hat{\omega}(k)t} - e^{i\tilde{\omega}(k)t})\hat{A}_0(\frac{k-k_0}{\varepsilon})| dk \\
\leq\ & \varepsilon^{\alpha} \sup_{K\in\mathbb{R}} |\frac{e^{i\hat{\omega}(k_0+\varepsilon K)t} - e^{i\tilde{\omega}(k_0+\varepsilon K)t}}{(1+|K|^3)}| \int |\hat{A}_0(K)|(1+|K|^3) dK \\
\leq\ & C\varepsilon^{\alpha} \sup_{K\in\mathbb{R}} |\frac{(\hat{\omega}(k_0+\varepsilon K) - \tilde{\omega}(k_0+\varepsilon K))t}{(1+|K|^3)}| \int |\hat{A}_0(K)|(1+|K|^3) dK \\
\leq\ & C\varepsilon^{\alpha+3} t \int |\hat{A}_0(K)|(1+|K|^3) dK \\
\leq\ & C\varepsilon^{\alpha+1} T_0 \int |\hat{A}_0(K)|(1+|K|^3) dK
\end{aligned}
$$

for all $t \in [0, T_0/\varepsilon^2]$, with constants C independent of $\varepsilon \in (0,1)$. As a consequence, the error $\mathcal{O}(\varepsilon^{\alpha+1})$ is smaller than the solution and the approximation which are both $\mathcal{O}(\varepsilon^{\alpha})$, i.e. u_{app} is a good approximation of u. The existence of the last integral with $\hat{A}_0$ in it implies that in physical space A_0 is three times differentiable.

Since $e^{\pm ik_0x} \cdot e^{\pm ik_0x} \notin \{e^{-ik_0x}, e^{ik_0x}\}$ there can be no quadratic terms in the nonlinear equation for A. Since $(e^{ik_0x})^2 \cdot e^{-ik_0x} = e^{ik_0x}$ the lowest order nonlinear terms in the nonlinear equation for A must be a multiple of $A|A|^2$.

Exactly for $\alpha = 1$ the linear terms $\varepsilon^{\alpha+2}\partial_T A$, $\varepsilon^{\alpha+2}\partial_X^2 A$, and the nonlinear terms $\varepsilon^{3\alpha} A|A|^2$ are of the same order and this sets the amplitude scale which leads to the nonlinear Schrödinger equation.

The KdV equation is derived for a basic wave number k_0. Since the functions $k \mapsto \hat{\omega}_{1,2}(k)$ are odd functions we have

$$\hat{\omega}(k) = \hat{\omega}'(0)k + \hat{\omega}'''(0)k^3/6 + \mathcal{O}(k^5).$$

The same analysis as for the NLS equation leads to an *Ansatz*

$$u(x,t) = \varepsilon^{\alpha} A(\varepsilon(x + \hat{\omega}'(0)t), \varepsilon^3 t)$$

where A solves the linearized KdV equation

$$\partial_T A = -\hat{\omega}'''(0)\partial_X^3 A/6$$

with $T = \varepsilon^3 t$. Since $e^{i0x} \cdot e^{i0x} = e^{i0x}$ the lowest order nonlinear terms in the nonlinear equation for A are quadratic terms. But since a spatially constant solution in the nonlinear equation for A corresponds to a flat top surface, i.e. η a constant, spatially constant solutions of the nonlinear equation for A must be preserved in time, i.e the nonlinear quadratic terms must contain a derivative, i.e. $\partial_X(A^2)$ instead of A^2.

Exactly for $\alpha = 2$ the linear terms $\varepsilon^{\alpha+3}\partial_T A$, $\varepsilon^{\alpha+3}\partial_X^3 A$, and the nonlinear terms $\varepsilon^{1+2\alpha}\partial_X(A^2)$ are of the same order.

Choosing one of the curves $\hat{\omega}_{1,2}$ gives unidirectional waves described by a KdV equation. Such waves are discussed in Section 4. In the general long wave limit, discussed in Section 5, the Fourier modes of η and w are concentrated at the wave number $k = 0$. These initial conditions are split to the two curves $\hat{\omega}_1$ and $\hat{\omega}_2$ leading to two KdV equations. In general these KdV equations will be coupled by nonlinear terms. However, as explained in Section 5.3 the influence of the coupling on the dynamics can be neglected in lowest order such that the general long wave limit can be described by two decoupled KdV equations.

4 Unidirectional long waves

4.1 Some historical remarks

In 1834, Russell observed a solitary wave created by a boat in a canal between Edinburgh and Glasgow of 40 cm height and 10 m length which traveled with a speed of 16km/h for many kilometers through the canal without changing drastically its shape [11]. Motivated by his observation in the following years he made a number of experiments. After the publication of his results [30], a vigorous scientific debate followed over whether or not such a wave of permanent form could exist. Airy [1] argued that even if dissipation, i.e. the loss of energy, is neglected, dispersion, i.e. the spreading of energy which is concentrated in the middle of the solitary wave, will destroy the solitary wave. It was finally accepted that such waves exist when Boussinesq [5] and Rayleigh [29] found approximations to such a wave by using a perturbation analysis which took into account the nonlinear nature of the problem. In 1954, Friedrichs and Hyers [14] showed rigorously the existence of solitary waves for the 2D water wave problem. For an overview of results about the existence of permanent solitary waves including surface tension effects see [18].

4.2 Long wave models for unidirectional waves

In contrast to Airy, Boussinesq recognized that the existence of solitary waves is a nonlinear phenomenon. By making the *Ansatz* that

$$\eta(x,0) = \varepsilon^2\phi_1(\varepsilon x, \varepsilon t), \qquad w(x,0) = \varepsilon^2\phi_2(\varepsilon x, \varepsilon t),$$

with ϕ_1 and ϕ_2 satisfying an additional condition which insures that the motion is unidirectional, Boussinesq identified the solitary wave with a stationary solution of the KdV equation [22]

$$\partial_\tau A = \partial_X A + \varepsilon^2 \partial_X^3 A + \varepsilon^2 A \partial_X A, \tag{11}$$

where $A = A(X,T) \in \mathbb{R}$. Eqn. (11) arises by expanding the water wave problem in the small parameter ε and retaining only the lowest order nonlinear terms in ε. Since the KdV equation is a quasilinear equation and since $\partial_\tau A = \partial_X A + \mathcal{O}(\varepsilon^2)$ from a perturbation point of view the so called regularized long wave equation

$$\partial_\tau A = \partial_X A + \varepsilon^2 \partial_\tau \partial_X^2 A + \varepsilon^2 A \partial_X A \tag{12}$$

is equivalent to the KdV equation up to a formal error $\mathcal{O}(\varepsilon^4)$. In a number of papers [2, 3] it was argued that (12) has the advantage that due to its semilinear character numerical simulations are easier.

However, note that the semilinearity can only be effectively exploited if ε is not too small, whereas in the water wave problem we expect (11) and (12) to be an accurate approximation only in the limit of small ε. Furthermore, in the asymptotic limit $\varepsilon \to 0$ the KdV equation is distinguished among all unidirectional long wave models. By introducing new coordinates $X \to X + \tau$, $T = \varepsilon^2 \tau = \varepsilon^3 t$ we obtain the well known version of the KdV equation

$$\partial_T A = \partial_X^3 A + A \partial_X A \tag{13}$$

which is independent of the small perturbation parameter ε. In contrast the regularized long wave equation transforms into

$$\partial_T A = \partial_X^3 A + A \partial_X A + \varepsilon^2 \partial_X^2 \partial_T A$$

which still depends on the perturbation parameter ε and which converges (at least formally) towards the KdV equation for $\varepsilon \to 0$.

Note that from the multiple time scales point of view, one would expect that the models (11) and (12) are formally only valid on time scales of length $\mathcal{O}(1/\varepsilon)$ (since they represent expansions with respect to the time scale $\tau = \varepsilon t$), in contrast to (13) which is formally valid on time scales of length $\mathcal{O}(1/\varepsilon^3)$. In particular, a justification of (11) and (12) on a time scale $\mathcal{O}(1/\varepsilon)$ as in [21] is only a justification of the transport equation

$$\partial_\tau A = \partial_X A, \tag{14}$$

since over this time scale the nonlinear dynamics of the KdV-equation (13) is invisible due to the fact that the elapsed time $T = \mathcal{O}(\varepsilon^2)$ is too short for the nonlinear terms to have a significant effect. For these reasons we feel that (13) is the most appropriate model for the study

of unidirectional longwaves, and in [33] we proved that it gives an accurate approximation to the motion of such waves over the correct time scale (i.e. $\mathcal{O}(1/\varepsilon^3)$).
Although (11) is mathematically equivalent to (16), we feel that it is less suitable for either rigorous or numerical study since the long times over which the wave must be followed to observe the KdV phenomena make it necessary to work with a very large spatial domain. In addition, the fact that the model still depends on the perturbation parameter makes it somewhat ungainly from the point of view of perturbation theory. The same disadvantages apply to (12), though we note that in [4], it is proven that if one takes identical small amplitude, long-wavelength initial data for (11) and (12) the resulting solutions remain close to each other over a time scale of $\mathcal{O}(1/\varepsilon^3)$ for t or $\mathcal{O}(1/\varepsilon^2)$ for τ, so that if one can solve (11) accurately over the very long space and time scales necessary to reproduce the solution of (13), then (12) gives a solution of the same qualitative order of accuracy.

4.3 Approximate interaction of solitary waves

For a direct derivation of the KdV equation (13) without using (11) we make the *Ansatz* that the solution has the form

$$\begin{pmatrix} \eta(x,t) \\ w(x,t) \end{pmatrix} = \varepsilon^2 A(\varepsilon(x-t), \varepsilon^3 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathcal{O}(\varepsilon^3), \tag{15}$$

where we recall that w is the horizontal component of the fluid velocity at the free surface. Inserting this hypothesized form for the solution into the water wave equation and retaining only the lowest order terms in ε, we find that the amplitude satisfies the KdV equation.
In its normalized form the KdV equation is given by

$$\partial_T A = -\partial_X^3 A + 6A\partial_X A, \tag{16}$$

where $A(X,T) \in \mathbb{R}$, $T = \varepsilon^3 t$, and $X = \varepsilon(x-t)$. The important point about (16) is that the nonlinear terms are in some equilibrium with the dispersion, and so (16) possesses a family of solitary waves

$$A(X,T) = \tilde{A}_\beta(X - \beta^2 T) = -\frac{\beta^2}{2}\operatorname{sech}^2(\frac{1}{2}\beta(X - \beta^2 T))$$

as solutions.

In 1965, Zabusky and Kruskal [44] observed that the KdV equation possesses N-solitons, i.e. waves of permanent form which go through each other without changing their shape asymptotically, i.e.

$$A_{N-Sol.}(X,T) \sim \sum_{j=1}^{N} \tilde{A}_{\beta_j}(X - \beta_j^2 T \pm \gamma_j + \delta_j),$$

for $T \to \pm\infty$, with $\beta_1 > \ldots > \beta_N$ and $\gamma_j, \delta_j \in \mathbb{R}$ some phase-shifts.

In [33] we showed that over very long, but finite, time intervals the water wave problem also possesses N-soliton like solutions, i.e. waves which go through each other without changing their shape very much, except at the time of nonlinear interaction. We did so by proving exact estimates between the approximations (15) obtained via the KdV equation (16) and true solutions of the water wave problem (5)-(8).

NOTATION. To state more precisely what our results are we introduce some further notation. Denote Fourier transform by

$$(\mathcal{F}u)(k) = \hat{u}(k) = \frac{1}{2\pi}\int u(x)e^{-ikx}dx.$$

The Sobolev space H^s is equipped with the norm $\|u\|_{H^s} = (\int |\hat{u}(k)|^2(1+|k|^2)^s dk)^{1/2}$. Moreover, let $\|u\|_{H^m(n)} = \|u\rho^n\|_{H^m}$, where $\rho(x) = (1+x^2)^{1/2}$, and let $\|u\|_{C_b^n} = \sum_{j=0}^n \|\partial_x^j u\|_{C_b^0}$, where $\|u\|_{C_b^0} = \sup_{x\in\mathbb{R}} |u(x)|$. Then the following holds.

Theorem 4.1 *Fix $s \geq 4$. For all C_1, $T_0 > 0$ there exist C_2, $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following is true. Let $A_1 \in C([0,T_0], H^{s+6}(2))$ be a solution of the KdV equation*

$$2\partial_T A_1 = -\frac{1}{3}\partial_X^3 A_1 - \frac{3}{2}A_1\partial_X A_1$$

with

$$\sup_{T\in[0,T_0]} \|A_1(T)\|_{H^{s+6}(2)} \leq C_1.$$

Assume further that the initial conditions for the water wave problem have the form

$$\begin{pmatrix} \eta \\ w \end{pmatrix} = \varepsilon^2 A_1(\varepsilon x, 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then there exists a unique solution of the 2D water wave problem (5)-(8) with these initial conditions and this solution satisfies the estimates:

$$\sup_{t\in[0,T_0/\varepsilon^3]} \| \begin{pmatrix} \eta \\ w \end{pmatrix} - \psi_{\mathrm{KdV}}(\varepsilon,\cdot,t)\|_{(C_b^{s-1})^2} \leq C_2\varepsilon^{5/2},$$

where

$$\psi_{\mathrm{KdV}}(\varepsilon,x,t) = \varepsilon^2 A_1(\varepsilon(x-t),\varepsilon^3 t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since the error $\mathcal{O}(\varepsilon^{5/2})$ is small compared with $\mathcal{O}(\varepsilon^2)$ of the approximation and the solution of the 2D water wave problem for all $T \in [0, T_0]$, this means that we can find the dynamics known from the KdV equation in the water wave problem, too. Corresponding to the N-soliton solution of the KdV equation is the approximate interaction of the associated solitary waves in the 2D water wave problem.

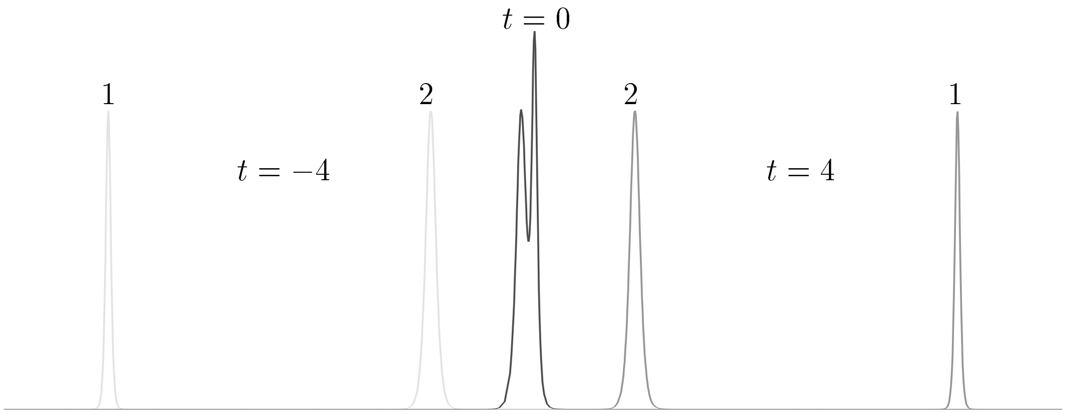

Figure 3: Sketch of a solitary wave interaction of solitary waves 1 and 2 in the 2D water wave problem approximately given by the 2-soliton solution in the KdV equation. See [38] for higher order corrections.

The difficulty in proving this result stems from the fact that the solutions are of order $\mathcal{O}(\varepsilon^2)$, but the time scale over which we must prove the approximation result is of order $\mathcal{O}(\varepsilon^{-3})$, i.e. much longer than $\mathcal{O}(\varepsilon^{-2})$ which is what a simple application of Gronwall's inequality would give.

We note that the first approximation theorem which rigorously related solutions of the KdV equation to the full water wave problem over the correct qualitative time scale was proved by Craig in [7]. However, the result of that reference did not in general allow one to study the interaction of solitary waves as Theorem 4.1 does.

4.4 High speed ferries

The following example from [17] shows that the 1-soliton of the KdV equation can be used to qualitatively understand technological problems of societal and economic importance.

In recent years a new generation of high speed ferries has caused serious problems, especially those that cross the English Channel between England and France. The waves created by these ferries damaged the shore, property and boats along the shore, and even resulted in one fatality.

If we repeat the linear analysis of Section 3.2, retaining the physical parameters h, the canal depth, and g the gravitational acceleration, we find that the linear dispersion relation in this case is:

$$\omega^2 = gk\tanh(hk) \ .$$

Recalling from that discussion that the velocity of long waves is given approximately by the derivative of the dispersion relation at $k = 0$, we see that

$$c = \partial_k \omega|_{k=0} = \sqrt{gh}. \tag{17}$$

The depth of the southern North Sea is about 30-40m, and so the velocity of the ferries of about 80 km/h is close to the approximate speed where solitary waves are created. Since these waves can propagate without loss of energy over large distances, they retain the potential to create enormous havoc when they come ashore and as a consequence of the fatal accident and other damage there are now speed limits for these ferries.

5 General long waves

In this section we give up the assumption of unidirectionality and consider general long wave limit models, i.e. we consider models for initial conditions of the form

$$\eta(x,0) = \varepsilon^\alpha \phi_1(\varepsilon x), \qquad w(x,0) = \varepsilon^\alpha \phi_2(\varepsilon x)$$

with $\alpha \geq 2$. We now explain some mathematical results which can help to choose the best of the host of formal models that have been proposed to treat this problem.

5.1 The linear wave equation

Inserting the *Ansatz*,

$$\eta(x,t) = \varepsilon^\alpha A(\varepsilon x, \varepsilon t), \qquad w(x,t) = \varepsilon^\alpha B(\varepsilon x, \varepsilon t),$$

with $\alpha > 2$ into the water wave equations, expanding formally in ε and retaining only the lowest order terms results in the linear system

$$\partial_\tau A = -\partial_X B, \qquad \partial_\tau B = -\partial_X A$$

which can be brought into the form of a linear wave equation

$$\partial_\tau A^2 = \partial_X^2 A.$$

For the 3D water wave problem the same analysis leads to $\partial_\tau A^2 = \Delta A$. This is the simplest and most used long wave limit model. For example, it has been applied to the numerical simulation of the dynamics of tsunamis and in the construction process of new harbors in order to avoid resonances. A version with X-dependent coefficients modeling the variable depth of the fluid (see formula (17)) has been used in the computation of the eigenmodes of Lake Constance in order to describe some documented time-periodic flooding in 1546 [39].

5.2 Nonlinear models

If one takes the exponent $\alpha = 2$ in the above *Ansatz*, one finds that the lowest order nonlinear and dispersive terms are of comparable order. At a first sight the situation for general long waves seems to be much more complicated than the unidirectional case. One indication of this is the much larger number of formal models that have been proposed to analyze this situation. The most famous examples are the Boussinesq equation [5]

$$\partial_\tau^2 Z = \partial_X^2 Z + \varepsilon^2 \partial_X^4 Z + \varepsilon^2 \partial_X^2 (Z^2),$$

and the regularized Boussinesq equation

$$\partial_\tau^2 Z = \partial_X^2 Z + \varepsilon^2 \partial_\tau^2 \partial_X^2 Z + \varepsilon^2 \partial_X^2 (Z^2)$$

which can be obtained from the Boussinesq equation by using the relation $\partial_\tau^2 Z = \partial_X^2 Z + \mathcal{O}(\varepsilon^2)$. (cf. [28]). Although it is sometimes stated that these models follow only from an assumption that the fluid is undergoing unidirectional motion, this assumption is not necessary to derive these equations. For a derivation from the Lagrangian formulation of the 2D water wave problem see [7] or [33].

There are also a number of long wave systems. Starting with Boussinesq's coupled equations

$$\begin{aligned} \partial_\tau A &= -\partial_X B - \varepsilon^2 \partial_X (BA) - \varepsilon^2 \frac{1}{6} \partial_X^3 B, \\ \partial_\tau B &= -\partial_X A - \varepsilon^2 \partial_X (B^2) - \varepsilon^2 \frac{1}{6} \partial_X^3 A, \end{aligned}$$

one obtains the BBM-system [2, 3]

$$\begin{aligned} \partial_\tau A &= -\partial_X B - \varepsilon^2 \partial_X (BA) + \varepsilon^2 \frac{1}{6} \partial_\tau \partial_X^2 A, \\ \partial_\tau B &= -\partial_X A - \varepsilon^2 \partial_X (B^2) + \varepsilon^2 \frac{1}{6} \partial_\tau \partial_X^2 B, \end{aligned}$$

using the relations $\partial_\tau A = -\partial_X B + \mathcal{O}(\varepsilon^2)$ and $\partial_\tau B = -\partial_X A + \mathcal{O}(\varepsilon^2)$. It is obvious that with this procedure infinitely many different models can be derived.

As with the regularized long wave equation (12) all these models possess the disadvantage of containing the small perturbation parameter ε. There is an ongoing scientific discussion (cf. [2, 6, 23, 24, 25]), going back to [5] about the question which of these models provide a good description of the dynamics of the water wave problem in the long wave limit.

5.3 Using mathematics to differentiate among these models

For a suitable set of initial conditions one can go beyond these formal arguments and use rigorous mathematics to help make a decision about which is the best of the many models which formally describe the long wave limit. Our answer is as follows:

The solutions of the water wave problem in the long wave limit split up into two wave packets, one moving to the right and one to the left, where each of these wave packets evolves independently as a solution of a KdV equation. To lowest order there is *no* interaction of the counter propagating wave packets. See Figure 4.

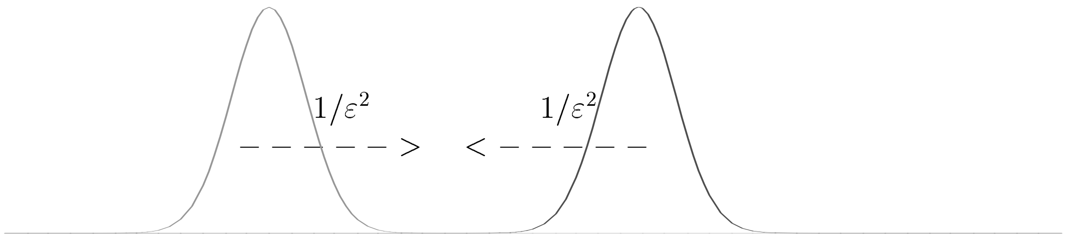

Figure 4: Two counter propagating wave packets do not interact in the long wave limit. In the KdV scaling the amplitudes and the widths are $\mathcal{O}(1)$, but since the speed is of order $\mathcal{O}(\varepsilon^{-2})$ the interaction time is only of order $\mathcal{O}(\varepsilon^2)$ and so the overall interaction is of order $\mathcal{O}(\varepsilon^2)$ and has no influence on the lowest order $\mathcal{O}(1)$ terms. This heuristic argument has been made rigorous in [33]

More precisely, we have shown in [33]:

Theorem 5.1 *Fix $s \geq 4$. For all C_1, $T_0 > 0$ there exist C_2, $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following is true.*

Let

$$\eta|_{t=0}(x) = \varepsilon^2 \Phi_1(\varepsilon x) \quad \text{and} \quad w|_{t=0}(x) = \varepsilon^2 \Phi_2(\varepsilon x) \tag{18}$$

with $\|(\Phi_1, \Phi_2)\|_{H^{s+6}(2) \cap H^{s+11}(0)} \leq C_1$. Split the initial conditions into

$$2A_1|_{T=0} = \Phi_2 + \Phi_1 \quad \text{and} \quad 2A_2|_{T=0} = \Phi_2 - \Phi_1$$

and let the amplitudes $A_1 = A_1(X,T)$ and $A_2 = A_2(X,T)$ satisfy

$$2\partial_T A_1 = -\frac{1}{3}\partial_X^3 A_1 - \frac{3}{2} A_1 \partial_X A_1 \text{ and } \quad 2\partial_T A_2 = \frac{1}{3}\partial_X^3 A_2 + \frac{3}{2} A_2 \partial_X A_2.$$

Then there exists a unique solution of the 2D water wave problem (5)-(8) with initial conditions (18) and this solution satisfies

$$\sup_{t \in [0, T_0/\varepsilon^3]} \| \begin{pmatrix} \eta \\ w \end{pmatrix} - \psi_{\mathrm{LWL}}(\varepsilon, \cdot, t) \|_{(C_b^{s-1})^2} \leq C_2 \varepsilon^{5/2},$$

where

$$\psi_{\mathrm{LWL}}(\varepsilon, x, t) \;=\; \varepsilon^2 A_1(\varepsilon(x-t), \varepsilon^3 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \varepsilon^2 A_2(\varepsilon(x+t), \varepsilon^3 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

As a consequence, in the long-wave approximation, it is sufficient to consider two decoupled KdV equations to describe the lowest order dynamics completely. While other more complicated, formally equivalent, models could also be expected to be correct in the sense of Theorem 4.1, we feel that the KdV description has the advantage of being simple, exactly solvable, and independent of ε, while other models as we have seen still depend on ε. All remarks which we made about the regularized long wave equation (12) are also true for the models from Section 5.2. In experimental or numerical investigations of the water wave problem for finite values of ε, one does see some interaction between the left and right moving wave train. However, the theorem above tells us that the coupling of the two counter propagating wave packets only occurs in the next order approximation – i.e. as a higher order correction to the KdV approximation. For a rigorous analysis of the next order correction to the KdV approximation to the long wave limit of a Boussinesq model see [38].

5.4 KdV aspects of Tsunamis

Tsunamis are large waves caused by earthquakes or landslides under the sea. The waves created by these events are almost unobservable in the open sea as their height is only a few meters and their length up to 100 kilometers. In the Pacific Ocean the average depth is around 5000m which leads with above formula (17) for the wave speed of long waves to an incredible velocity of around 700 km/h. If such a wave encounters an island or coastline, the waves steepen as they come ashore and may reach heights of 20m and cause enormous damage.

If the earthquake happens in a crack along a continent it creates almost parallel initial conditions. In this approximately two dimensional situation, we may apply Theorem 5.1. This theorem gives us an almost complete description of the long time behavior of the water wave problem in the long wave limit. On a time scale $\mathcal{O}(1/\varepsilon)$ the solutions split up into the two wave packets one moving to the right and one to the left. These wave packets evolve independently as solutions of the KdV equations. The long-time behavior for the solutions of the KdV equations can also be computed explicitly with the help of the inverse scattering transform. Some solitons which are ordered with respect to their height evolve out of a dispersive remainder (cf. [12]). As illustrated in Figure 5, Theorem 5.1 implies that the same behavior will be observed in the water wave problem in the long wave limit.

For the tsunamis this means that if the distance between the epicenter of the earthquake and the island is relatively small, the tsunami appears as a wave train (time scale $\mathcal{O}(1/\varepsilon)$), but if the earthquake and the island are far away from each other, as for the 1946 tsunami on Hawaii caused by an earthquake at the Aleutian islands $5000\mathrm{km}$ away, the tsunami can appear as a number of well separated solitary waves (time scale $\mathcal{O}(1/\varepsilon^3)$).

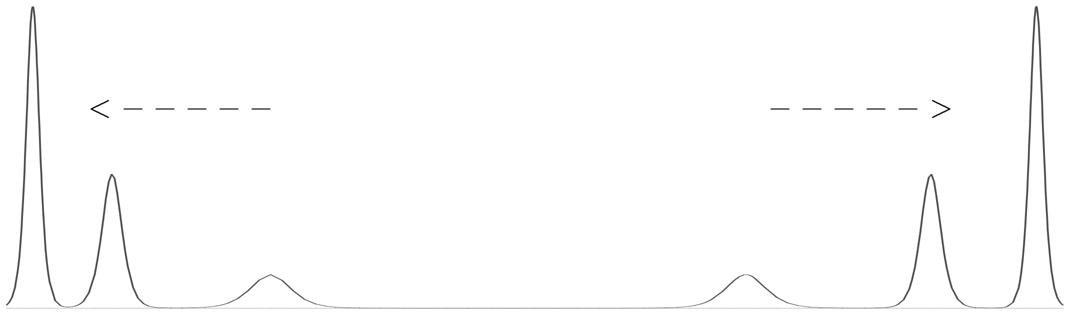

Figure 5: The long time behavior for the water wave problem in the long wave limit for $t = T_0/\varepsilon^3$ with T_0 large. The solutions of the water wave problem in the long wave limit splits up into two wave packets, one moving to the right and one to the left, where each of these wave packets evolves independently as a solution of a KdV equation. For large T_0 solitons evolve out of a dispersive remainder.

6 Oscillating wave packets

Now we come to the second universal modulation equation for the 2D water wave problem. In 1968 V.E. Zakharov [45] derived the NLS equation

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2, \tag{19}$$

with $T \in \mathbb{R}$, $X \in \mathbb{R}$, $A(X,T) \in \mathbb{C}$, and coefficients $\nu_j = \nu_j(k_0) \in \mathbb{R}$ from the equations of the 2D water wave problem in order to describe slow spatial and temporal modulations of a spatially and temporarily oscillating wave packet with a basic spatial wavenumber $k_0 \neq 0$.

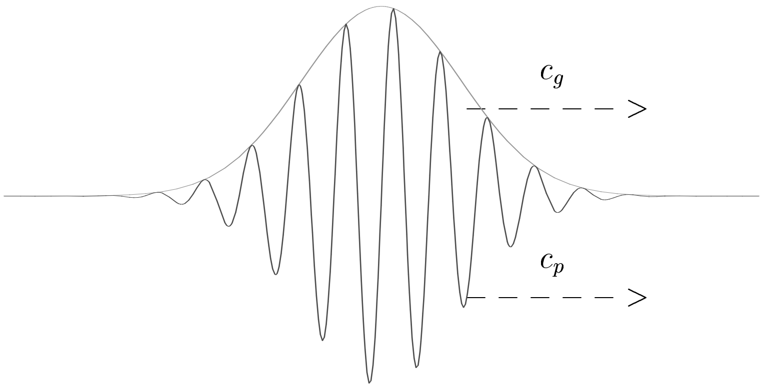

Figure 6: The envelope advancing with the group velocity c_g modulating the oscillating wave packet which advances with the phase velocity c_p is described by the amplitude A solving the NLS equation.

To put this in the context of the approximation results we have discussed above, we make the

Ansatz

$$\begin{aligned}\begin{pmatrix}\eta\\ w\end{pmatrix} &\approx \varepsilon\psi_{\mathrm{NLS}}(x,t)\\ &= \varepsilon A(\varepsilon(x-c_g t),\varepsilon^2 t)e^{ik_0(x-c_p t)}\varphi_1 + c.c,\end{aligned}$$

where φ_1 is an explicitly computable two dimensional complex-valued vector.
On the basis of preliminary investigations, we expect the following result to hold:

Claim 6.1 *Fix s sufficiently large. Then for all $C_1, T_0 > 0$ there exist $C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following is true.*

Let $A \in C([0,T_0], H^{s+3}(\mathbb{R},\mathbb{C}))$ be a solution of the NLS equation (19) satisfying

$$\sup_{T\in[0,T_0]} \|A(T)\|_{H^{s+3}} \leq C_1.$$

Then there are solutions of the 2D water wave problem (5)-(8) with

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\begin{pmatrix}\eta\\ w\end{pmatrix}(x,t) - \varepsilon\psi_{\mathrm{NLS}}(x,t)\|_{(C_b^{s-1})^2} \leq C_2\,\varepsilon^{3/2}.$$

The proof of this claim will be the subject of a forthcoming paper. Again the difficulty is primarily due to the fact that the solutions are of order $\mathcal{O}(\varepsilon)$, which if one naively estimates the error leads only to an approximation result over the time scale $\mathcal{O}(\varepsilon^{-1})$. However, in order to observe the dynamics of the NLS equation in the water wave problem we need to control the error over the much longer time scale $\mathcal{O}(\varepsilon^{-2})$ as stated in the theorem. In order to obtain this long time scale a so called normal form transform [19] is applied to eliminate the quadratic terms. The proof of the claim will be based on [31] in which for a model problem a method has been developed to handle the resonance at the wavenumber $k = 0$. Estimates for the so called residual, i.e. the terms which do not cancel after inserting the approximation into the equations, can be found in [8].

7 Discussion

Although the local existence and uniqueness theory is completely different if surface tension is included, the long wave limit results, Theorem 4.1 and Theorem 5.1, have been transfered also to this case [34]. The surface tension term $\mu\partial_x[\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}]$ has to be added to the right hand side of (8), where μ is proportional to the strength of surface tension.

The justification of the NLS equation will be much more complicated in case of surface tension since additional resonances occur. We expect that for different magnitudes μ of the surface tension there are wave numbers k_0 for which the approximation property, as stated in Claim 6.1, does not hold.

Models for the prediction of the amplitude of wind induced waves which are still dangerous for large ships make essential use of resonance effects between different wave packets. For wind induced waves on lakes or rivers small surface tension effects play a role. In this case with the *Ansatz*

$$\begin{aligned}\begin{pmatrix}\eta\\ w\end{pmatrix} \approx \varepsilon\psi_{\mathrm{TWI}}(x,t) &= \varepsilon A_1(\varepsilon x,\varepsilon t)e^{i(k_1 x+\omega_1 t)}\varphi_1\\ &\quad +\varepsilon A_2(\varepsilon x,\varepsilon t)e^{i(k_2 x+\omega_2 t)}\varphi_2\\ &\quad +\varepsilon A_3(\varepsilon x,\varepsilon t)e^{i(k_3 x+\omega_3 t)}\varphi_3 + c.c.\end{aligned}$$

the TWI equations (Three wave interaction)

$$\begin{aligned}\partial_T A_1 &= c_g(k_1)\partial_X A_1 + \gamma_1\overline{A_2A_3},\\ \partial_T A_2 &= c_g(k_2)\partial_X A_2 + \gamma_2\overline{A_1A_3},\\ \partial_T A_2 &= c_g(k_3)\partial_X A_3 + \gamma_3\overline{A_1A_2}\end{aligned} \tag{20}$$

with $\phi_j \in \mathbb{C}^2$ for three resonant wave packets satisfying

$$k_1+k_2+k_3=0, \qquad \omega_1+\omega_2+\omega_3=0 \tag{21}$$

can be derived (cf. [10]). In [34] the following approximation result has been proven

Theorem 7.1 *Fix $s \geq 6$. Then for all $C_1, T_0 > 0$ there exist $C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the following is true.*

Let $A_1, A_2, A_3 \in C([0,T_0],(H^{s+2}(\mathbb{R},\mathbb{C}))^3)$ be solutions of the TWI system (20) satisfying

$$\sup_{T\in[0,T_0]} \|A_j(T)\|_{H^{s+2}} \leq C_1.$$

Then there are solutions of the 2D water wave problem (5)-(8) with

$$\sup_{t\in[0,T_0/\varepsilon]} \|\begin{pmatrix}\eta\\ w\end{pmatrix}(x,t) - \varepsilon\psi_{\mathrm{TWI}}(x,t)\|_{(C_b^{s-1})^2} \leq C_2\,\varepsilon^{3/2}.$$

For wind induced waves on the sea, surface tension effects can be neglected. Then there are no three wave packets satisfying the resonance condition (21). Then a system of four such equations associated to four wave packets satisfying

$$k_1+k_2+k_3+k_4=0, \qquad \omega_1+\omega_2+\omega_3+\omega_4=0 \tag{22}$$

has been considered (cf. [10]). The justification of this system of equations should follow in a fashion similar to the justification of the NLS equation.

Another situation of great practical interest concerns waves in a canal with a variable bottom topography. While a number of papers have developed formal models to describe the waves in this situation (cf. [16]), to our knowledge there are no rigorous approximation results in the literature which treat this problem.

We close the paper with some remarks about models for the 3D water wave problem. In three space dimensions the two curves of eigenvalues have to be rotated around the ω axis leading to two rotationally symmetric surfaces. In the long wave limit we have to expand the surfaces of eigenvalues at the wave vector $k = 0$. This is a highly degenerate situation since the two surfaces form a cone at the wave vector $k = 0$. Making the *Ansatz*

$$\psi_{\mathrm{KP}}(\varepsilon, x_1, x_3, t) = \varepsilon^2 A_1(\varepsilon(x_1 - t), \varepsilon^2 x_3, \varepsilon^3 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where x_3 is the new variable in the second unbounded space direction along the top surface, one can formally derive the Kadomtsev-Petviashvili (KP) equation

$$\partial_X \partial_T A = \partial_X (\partial_X^3 A + A \partial_X A) + \partial_Y^2 A,$$

with $X = \varepsilon(x_1 - t)$, $Y = \varepsilon^2 x_3$, The solutions of this equation describe unidirectional waves slowly modulated in the direction normal to the direction of propagation. For a model problem it has been pointed out in [15] that the approximation property for the KP-equation depends strongly on the chosen initial condition of the KP-equation.

The natural generalization of the NLS equation is the Davey-Stewartson system

$$\begin{aligned} i\partial_T A &= \partial_X^2 A + \partial_Y^2 A + A|A|^2 + A\partial_X B \\ 0 &= \partial_X^2 B + \partial_Y^2 B + |A|^2 \end{aligned}$$

with $X = \varepsilon(x_1 - c_g t)$, $Y = \varepsilon x_3$, and where the function B corresponds to the wave vector $k = 0$. Estimates for the residual can be found in [9].

So far there are no approximation results for these equations for the 3D water wave problem due to the fact that so far there is no local existence and uniqueness theory for the 3D water wave problem for canals of finite depth, and such existence results are essential for the justification of any such approximate models. However, we believe that if some existence theorems were forthcoming the general approach to proving approximation results for two dimensional water waves which was described in this survey should carry over to the three dimensional situation as well.

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