p-Adic deformations of arithmetic cohomology

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Contents

1 Introduction

Let $\mathbf{G}_{/\mathbb{Q}}$ be a connected reductive algebraic group defined over $\mathbb{Q}$, let p be a prime for which $\mathbf{G}$ is split over $\mathbb{Q}_p$, and let $\mathbf{T}_{/\mathbb{Q}_p} \subseteq \mathbf{G}_{/\mathbb{Q}_p}$ be a $\mathbb{Q}_p$-split torus of maximal rank. Let $\mathcal{X}$ be the $\mathbb{Q}_p$-rigid analytic "weight space" that parametrizes $\mathbb{Q}_p$-Frechet-algebra-valued characters of $\mathbf{T}(\mathbb{Z}_p)$ – see §3.5.

Our guiding problem in this paper is to construct a universal eigenvariety over $\mathcal{X}$ that parametrizes analytic families of packets of Hecke eigenvalues that occur in OC cohomology. By OC (overconvergent) cohomology, we mean the cohomology of an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ with coefficients in a module of distributions on the big cell associated to $\mathbf{G}_{/\mathbb{Q}_p}$. This concept generalizes that of the oveconvergent modular forms of Katz.

We attain this goal to a large extent for in OC cohomology of finite slope, obtaining much of what Hida gets in his theory of ordinary modular forms. However, in this paper, we will only construct a universal eigenvariety locally over $\mathcal{X}$. We expect that these local constructions will patch together, but there are some technical problems we haven't yet considered.

Before entering into technicalities, let us state a rough version of our results. In the remainder of the introduction we will give a more accurate account of the contents.

For certain characters c of $\mathbf{T}(\mathbb{Z}_p)$, in §3.7 we construct the universal highest weight module $\mathcal{D}_c$ with highest weight c. It is the topological completion of a p-adic analogue of a Verma module. If we choose for c the universal character (Theorem 3.5.4), we obtain the module $\mathcal{D}$. For any admissible open $\Omega \subset \mathcal{X}$, define $\mathcal{D}_\Omega = \mathcal{O}_\mathcal{X}(\Omega)\hat{\hat{\otimes}}_{\mathcal{O}_\mathcal{X}(\mathcal{X})}\mathcal{D}$. For each $k \in \mathcal{X}$, we obtain the module $\mathcal{D}_k$. These are all modules of distributions on the big cell of $\mathbf{G}_{/\mathbb{Q}_p}$, which is a p-adic manifold.

There is a unique continuous map $b_k : \mathcal{D}_\Omega \to \mathcal{D}_k$ for any $k \in \Omega$, taking the maximal vector to the maximal vector (Theorem 3.7.2). If $k \in \mathcal{X}$ is dominant integral, we have the finite dimensional irreducible representation V_k with highest weight k. There is a unique continuous map $a_k : \mathcal{D}_k \to V_k$, taking the maximal vector to the maximal vector (Theorem 3.7.3).

Let (Γ, S) be a congruence Hecke pair in $\mathbf{G}(\mathbb{Q})$ modeled at p on the Iwahori subgroup I or a subgroup of finite index in I, as explained further in §1.2. Let $\mathcal{H} = \mathcal{H}(\Gamma, S)$ be the Hecke algebra, assumed commutative. (In actual fact, we work adelically.) We fix a strictly positive $\pi \in S$ (§1.2) and let U denote the Hecke operator $\Gamma\pi\Gamma$. For any p-adic module M on which U acts, and any nonnegative rational number h, we let M_h be the subset of M on which U acts with slopes $\leq h$. If M is an S-module, define the $\mathcal{H}$-module

$$H(M) := \oplus_i H^i(\Gamma, M).$$

We summarize our main results in the following three theorems:

Theorem 1. For dominant integral k, let $m(k)$ be the positive piecewise linear function defined in §3.11 (21). Then if $h < m(k)$,

$$a_k^* : H(\mathcal{D}_k)_h \xrightarrow{\sim} H(V_k)_h$$

is an isomorphism.

This is Theorem 6.4.1.

Theorem 2. For any $k \in \mathcal{X}$, any $h \geq 0$, there exists an admissible affinoid open neighborhood Ω of k such that $H(\mathcal{D}_\Omega)_h$ is finitely generated over $\mathcal{O}_\mathcal{X}(\Omega)$. Moreover, for any $k' \in \Omega$, specialization induces a map

$$b_k^* : H^*(\mathcal{D}_\Omega)_h \longrightarrow H^*(\mathcal{D}_{k'})_h$$

such that any system of $\mathcal{O}_{\mathcal{X}}(\Omega)$-valued Hecke eigenvalues occurring in $H(\mathcal{D}_{\Omega_h})_h$ specializes at k' to a system of eigenvalues occurring in $H(\mathcal{D}_{k'})_h$, or to zero.

This follows immediately from Theorem 6.2.1.

A converse of Theorem 2 holds. The best way to express this is to make the following definition: Let the reduced $\mathcal{O}_{\mathcal{X}}(\Omega)$-algebra $\mathcal{R}(\Omega, h)$ be the image of $\mathcal{H}$ in the endomorphisms of $H(\mathcal{D}_\Omega)_h$, modulo nilpotents. Denote the tautological map by $\lambda : \mathcal{H} \to \mathcal{R}(\Omega, h)$.

Theorem 3. For any $k \in \mathcal{X}$, any h, take the admissible affinoid open neighborhood Ω of k given by Theorem 2. The structure map of rings $\mathcal{O}_{\mathcal{X}}(\Omega) \to \mathcal{R}(\Omega, h)$ is a finite morphism. Let $\kappa : X(\Omega, h) \to \Omega$ be the associated affinoid space morphism. Then

- For any $P \in X(\Omega, h)$, the specialization $\lambda_P : \mathcal{H} \to \mathbb{C}_p$ is a system of Hecke eigenvalues occurring in $H(\mathcal{D}_{\kappa(P)})_h$.
- $X(\Omega, h)$ is a universal object in the category of reduced rigid analytic spaces $\mathcal{V}$ equipped with morphisms $\kappa' : \mathcal{V} \to \Omega$ and $\lambda' : \mathcal{H} \to \mathcal{O}(\mathcal{V})$ such that for all P in some Zariski dense subset of $\mathcal{V}$, the specialization λ'_P is a system of Hecke eigenvalues occurring in $H(\mathcal{D}_{\kappa'(P)})_h$.

This theorem follows immediately from the more general Theorem 4 given below in §1.7.

1.1 OC and Automorphic Cohomology

We say that a system of Hecke eigenvalues is "OC" of weight $k \in \mathcal{X}$, if it occurs in the cohomology $H^*(S_K, \widetilde{\mathcal{D}}_k)$, where S_K is the Shimura manifold associated to some open compact subgroup $K_{\mathbb{A}_f}$ and $\widetilde{\mathcal{D}}_k$ is the local system on S_K associated to $\mathcal{D}_k$ – see §2.1.

If $k \in \mathcal{X}$ can be written as $k = \psi + \epsilon$ where $\psi = \psi_k$ is a dominant highest weight for $\mathbf{G}(\mathbb{Q}_p)$ and ϵ is a finite order character, we say that k is "arithmetic". We denote by $\mathcal{X}^+$ the set of arithmetic weights. If k is arithmetic, we say that a system of Hecke eigenvalues is "automorphic" of weight k, if it occurs in the cohomology $H^*(S_K, \widetilde{V}_k)$, and $\widetilde{V}_k$ is the local system on S_K associated to V_k.

1.2 Graded Hecke Pairs and Algebras

We let I_p be an Iwahori subgroup of $\mathbf{G}(\mathbb{Z}_p)$ and Λ a certain free finitely generated abelian subgroup of $\mathbf{T}(\mathbb{Q}_p)$ which generalizes $\{\text{diag}\,(p^{a_1}, \ldots, p^{a_n}) \mid a_1, \ldots, a_n \in \mathbb{Z}\}$ when $\mathbf{G} = GL(n)$ – see §2.5. The subsemigroup Λ^+ consists of those $x \in \Lambda$ on which the positive roots have a nonnegative p-adic ord. We say x is "strictly positive" if all these p-adic ords are positive.

We set Σ_p to be the semigroup generated by I_p and Λ^+. The triple (I_p, Σ_p, Λ) is an example of what we call "a graded Hecke pair" (Definition 2.5.2). If R is any ring, the Hecke algebra $\mathcal{H}_p$ of double cosets $I_p \backslash \Sigma_p / I_p$ is naturally isomorphic to $R[\Lambda^+]$. We fix a strictly positive $\pi \in \Lambda^+$ and denote by U the corresponding Hecke operator.

In fact, we work with arbitrarily deep level structures inside I_p. For $s \in \Lambda^+$, let $I^s = s^{-1} I_p s \cap I_p$ and $\Sigma^s = s^{-1} \Sigma_p s \cap \Sigma_p$ – see §3.1.

We fix a Hecke pair $(K_{\mathbb{A}_f}, \Sigma_{\mathbb{A}_f})$ in $\mathbf{G}(\mathbb{A}_f)$ with $K_{\mathbb{A}_f}$ a compact open subgroup and we suppose this data factors as:

$$K_{\mathbb{A}_f} := \prod_{\ell < \infty} K_\ell, \quad \text{and} \quad \Sigma_{\mathbb{A}_f} := \prod_{\ell < \infty}{}' \, \Sigma_\ell,$$

where each $K_\ell \subseteq \mathbf{G}(\mathbb{Q}_\ell)$ is a compact open subgroup and Σ_ℓ is a subsemigroup of $\mathbf{G}(\mathbb{Q}_\ell)$ containing K_ℓ and at p we assume the pair (K_p, Σ_p) is the Hecke pair (I^s, Σ^s) for some $s \in \Lambda^+$.

For any ring R we define the abstract Hecke algebra

$$\mathcal{H}_R := \mathcal{H}_R(K_{\mathbb{A}_f}, \Sigma_{\mathbb{A}_f})$$

as the double coset algebra with coefficients in R. We assume the above data has been chosen so that $\mathcal{H}_R$ is commutative. Let $\mathcal{H} = \mathcal{H}_{\mathbb{Q}_p}$.

1.3 Eigenvarieties

Let $\Omega \subseteq \mathcal{X}$ be an arbitrary open $\mathbb{Q}_p$-analytic subvariety of $\mathcal{X}$. An "eigenvariety" over Ω is a triple $(\mathcal{V}, k, \lambda)$ consisting of a reduced $\mathbb{Q}_p$-rigid analytic variety $\mathcal{V}$, a locally finite morphism

$$\begin{array}{c} \mathcal{V} \\ \kappa \downarrow \\ \Omega \end{array}$$

and a ring homomorphism

$$\lambda : \mathcal{H} \longrightarrow \mathcal{O}(\mathcal{V}),$$

where $\mathcal{O}(\mathcal{V})$ is the ring of global $\mathbb{Q}_p$-rigid analytic functions on $\mathcal{V}$.

An eigenvariety $(\mathcal{V}, \kappa, \lambda)$ over Ω is said to be "automorphic" (resp. "OC") of level $K_{\mathbb{A}}$ if there is a set S of arithmetic points on $\mathcal{V}$ such that

1. S is Zariski dense in $\mathcal{V}$, and
2. for every $P \in S$, the specialization $\lambda_P : \mathcal{H} \to \mathbb{C}_p$ of λ at P is automorphic (resp. OC) of weight $\kappa(P)$.

The standard example of an eigenvariety is the Coleman-Mazur eigencurve for $GL(2)$ of tame level 1, which is defined over the full weight space $\mathcal{X}$.

There is an obvious way to define morphisms so that we have a category of eigenvarieties over Ω. We wish to construct an automorphic (resp. OC) eigenvariety $(\mathcal{V}, \kappa, \lambda)$ over Ω which is *universal* in the sense that any other automorphic (resp. OC) eigenvariety $(\mathcal{V}_0, \kappa_0, \lambda_0)$ admits a unique morphism to $(\mathcal{V}, \kappa, \lambda)$.

1.4 Universal $\mathcal{S}$-Eigenvarieties

In this paper we construct local universal eigenvarieties in the presence of a certain finite slope hypothesis on λ. To explain this slope hypothesis, consider more generally a multiplicative subset $\mathcal{S}$ of $\mathcal{H}_{A(\Omega)}$, where $A(\Omega)$ denotes the ring of rigid analytic functions on Ω. Typically, $\mathcal{S}$ will consist of certain polynomials in $A(\Omega)[U]$.

An $\mathcal{S}$-eigenvariety over Ω is an eigenvariety $(\mathcal{V}, \kappa, \lambda)$ over Ω such that for every point $P \in \mathcal{V}$, there exists an $s \in \mathcal{S}$ such that $\lambda_P(s) = 0$.

An OC $\mathcal{S}$-eigenvariety over Ω is an OC $\mathcal{S}$-eigenvariety over Ω that is OC. An automorphic $\mathcal{S}$-eigenvariety over Ω is an automorphic $\mathcal{S}$-eigenvariety over Ω that is automorphic and such that there is a Zariski dense subset $S \subset \mathcal{V}$ of automorphic points with the property that for every $P \in S$, $H(\mathcal{D}_\kappa(P))_{\mathcal{S}} \to H(V_\kappa(P))_{\mathcal{S}}$ is an isomorphism. The concept of universal automorphic or OC $\mathcal{S}$-eigenvariety is clear.

If $h \in \mathbb{Q}^{\geq 0}$, we define a certain $\mathcal{S}_h$ with the property that for any P and s, $\lambda_P(s) = 0$ implies that the Hecke eigenvalue $\lambda(U)$ has p-adic ord $\leq h$ – see §1.6 below for more details.

One of our main results can be phrased as asserting the following: For any $h \in \mathbb{Q}^{\geq 0}$ and $k \in \mathcal{X}$ there exists an admissible affinoid open neighborhood Ω of k and a universal OC $\mathcal{S}_h$-eigenvariety over Ω. We prove this by constructing it, as follows.

1.5 Construction Using Universal Highest Weight Modules.

Our approach to constructing a universal eigenvariety is to use the cohomology of the universal highest weight modules $\mathcal{D}_\Omega$. These modules are Frechet spaces. In fact they are projective limits of orthonormalizable Banach modules over $\mathcal{O}_\mathcal{X}(\Omega)$. Since $\mathcal{D}_\Omega$ consists of distributions on the big cell, it is also endowed with a right action of $\Sigma_{\mathbb{A}_f}$ (for $s \in \Lambda^+$ sufficiently large, depending on Ω.) The cohomology thus acquires a Hecke action.

In fact, $\mathcal{D}_\Omega$ can be naturally embedded into a certain induced module from I_p to Σ_p in a way compatible with its being a projective limit of Banach modules – see §5.5 and Proposition 5.6.1. The machinery in Chapter 5 is developed in order to keep track of the Hecke actions on the cohomology of all these modules.

Before continuing, let us make note of three technical challenges (that do not occur when dealing with classical modular forms) which must be overcome when working with a general group $\mathbf{G}$.

First, the coboundaries may not be closed in the cochains. Earlier authors have avoided this by working in situations in which the coboundaries are 0. To handle nontrivial coboundaries, we lift U to the level of orthonormalizable cochains, work there, and then pass to cohomology – see §2.6 and §2.7.

Second, for a fixed i, H^i is not an exact functor on coefficient modules. We deal with this by using the ring-theoretic Theorem 6.1.1 to keep track of Hecke eigenpackets. The proof of this theorem exploits the long exact sequence of cohomology.

Third, we have to introduce some new ideas in the higher rank case to factor the Fredholm determinant of the U operator on the cohomology, so that we can apply Coleman's method of Riesz factorizations of Banach modules. These involve the "factorization" of Newton polygons as well as of power series – see Chapter 4, especially Theorems 4.4.2 and 4.5.1.

We may also remark here that our method, pursued always over $\mathbb{Q}_p$, does not enable us to keep track of p-torsion in the cohomology.

1.6 $\mathcal{S}$ Decompositions

The cohomology of the Shimura manifold S_K with coefficients in $\mathcal{D}_\Omega$ is presumably of infinite rank over $\mathcal{O}_\mathcal{X}(\Omega)$. To find Hecke eigenpackets occurring in the cohomology, we must cut down the cohomology to something of finite rank. This is done using slope decompositions with respect to U. However, as mentioned in §1.5, we have to work with a non-unique lift of U to the cochain level. To show that the slope decomposition we get doesn't depend on the lift, we use a purely algebraic concept of $\mathcal{S}$-decompositions, defined in §4.1 and with properties contained in Proposition 4.1.2.

We believe $\mathcal{S}$-decompositions will be useful in many situations. When $\mathcal{S}$-decompositions exist, most of Hida's ring theoretic lemmas can be proved.

Let R be a commutative noetherian ring, $\mathcal{R}$ a commutative R-algebra, and $\mathcal{S} \subseteq \mathcal{R}$ is a multiplicative subset. For any $\mathcal{R}$-module H define $H_\mathcal{S} := \{ h \in H \mid \exists \alpha \in \mathcal{S} \text{ such that } \alpha h = 0 \}$. An $\mathcal{S}$-decomposition of H is an $\mathcal{R}$-module decomposition $H = H_\mathcal{S} \oplus H'$, such that $H_\mathcal{S}$ is finitely generated as R-module; and H' is an $\mathcal{R}$-submodule of H on which every element of $\mathcal{S}$ acts invertibly (i.e. has a two-sided inverse in $\mathrm{End}_\mathcal{R}(H')$).

Note that we build into the definition the finite generation of $H_{\mathcal{S}}$ over R.

Now assume that R is a Banach ring and $\mathcal{R} = R[U]$ for some endomorphism U of H. For any polynomial $Q(T)$, let $Q^*(T) = T^{\deg Q}Q(T^{-1})$. Let $\mathcal{S}_h$ be the multiplicative subset of $\mathcal{R}$ consisting of $Q^*(U)$ where Q runs over all polynomials in $R[T]$ satisfying: (a) the leading coefficient of Q is a multiplicative unit, and (b) Q has slope $\leq h$. Then Lemma 4.6.4 says that a "slope $\leq h$ decomposition" of H is exactly a $\mathcal{S}_h$-decomposition of H.

When Ω is an open affinoid in $\mathcal{S}$ and $R = A(\Omega)$, we obtain slope decompositions by factoring the Fredholm determinant $\det(1 - TU)$ of U acting on cochains C with values in $\mathcal{D}_\Omega$. However, in the proof of Theorem 4.5.1, we have to shrink Ω to a subaffinoid open Ω_0 to get the appropriate factorization.

Since $C_{\mathcal{S}_h}$ is finitely generated over $A(\Omega)$, we can get control of the Hecke eigenpackets occurring in it. To do this, we use the ring theoretic construction, which is true in a very general situation as follows:

For any ring A, define A_{red} to be A modulo its nilradical. Let R be a noetherian ring, (Γ, Σ) a Hecke pair, and denote the Hecke algebra over R as $\mathcal{H}_R := \mathcal{H}(\Gamma, \Sigma) \otimes R$. We assume $\mathcal{H}_R$ is commutative. Let $\mathcal{S}$ be a multiplicative subset of $\mathcal{H}_R$. Let M be an $R[\Sigma]$-module, so that the cohomology $H(M) := \bigoplus H^*(\Gamma, M)$ is an $\mathcal{H}_R$-module.

Let $\mathcal{I}$ be an ideal in R that is generated by a finite M-regular sequence. We assume that $H(M)$ has an $\mathcal{S}$-decomposition, from which it follows easily that $H(M/\mathcal{I}M)$ has one too. For any module $R[\Sigma]$-module N, define $\widetilde{\mathcal{R}}(N) = \text{Im }(\mathcal{H}_R \to \text{End}_R(H(N)_{\mathcal{S}})$ and $\mathcal{R}(N) = \widetilde{\mathcal{R}}(N)_{\text{red}}$. When we need to include $\mathcal{S}$ in the notation, we will write $\mathcal{R}(N, \mathcal{S})$.

We call the natural map $\mathcal{H}_R \to \mathcal{R}(N)$ the "tautological map." Note that ring homomorphisms from $\mathcal{R}(N)$ to a field L correspond to L-valued Hecke eigenpackets that occur in $H(N)$.

Then we prove Theorem 6.1.1: there is a natural isomorphism $(\mathcal{R}(M)/\mathcal{I}\mathcal{R}(M))_{\text{red}} \cong \mathcal{R}(M/\mathcal{I})$. This enables us to compare Hecke eigenpackets occurring in the cohomology of $\mathcal{D}_\Omega$ with those occurring in the cohomology of $\mathcal{D}_k$, since $\mathcal{D}_k \approx \mathcal{D}_\Omega/\mathcal{I}\mathcal{D}_\Omega$ where $\mathcal{I}$ is the ideal of functions in $A(\Omega)$ that vanish at k. The isomorphism of Theorem 6.2.1,

$$(\mathcal{R}(\mathcal{D}_\Omega)/\mathcal{I}\mathcal{R}(\mathcal{D}_\Omega))_{\text{red}} \cong \mathcal{R}(\mathcal{D}_k),$$

which holds for sufficiently small Ω containing k, lies at the core of the universality of $\mathcal{R}(\mathcal{D}_\Omega)$.

1.7 Universality

Theorem 4. Let $\Omega \subseteq \mathcal{X}$ be open and $\mathcal{S}$ a multiplicative subset of $\mathcal{H}_{A(\Omega)}$. Suppose $H(\mathcal{D}_\Omega)$ has an $\mathcal{S}$-decomposition. Let $\mathcal{R}(\Omega, \mathcal{S}) = \mathcal{R}(\mathcal{D}_\Omega, \mathcal{S})$ and denote the tautological map by $\lambda : \mathcal{H}_{A(\Omega)} \to \mathcal{R}(\Omega, \mathcal{S})$.

Then

- The natural map of rings $A(\Omega) \to \mathcal{R}(\Omega, \mathcal{S})$ is a finite morphism, so that $\mathcal{R}(\Omega, \mathcal{S})$ is the ring of rigid analytic functions on its associated affinoid space $X(\Omega, \mathcal{S})$.
- Let c be the natural map $X(\Omega, \mathcal{S}) \to \Omega$. Then $(X(\Omega, \mathcal{S}), c, \lambda)$ is an OC eigenvariety over Ω of type $\mathcal{S}$. (Note that c need not be surjective.)
- $(X(\Omega, \mathcal{S}), c, \lambda)$ is universal for all OC eigenvarieties over Ω of type $\mathcal{S}$.
- Suppose moreover that $(X(\Omega, \mathcal{S}), c, \lambda)$ is an automorphic eigenvariety. Then $(X(\Omega, \mathcal{S}), c, \lambda)$ is universal for all automorphic eigenvarieties over Ω of type $\mathcal{S}$.

Proof. The first two items follow from the definitions. For the next one, let $(\mathcal{V}, \kappa, \mu)$ be an OC eigenvariety over Ω of type $\mathcal{S}$. We may assume that $\mathcal{V}$ is affinoid. We must find a ring homomorphism $\alpha : \mathcal{R}(\Omega, \mathcal{S}) \to \mathcal{O}(\mathcal{V})$ such that $\mu = \lambda \circ \alpha$. Since λ is surjective, α is uniquely determined, if it exists.

So we must show that $\mathrm{Ker}(\lambda) \subset \mathrm{Ker}(\mu)$. Suppose $h \in \mathcal{H}$ such that $\lambda(h) = 0$. We will show $\mu(h) = 0$ point by point on $\mathcal{V}$.

Choose $P \in \mathcal{V}$. We want to show $\mu_P(h) = 0$. It suffices to do this for a Zariski dense set of P, so we may assume that μ_P is the Hecke eigenpacket of some OC cohomology class of type $\mathcal{S}$. That is, μ_P is a Hecke eigenpacket occurring in $H(\mathcal{D}_{\kappa(P)})_{\mathcal{S}}$ and therefore μ_P factors through some algebra homomorphism $\beta : \mathcal{R}(\mathcal{D}_{\kappa(P)}, \mathcal{S}) \to \mathbb{C}_p$.

Let $\gamma_P : \mathcal{H}_{A(\Omega)} \to \mathcal{R}(\mathbb{D}_{\kappa(P)}, \mathcal{S})$ be the tautological map. Then $\mu_P(h) = \beta(\gamma_P(h)) \in \mathbb{C}_p$. By the ring-theoretic Theorem 6.1.1, $\mathcal{R}(\mathcal{D}_{\kappa(P)}, \mathcal{S})$ is a quotient of $\mathcal{R}(\Omega, \mathcal{S})$. Therefore, we can pull β back to a homomorphism $\beta^\dagger : \mathcal{R}(\Omega, \mathcal{S}) \to \mathbb{C}_p$. It follows that $\mu_P(h) = \beta^\dagger(\lambda(h)) = 0$.

The proof of the last item is the same, with "OC" replaced by "automorphic".

1.8 Other Constructions of Eigenvarieties

Other approaches to constructing eigenvarieties for general G split at p are due to Emerton [Em] and Urban [U]. The introduction of Emerton's paper gives an excellent overview of the situation from his point of view.

1.9 Acknowledgements

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In the whole paper, we fix a prime number p and all references to Banach spaces, Banach algebras, etc. will be to p-adic Banach spaces, p-adic Banach algebras, etc.

2 Generalities on cohomology and Hecke algebras

In this chapter, we recall the adelic construction of arithmetic locally symmetric spaces and their cohomology. We can also view this cohomology as group cohomology for the corresponding arithmetic groups. The adelic construction is more convenient for handling the global Hecke algebra, whose action on the cohomology we recall.

In section 2.5 we construct the local Hecke algebra at a prime p which we will use in the rest of the paper.

In the last two sections, we assume that the coefficient module we use for the cohomology is an ON-able Banach space. We consider resolutions of $\mathbb{Z}$ such that the cochains inherit a structure of ON-able Banach space and that certain Hecke operators u can be lifted to operators U that act completely continuously on the cochains.

In the last section we construct the Fredholm power series of U on the cochains. It depends on various choices we have made.

2.1 Shimura manifolds.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{Q}$ and choose a decomposable compact open subgroup $K_{\mathbb{A}_f} := \prod_\ell K_\ell$ of $\mathbf{G}(\mathbb{A}_f)$. Let $\mathbf{G}(\mathbb{R})^{\mathrm{o}}$ denote the connected component of $\mathbf{G}(\mathbb{R})$ and fix a maximal compact subgroup K_∞ of $\mathbf{G}(\mathbb{R})^{\mathrm{o}}$. Then $\mathbf{H} := \mathbf{G}(\mathbb{R})^{\mathrm{o}}/K_\infty$ is the symmetric space associated to $\mathbf{G}$. Set

$$K_{\mathbb{A}} := K_\infty \times K_{\mathbb{A}_f}, \qquad \text{and} \qquad \mathcal{K} := \mathbf{G}(\mathbb{R})^{\mathrm{o}} \times K_{\mathbb{A}_f},$$

and define

$$\mathbf{M}_K := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_{\mathbb{A}}. \tag{1}$$

For each $x \in \mathbf{G}(\mathbb{A})$ we let $\mathbf{M}^{\mathrm{o}}_K(x)$ be the connected component of $\mathbf{M}_K$ containing the image of x under the natural map $\mathbf{G}(\mathbb{A})\longrightarrow\mathbf{M}_K$. Since the double coset space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/\mathcal{K}$ is finite, we can choose a finite set $\{x_i\}$ of representatives in $\mathbf{G}(\mathbb{A})$ and then

$$\mathbf{M}_K = \coprod_i \mathbf{M}^{\mathrm{o}}_K(x_i) \tag{2}$$

is the decomposition of $\mathbf{M}_K$ into its finitely many connected components. For any $x \in \mathbf{G}(\mathbb{A})$ the group

$$\Gamma(x) := \mathbf{G}(\mathbb{Q}) \cap \left(x\mathcal{K}x^{-1}\right)$$

is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ and the map $pr_x : \mathbf{H}\longrightarrow\mathbf{M}^{\mathrm{o}}_K(x)$ defined by

$$\begin{array}{cccccc} pr_x : & \mathbf{H} & \xrightarrow{\sim} & \mathcal{K}/K_{\mathbb{A}} & \longrightarrow & \mathbf{M}^{\mathrm{o}}_K(x) \\ & zK_\infty & \longmapsto & zK_{\mathbb{A}} & \longmapsto & \mathbf{G}(\mathbb{Q})xzK_{\mathbb{A}} \end{array}$$

descends to a homeomorphism

$$\Gamma(x)\backslash\mathbf{H} \xrightarrow{\sim} \mathbf{M}^{\mathrm{o}}_K(x).$$

In what follows we will always impose the following assumption:

$$\Gamma(x) \text{ is torsion-free for all } x \in \mathbf{G}(\mathbb{A}). \tag{3}$$

We note that this is a condition on the compact open subgroup $\mathbb{K}_{\mathbb{A}_f}$ and that this condition is satisfied for all sufficiently small choices of $\mathbb{K}_{\mathbb{A}_f}$. Under assumption (3), $\mathbf{M}_K$ is clearly a manifold. We call $\mathbf{M}_K$ the Shimura manifold of level K.

2.2 Local coefficient systems on Shimura manifolds.

Let R be a commutative ring and consider the category of R-modules endowed with a right $\mathcal{K}$-module structure and also a left $\mathbf{G}(\mathbb{Q})$-module structure, both actions commuting with each other. We will suppress mention of R and refer to any object in this category as a $\mathbf{G}(\mathbb{Q}) \times \mathcal{K}$-module.

Let $\mathbb{D}$ be any right $R[\mathcal{K}]$-module. We regard $\mathbb{D}$ as a $\mathbf{G}(\mathbb{Q}) \times \mathcal{K}$-module by letting $\mathbf{G}(\mathbb{Q})$ act trivially on the left. On the space $\mathbf{G}(\mathbb{A}) \times \mathbb{D}$ we let $K_{\mathbb{A}}$ act diagonally on the right, and let $\mathbf{G}(\mathbb{Q})$ act diagonally on the left and form the quotient

$$\widetilde{\mathbb{D}} := \mathbf{G}(\mathbb{Q})\backslash\big(\mathbf{G}(\mathbb{A}) \times \mathbb{D}\big)/K_{\mathbb{A}}.$$

Projection to the first factor gives us a natural projection

$$\xi : \widetilde{\mathbb{D}} \longrightarrow \mathbf{M}_K.$$

It is easy to see that $(\widetilde{\mathbb{D}}, \xi)$ is a local coefficient system on $\mathbf{M}_K$. Indeed, for each $x \in \mathbf{G}(\mathbb{A})$ we let $\mathbb{D}(x)$ be the left $R[\Gamma(x)]$-module whose underlying R-module is $\mathbb{D}$ with $\Gamma(x)$ acting by the formula

$$\gamma \cdot \mu := \mu|(x^{-1}\gamma^{-1}x)$$

for any $\gamma \in \Gamma(x)$ and $\mu \in \mathbb{D}$. Let

$$\widetilde{\mathbb{D}}(x) := \Gamma(x)\backslash(\mathbf{H} \times \mathbb{D}(x)) \longrightarrow \mathbf{M}_K^{\circ}(x)$$

be the associated coefficient system on $\mathbf{M}_K^{\circ}(x)$. Then we have a well-defined map

$$\begin{array}{ccc} \widetilde{\mathbb{D}}(x) & \longrightarrow & \widetilde{\mathbb{D}} \\ \Gamma(x) \cdot (zK_{\mathbb{A}}, \mu) & \longmapsto & \Big(\mathbf{G}(\mathbb{Q})xzK_{\mathbb{A}}, \mu|z^{-1}\Big) \end{array}$$

where $zK_{\mathbb{A}} \in \mathcal{K}/K_{\mathbb{A}} = \mathbf{H}$ and $\mu \in \mathbb{D}(x) = \mathbb{D}$. A straightforward verification shows that this map identifies $\widetilde{\mathbb{D}}(x)$ with the restriction of $\widetilde{\mathbb{D}}$ to $\mathbf{M}_K^{\circ}(x)$.

2.3 The cohomology of $\widetilde{\mathbb{D}}$.

Having fixed representatives $\{x_i\}$ for the double cosets $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/\mathcal{K}$, we could study the cohomology of the local system $\widetilde{\mathbb{D}}$ on $\mathbf{M}_K$ using the canonical isomorphism

$$H^*(\mathbf{M}_K, \widetilde{\mathbb{D}}) \cong \bigoplus_i H^*(\Gamma(x_i), \mathbb{D}(x_i)) \tag{4}$$

and applying the theory of arithmetic groups to the right hand side. In this section we introduce an adelic description of the cohomology.

Let $S_*(\mathbf{H})$ be the complex of singular chains on $\mathbf{H}$ endowed with the natural left action of $\mathbf{G}(\mathbb{Q})$ induced by the action of $\mathbf{G}(\mathbb{Q})$ on $\mathbf{H}$. Since $\mathbf{H}$ is simply connected, we have a canonical exact sequence

$$\longrightarrow S_{n+1}(\mathbf{H}) \longrightarrow S_n(\mathbf{H}) \longrightarrow \cdots \longrightarrow S_0(\mathbf{H}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

in the category of $\mathbf{G}(\mathbb{Q})$-modules. Moreover, $S_*(\mathbf{H})$ is, in fact a *free* $\mathbb{Z}[\Gamma]$-resolution of $\mathbb{Z}$ for any torsion-free arithmetic subgroup $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$. In particular, we may use this resolution to compute the $\Gamma(x)$-cohomology of $\mathbb{D}(x)$.

To facilitate the adelic point of view, we define the complex

$$S_* := S_*(\mathbf{H}) \otimes \mathbb{Z}[\mathbf{G}(\mathbb{A})]$$

which we regard as a complex of $\mathbf{G}(\mathbb{Q}) \times \mathcal{K}$-modules by letting $\mathbf{G}(\mathbb{Q})$ act diagonally on the left and letting $\mathcal{K}$ acting diagonally on the right, where the right action of $\mathcal{K}$ on $S_*(\mathbf{H})$ is taken to be the trivial action. Since $\mathbb{Z}[\mathbf{G}(\mathbb{A})]$ is a free $\mathbb{Z}[\mathbf{G}(\mathbb{Q})]$-module, it follows that the complex

$$\longrightarrow S_{n+1} \longrightarrow S_n \longrightarrow \cdots \longrightarrow S_0 \longrightarrow \mathbb{Z}[\mathbf{G}(\mathbb{A})] \longrightarrow 0$$

is an exact sequence of free $\mathbb{Z}[\mathbf{G}(\mathbb{Q})]$-modules endowed with a right action of $\mathcal{K}$.

Now let $\mathbb{D}$ be a right $\mathcal{K}$-module and regard $\mathbb{D}$ as a $\mathbf{G}(\mathbb{Q}) \times \mathcal{K}$-module as before. Define the complex $C^*(\mathbb{D}) := \mathrm{Hom}_{\mathbf{G}(\mathbb{Q})\times\mathcal{K}}(S_*, \mathbb{D})$.

Proposition 2.3.1 *There is a canonical isomorphism*

$$H^*(\mathbf{M}_K, \widetilde{\mathbb{D}}) \cong H\big(C^*(\mathbb{D})\big).$$

Proof. First, we note that for any left $\mathbf{G}(\mathbb{Q})$-module A with trivial right action by $\mathcal{K}$, the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{G}(\mathbb{Q})\times\mathcal{K}}\big(A \otimes \mathbb{Z}[\mathbf{G}(\mathbb{A})], \mathbb{D}\big) & \longrightarrow & \bigoplus_i \mathrm{Hom}_{\Gamma(x_i)}(A, \mathbb{D}(x_i)) \\ \Phi & \longmapsto & (\varphi_i)_i \end{array}$$

defined by $\varphi_i(a) := \Phi(a \otimes x_i)$ is an isomorphism. In particular we have

$$C^*(\mathbb{D}) = \bigoplus_i \mathrm{Hom}_{\Gamma(x_i)}(S_*(\mathbf{H}), \mathbb{D}(x_i)).$$

Since $S_*(\mathbf{H})$ is a free $\Gamma(x_i)$-resolution of $\mathbb{Z}$ for each i, we have canonical isomorphisms

$$H(C^*(\mathbb{D})) \cong \bigoplus_i H^*(\Gamma(x_i), \mathbb{D}(x_i)) \cong H^*(\mathbf{M}_K, \widetilde{\mathbb{D}})$$

and the proposition is proved.

2.4 Hecke algebras and their action on the cohomology of $\widetilde{\mathbb{D}}$

Let $\Sigma_{\mathbb{A}_f} \subseteq \mathbf{G}(\mathbb{A}_f)$ be a semigroup containing $K_{\mathbb{A}_f}$ and let $\mathbb{D}$ be a right $\Sigma_{\mathbb{A}_f}$-module. In particular, we let $\mathcal{K}$ act on $\mathbb{D}$ via the natural homomorphism $\mathcal{K} \longrightarrow \Sigma_{\mathbb{A}_f}$ given by the composition $\mathcal{K} \to K_{\mathbb{A}_f} \hookrightarrow \Sigma_{\mathbb{A}_f}$ where the first map is the natural projection and the second is the natural inclusion. So we may form the cohomology $H^*(\mathbb{D}) := H^*(\mathbf{M}_K, \widetilde{\mathbb{D}})$ as in the last section. The additional structure on $\mathbb{D}$ given by the action of $\Sigma_{\mathbb{A}_f}$ allows us to define Hecke operators on $H^*(\mathbb{D})$. More precisely, the pair $(K_{\mathbb{A}_f}, \Sigma_{\mathbb{A}_f})$ is a Hecke pair. So we may form the Hecke algebra

$$\mathcal{H}_R := \mathcal{H}_R(K_{\mathbb{A}_f}, \Sigma_{\mathbb{A}_f})$$

in the usual way as a convolution algebra of double cosets over the base ring R.

The algebra $\mathcal{H}_R$ acts naturally on $C^*(\mathbb{D})$ as follows. First, let $\Sigma_{\mathbb{A}_f}$ act on $\mathrm{Hom}_{\mathbf{G}(\mathbb{Q})}\big(S_*, \mathbb{D}\big)$ by the formula $\sigma : \Phi \longmapsto \Phi|\sigma$, where $\Phi|\sigma$ is defined by

$$(\Phi|\sigma)(x) = \Phi(x\sigma^{-1})|\sigma, \quad (x \in S_*)$$

for $\sigma \in \Sigma_{\mathbb{A}_f}$ and $\Phi \in \mathrm{Hom}_{\mathbf{G}(\mathbb{Q})}\big(S_*, \mathbb{D}\big)$. Thus

$$C^*(\mathbb{D}) = \left\{ \Phi \in \mathrm{Hom}_{\mathbf{G}(\mathbb{Q})}\big(S_*, \mathbb{D}\big) \;\middle|\; \Phi|k = \Phi \text{ for all } k \in K_{\mathbb{A}_f} \right\}$$

and $\mathcal{H}_R$ acts in the usual way. In particular, if $h_\sigma := [K_{\mathbb{A}_f}\sigma K_{\mathbb{A}_f}] \in \mathcal{H}_R$ is the element represented by the characteristic function of the double coset $K_{\mathbb{A}_f}\sigma K_{\mathbb{A}_f}$, then for any $\Phi \in C^*(\mathbb{D})$ we have

$$\Phi|h_\sigma = \sum_j \Phi|\sigma_j \tag{5}$$

where $K_{\mathbb{A}_f}\sigma K_{\mathbb{A}_f} = \coprod_j K_{\mathbb{A}_f}\sigma_j$ is the right coset decomposition.

The action of $\mathcal{H}_R$ on $C^*(\mathbb{D})$ commutes with the coboundary maps and therefore induces an action of $\mathcal{H}_R$ on the cohomology $H^*(\mathbb{D})$:

$$\mathcal{H}_R \longrightarrow \mathrm{End}_R\left(H^*(\mathbb{D})\right).$$

2.5 The Iwahori Hecke pair at p.

Recall that $\mathbf{G}$ is a connected, reductive, algebraic group defined over $\mathbb{Q}$. Let $G = \mathbf{G}(\mathbb{Q}_p)$. Inside $\mathbf{G}$ fix a maximal $\mathbb{Q}_p$-split torus $\mathbf{T}$. Corresponding to $\mathbf{T}(\mathbb{Q}_p)$ there is an apartment in the building for G and we fix a chamber in the apartment. Its stabilizer is an Iwahori subgroup I of G [C p. 140].

Let M be the centralizer of $\mathbf{T}(\mathbb{Q}_p)$ in G and P a parabolic subgroup of G such that $(P, \mathbf{T}(\mathbb{Q}_p))$ is a parabolic pair. Then $P = MN$ is a minimal $\mathbb{Q}_p$-parabolic subgroup of G, where $N = R_u(P)$ is the unipotent radical of P. Then M consists of the $\mathbb{Q}_p$-points of a connected and reductive algebraic group defined over $\mathbb{Q}_p$ and $P = MN$ is a Levi decomposition of P, so that $M \cap N = 1$ and M normalizes N. We let M^0 denote the unique maximal compact subgroup of M. ([C] pp. 127, 134-5).

Let $P^{\mathrm{opp}} = MN^{\mathrm{opp}}$ be the opposite parabolic subgroup and its Levi decomposition, where $N^{\mathrm{opp}} = R_u(P^{\mathrm{opp}})$. Note that $P^{\mathrm{opp}} \cap P = M$ ([C] p. 128). We set $N^+ = N \cap I$, $N^- = N^{\mathrm{opp}} \cap I$ and $T = \mathbf{T}(\mathbb{Q}_p) \cap I = \mathbf{T}(\mathbb{Z}_p)$.

Then $M^0 \cap \mathbf{T}(\mathbb{Q}_p) = T$, the maximal compact subgroup of $\mathbf{T}(\mathbb{Q}_p)$ and $M^0\mathbf{T}(\mathbb{Q}_p)$ has finite index in M ([C] p. 135). Define

$$\mathbf{T}^+ = \{t \in \mathbf{T}(\mathbb{Q}_p) \mid t^{-1}N^+t \subset N^+\}.$$

We set $M^+ = M^0T^+$.

We have the Iwahori decomposition : $I = N^-M^0N^+$ (and in fact this gives a unique factorization of each element of I). Moreover, if $m \in M^+$ then $mN^-m^{-1} \subset N^-$ and $m^{-1}N^+m \subset N^+$ ([C] p. 140 and [T] p. 50 last paragraph of 3.1.1 where Ω is taken to be an open facet).

Set $\Sigma = IT^+I$.

Lemma 2.5.1 *(I, Σ) is a Hecke pair.*

Proof. There are two things that need to be proved. First we must show that Σ is closed under multiplication. In fact, if $b, c \in T^+$ we will show that $IbIcI = IbcI$. The left hand side clearly contains the right hand side. So we must show that $bIc \subset IbcI$.

Let $x \in I$ and write $x = n^-mn^+$ with $n^\pm \in N^\pm$ and $m \in M^0$. Then

$$bxc = bn^-mn^+c = n_1^-bcmn_1^+ \in IbcI$$

for some $n_1^\pm \in N^\pm$.

To see that Σ commensurates I, it suffices to check that any $b \in T^+$ commensurates $I = N^-M^0N^+$. Now b centralizes M^0. As for its conjugation action on $N^\pm$, use the fact that $N^\pm$ is generated by root subgroups to see that b commensurates it.

We will need the following definition:

Definition 2.5.2 A *graded Hecke pair* in G is a triple (I, Σ, Λ) consisting of a Hecke pair (I, Σ) in G and a free finitely generated abelian subgoup $\Lambda \subset G$ satisfying the following properties:

(a) there is a (unique) partial ordering $\leq$ on Λ for which $\Lambda^+ := \Lambda \cap \Sigma$ is the monoid of non-negative elements of Λ;

(b) the canonical map $\Lambda^+ \longrightarrow I\backslash\Sigma/I$ is a bijection;

(c) the map $\delta : \Sigma \longrightarrow \Lambda^+$ defined by composing $\Sigma \longrightarrow I\backslash\Sigma/I$ with the inverse of (b) is a multiplicative homomorphism;

(d) for all $\sigma, \tau \in \Sigma$ with $\delta(\sigma) \le \delta(\tau)$, we have

$$\sigma\Sigma \cap \tau\Sigma \text{ is non-empty } \iff \sigma \in \tau\Sigma.$$

The condition (d) can be restated as follows. Let $\Sigma^{-1} := \{\sigma^{-1} \mid \sigma \in \Sigma\}$ and extend δ to a function $\delta : \Sigma\Sigma^{-1} \cup \Sigma^{-1}\Sigma \longrightarrow \Lambda$ by $\delta(\sigma\tau^{-1}) = \delta(\tau^{-1}\sigma) = \delta(\sigma) \cdot \delta(\tau)^{-1}$, for all $\sigma, \tau \in \Sigma$. Then condition (d) is equivalent to the assertion

$$\Sigma = \left\{ \sigma \in \Sigma\Sigma^{-1} \cap \Sigma^{-1}\Sigma \;\middle|\; \delta(\sigma) \ge 1 \right\}. \tag{6}$$

We want to extend (I, Σ) to a graded Hecke pair.

Let $X^*(\mathbf{T})$ denote the free abelian group of $\mathbb{Q}_p$-characters of $\mathbf{T}$ and $X_*(\mathbf{T}) = \mathrm{Hom}_{\mathbb{Z}}(X_*(\mathbf{T}), \mathbb{Z})$. The map ord : $\mathbf{T}(\mathbb{Q}_p) \to X_*(\mathbf{T})$ is defined by the equation

$$< \mathrm{ord}\,(z), \lambda >= \mathrm{ord}\,_p \lambda(z)$$

for all $z \in \mathbf{T}(\mathbb{Q}_p), \lambda \in X^*(\mathbf{T})$. The kernel of ord is T. We let $\Lambda(\mathbf{T})$ denote the image of ord ([C], pp. 134-5).

We choose a splitting of $\Lambda(\mathbf{T})$ back into $\mathbf{T}(\mathbb{Q}_p)$ and call the image Λ. So $\Lambda \subset \mathbf{T}(\mathbb{Q}_p)$ is a free abelian group. We define $\Lambda^+ = \Lambda \cap T^+$.

We will identify Λ to its image in M/M^0. Note that $M^0 = M \cap I$ and Λ has finite index in M/M^0 ([C] p. 140).

Clearly, $T^+ = \Lambda^+ T$, so that $\Sigma = IT^+I = I\Lambda^+I$.

Theorem 2.5.3 *(I, Σ, Λ) is a graded Hecke pair.*

Proof. We will check conditions (a)-(d).

(a) We must check that $\Lambda \cap \Sigma = \Lambda^+$. The left hand side clearly contains the right hand side. So suppose that $a \in \Lambda \cap \Sigma$. Then $a \in \Lambda$ and $a = xby$ for some $x, y \in I$ and $b \in \Lambda^+$.

The Bruhat-Tits decomposition ([C] p. 140 and [T] p. 51) says that G is the disjoint union of double cosets Iw_1I, where w_1 runs over the affine Weyl group $(M/M^0)W$ (where W is the usual Weyl group of G). Therefore $x = y = 1$ and $a = b \in \Lambda^+$.

(b) We have a map $\Sigma \to \Lambda^+$ defined by sending $\sigma = xby$ to b where $x, y \in I$ and $b \in \Lambda^+$. By the Bruhat-Tits decomposition, this map is well-defined and it descends to a surjective map $\delta : I\backslash\Sigma/I \to \Lambda^+$.

To show injectivity, suppose $I\sigma I$ and $I\sigma' I$ have the same image. Then $\sigma = xby$ and $\sigma' = x'by'$ with $x, x', y, y' \in I$ and $b \in \Lambda^+$. Then $\sigma' = x'x^{-1}\sigma y^{-1}y' \in I\sigma I$ and $I\sigma I = I\sigma' I$.

(c) We must show that δ is multiplicative. That is, if $b, c \in \Lambda^+$ we must show that $IbIcI = IbcI$. The left hand side clearly contains the right hand side. So we must show that $bIc \subset IbcI$. We proved this already in the proof of Lemma 2.5.1.

(d) Before we prove this, we need the following lemma.

Lemma 2.5.4 *Let $X = P^{opp}N$. (i) Any $\sigma \in X$ has a unique decomposition $\sigma = utv$ with $u \in N^{opp}$, $t \in M$ and $v \in N$. (ii) $\Sigma \subset X$. (iii) For any $\sigma = utv \in X$, $\sigma \in \Sigma$ if and only if $u \in N^-$, $t \in M^0\Lambda^+ = M^+$ and $v \in N^+$.*

Proof of Lemma 2.5.4. (i) We know that $P^{\mathrm{opp}} \cap N \subset P^{\mathrm{opp}} \cap P = M$, so that $P^{\mathrm{opp}} \cap N = M \cap N = 1$. Given $\sigma \in X$, we can certainly write it in some way as $\sigma = utv$ as claimed. If now (with obvious notation) $utv = u't'v'$ then $v'v^{-1} \in P^{\mathrm{opp}} \cap N$, and hence $v = v'$ and then $t = t'$ and $u = u'$.

(ii) First suppose that $\sigma = xby \in \Sigma$, where as usual $x, y \in I$ and $b \in \Lambda^+$. Then $x = n_1^- m_1 n_1^+$ and $y = n_2^- m_2 n_2^+$ with $n_i^{\pm} \in N^{\pm}$ and $m_i \in M^0$. We will use similar notation for other elements of $N^{\pm}$. Recall that M^0 normalizes both N^+ and N^- because M normalizes both N and N^{opp} and $M^0 \subset I$.

Then $\sigma = n_1^- m_1 n_1^+ b n_2^- m_2 n_2^+ = n_1^- m_1 b n_3^+ n_2^- m_2 n_2^+$. Since $n_3^+ n_2^- \in I$, we can rewrite it as $n_4^- m_4 n_4^+$. So $\sigma = n_1^- m_1 b n_4^- m_4 n_4^+ m_2 n_2^+ = n_5^- m_1 b m_4 m_2 n_5^+ \in N^- M^0 \Lambda^+ N^+ \subset X$.

(iii) One implication is obvious. For the other, suppose $\sigma = utv \in \Sigma$. Then also $\sigma = xby$ with $x \in N^-$, $b \in M^0\Lambda^+$ and $y \in N^+$ as we saw in (ii). By the uniqueness (i) we see that $u = x, t = b, v = y$.

Now we can prove (d) in the form of (6). We must show that if $g \in \Sigma\Sigma^{-1} \cap \Sigma^{-1}\Sigma$ and if $\delta(g) \geq 1$ then $g \in \Sigma$.

By the Lemma, any element of Σ can be written in the form $n^- m^0 b n^+$ with $n^{\pm} \in N^{\pm}$, $m^0 \in M^0$ and $b \in \Lambda^+$. We use the obvious notation for various such decompositions.

Write

$$g = n_1^- m_1^0 b_1 n_1^+ {b_2}^{-1} m_2^0 n_2^- = n_3^+ {b_3}^{-1} m_3^0 n_3^- b_4 m_4^0 n_4^+$$

and note that because $\delta(g) \geq 1$ we have that $b_1 {b_2}^{-1} = {b_3}^{-1} b_4 \in \Lambda^+$.

We need to show $g \in \Sigma$. Since Σ is closed under right or left multiplication by M^0, and M^0 normalizes N^+, N^- and centralies Λ^+, without loss of generality we may assume that $m_i^0 = 1$ for all i (cancel $m_1^0 m_2^0$ on the left and $m_3^0 m_4^0$ on the right.) Since Σ is closed under right multiplication by N^+ we may assume that $n_4^+ = 1$ and since Σ is closed under left multiplication by N^- we may assume that $n_1^- = 1$.

We then have

$$g = b_1 n_1^+ {b_2}^{-1} n_2^- = n_3^+ {b_3}^{-1} n_3^- b_4$$

and therefore

$$(n_3^+)^{-1} b_1 n_1^+ {b_2}^{-1} = {b_3}^{-1} n_3^- b_4 (n_2^-)^{-1}.$$

Using the fact that M normalizes N and N^{opp} we get that

$$b_1 {b_2}^{-1} [b_2 {b_1}^{-1} (n_3^+)^{-1} b_1 {b_2}^{-1}] (b_2 n_1^+ {b_2}^{-1}) = ({b_3}^{-1} n_3^- b_3) [{b_3}^{-1} b_4 (n_2^-)^{-1} {b_4}^{-1} b_3] {b_3}^{-1} b_4$$

are two decompositions of the same element in X in the form $N^{\mathrm{opp}} M N$. By the uniqueness of Lemma 2 (i), we get that

$$[b_2 {b_1}^{-1} (n_3^+)^{-1} b_1 {b_2}^{-1}] (b_2 n_1^+ {b_2}^{-1}) = b_2 {b_1}^{-1} (n_3^+)^{-1} b_1 n_1^+ {b_2}^{-1} = 1.$$

Therefore, $g = n_3^+ b_1 {b_2}^{-1} n_2^- \in I \Lambda^+ I = \Sigma$.

From now on we will fix a compact open subgroup $K_{\mathbb{A}_f}$ as in section 2 and a semigroup $\Sigma_{\mathbb{A}_f} \subseteq \mathbf{G}(\mathbb{A}_f)$ containing $K_{\mathbb{A}_f}$. We will assume that $K_p = I$ and that $\Sigma_{\mathbb{A}_f} = \Sigma_{\mathbb{A}_f}^{(p)} \times \Sigma$. Thus the local component of the corresponding Hecke algebra at p is the algebra of double cosets $I \backslash \Sigma / I$, which is commutative. We assume that the whole global Hecke algebra is also commutative.

2.6 Finite resolutions.

Recall that in chapter 2 we fixed representatives $\{x_i\}$ for the double cosets $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/\mathcal{K}$. We also had $S_*(\mathbf{H})$, the complex of singular chains on $\mathbf{H}$ endowed with the natural left action of $\mathbf{G}(\mathbb{Q})$, and a canonical exact sequence

$$\longrightarrow S_{n+1}(\mathbf{H})\longrightarrow S_n(\mathbf{H})\longrightarrow\cdots\longrightarrow S_0(\mathbf{H})\longrightarrow\mathbb{Z}\longrightarrow 0$$

in the category of $\mathbb{Z}[\mathbf{G}(\mathbb{Q})]$-modules. In particular, $S_*(\mathbf{H})$ is a free $\Gamma(x_i)$-resolution of $\mathbb{Z}$ for each i.

Let $\mathbb{D}$ be a module as in Section 2.2 and recall that $C^*(\mathbb{D}) = \mathrm{Hom}_{\mathbf{G}(\mathbb{Q})\times\mathcal{K}}(S_*, \mathbb{D})$.

From (§2.3(4)) and Proposition 2.3.1 we have the canonical isomorphisms:

$$H\big(C^*(\mathbb{D})\big) \cong H^*(\mathbf{M}_K, \widetilde{\mathbb{D}}) \cong \bigoplus_i H^*(\Gamma(x_i), \mathbb{D}(x_i)).$$

Also we showed that

$$C^*(\mathbb{D}) = \bigoplus_i \mathrm{Hom}_{\Gamma(x_i)}(S_*(\mathbf{H}), \mathbb{D}(x_i)).$$

For each i, we choose a finite resolution $F_*^{[i]}$ of $\mathbb{Z}$ by finitely generated, free, left $\mathbb{Z}[\Gamma(x_i)]$-modules, so that

$$\longrightarrow F_{n+1}^{[i]}\longrightarrow F_n^{[i]}\longrightarrow\cdots\longrightarrow F_0^{[i]}\longrightarrow\mathbb{Z}\longrightarrow 0$$

is exact. Such resolutions exist by a result of Borel and Serre, since we are assuming that $\Gamma(x_i)$ is torsion-free.

We also choose homotopy equivalences between each $F_*^{[i]}$ and $S_*(\mathbf{H})$. That is, we choose chain maps $f^{[i]} : F_*^{[i]} \to S_*(\mathbf{H})$ and $g^{[i]} : S_*(\mathbf{H}) \to F_*^{[i]}$ such that $f^{[i]} \circ g^{[i]}$ and $g^{[i]} \circ f^{[i]}$ are homotopy equivalent to the identity, for all i. They are unique up to a unique homotopy.

Define the cochains $\widetilde{C}^*(\mathbb{D}) := \bigoplus_i \mathrm{Hom}_{\Gamma(x_i)}(F_*^{[i]}, \mathbb{D}(x_i))$. They have the same cohomology as $C^*(\mathbb{D})$. In fact we have inverse homotopy equivalences between them, $f := \oplus(f^{[i]})^* : C^*(\mathbb{D}) \to \widetilde{C}^*(\mathbb{D})$ and $g := \oplus(g^{[i]})^* : \widetilde{C}^*(\mathbb{D}) \to C^*(\mathbb{D})$.

We use these homotopy equivalences to transfer the action of the Hecke algebra defined on the cochains $C^*(\mathbb{D})$ in Section 2.4 to the cochains $\widetilde{C}^*(\mathbb{D})$. Note that we do not necessarily get an action of the whole Hecke algebra simultaneously on $\widetilde{C}^*(\mathbb{D})$, but we do get for any single Hecke operator a formula that induces that Hecke operator on the cohomology.

Fix $\sigma \in \Sigma_{\mathbb{A}_f}$. Recall that for $\Phi \in C^*(\mathbb{D})$, $\Phi|\sigma$ was defined by

$$(\Phi|\sigma)(x) = \Phi(x\sigma^{-1})|\sigma.$$

For any $\Psi = \sum \Psi^{[i]} \in \widetilde{C}^*(\mathbb{D})$, define $\Psi\widetilde{|}\sigma$ by

$$\Psi\widetilde{|}\sigma = f(g(\Psi)|\sigma) = \sum_i \{(\sum_j \Psi^{[j]} \circ g^{[j]})|\sigma) \circ f^{[i]}\}, \tag{7}$$

or briefly $\widetilde{|}\sigma = f \circ |\sigma \circ g$.

Now if $h_\sigma \in \mathcal{H}_R$ is the element represented by the characteristic function of the double coset $K_{\mathbb{A}_f}\sigma K_{\mathbb{A}_f}$, then for any $\Psi \in \widetilde{C}^*(\mathbb{D})$ we define

$$\Psi|H_\sigma = \sum_j \Psi\widetilde{|}\sigma_j \tag{8}$$

where $K_{\mathbb{A}_f}\sigma K_{\mathbb{A}_f} = \coprod_j K_{\mathbb{A}_f}\sigma_j$ is the right coset decomposition.

Then H_σ induces h_σ on the cohomology.

More generally, suppose $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{F}$ are merely $K_{\mathbb{A}_f}$ modules, $a : \mathbb{D} \to \mathbb{E}$ is any linear map, and $b_j : \mathbb{E} \to \mathbb{F}$ are also linear maps, indexed by the same set that indexes the right cosets above. Write the map $a(x)$ as $x|\sigma$ and the maps $b_j(y)$ as $y|\kappa_j$. Then we can use the same formulas (7) and (8) to define a map $H_\sigma : C^*(\mathbb{D}) \to C^*(\mathbb{F})$. Thus we interpret $x|\sigma_j$ as $b_j(a(x))$ with $\sigma_j = \sigma\kappa_j$. Of course H_σ depends on a and the b_j, which we suppress from the notation.

2.7 ON-able cochains and completely continuous maps.

For definitions and the basic properties of orthonormalizable (ON-able) Banach modules and characteristic power series, see [B] or [Co].

Definition 2.7.1

(a) Let R be a flat $\mathbb{Z}_p$-algebra, which we assume is separated and complete in the p-adic topology. Let M, N be a R-modules and $\lambda : M \longrightarrow N$ be an R-module map. Then λ is said to be *completely continuous* over R if for every $n \geq 0$, the image of the composition

$$M \xrightarrow{\lambda} N \longrightarrow N/p^nN$$

is finitely generated as an R/p^nR-module.

(b) Let K be a finite extension of $\mathbb{Q}_p$ and now let R be a K-Banach algebra. We assume the values of R are the same as the values of K, i.e. $\|R\| = \|K\|$. Let M, N be R-Banach modules and let R^0, M^0, N^0 be the closed unit balls in R, M, N respectively. Note that M^0 and N^0 are R^0-modules. Let $\lambda : M \longrightarrow N$ be an R-Banach module map. Then λ is said to be *completely continuous* over R if $\lambda|M^0 : M^0 \to N^0$ is completely continuous over R^0.

Definition 2.7.2 An element $t \in \Lambda^+$ is said to be *strictly positive* if $\alpha(t) < 0$ for every positive root α of $\mathbf{G}(\mathbb{Q}_p)$. (See Section 3.1 below for the definition of the "positive" roots.)

Definition 2.7.3 Let Σ be a subsemigroup of $\Sigma_{\mathbb{A}_f}$ and M an R-Banach module endowed with a continuous action of the semigroup Σ. We say the action of Σ on M is completely continuous if

(a) for all $\sigma \in \Sigma$, $\|\sigma\|_M \leq 1$, i.e. $\sigma(M^0) \subseteq M^0$; and

(b) for all strictly positive $t \in \Lambda^+ \cap \Sigma$, the operator $t : M \longrightarrow M$ is a completely continuous R-morphism.

Let $\mathbb{Q}^+$ denote the set of nonnegative rational numbers. Recall that a polynomial $Q(T)$ is called "Fredholm" if $Q(0) = 1$.

Definition 2.7.4 Let M be a K-Banach space with a continuous action of Σ. Let $h \in \mathbb{Q}^+$ and t be a strictly positive element of $\Lambda^+ \cap \Sigma$. Then we say M has slope $> h$ with respect to t if

$$\|t\|_M < p^{-h}.$$

If $Q(T)$ is a polynomial of degree d, let $Q^*(T) := T^dQ(1/T)$.

Proposition 2.7.5 *Let $\mathbb{D}$ be a K-Banach space with a completely continuous action of Σ which satisfies the properties of $\mathbb{D}$ in Section 2.6. Suppose that $\mathbb{D}$ has slope $> h$ for some $h \in \mathbb{Q}^+$ with respect to t. Let U acting on the cochains $C^*(\mathbb{D})$ be a lift (as in §2.4(5)) of the Hecke operator h_t acting on the cohomology $H^*(C^*(\mathbb{D}))$. Let $Q \in K[T]$ be a Fredholm polynomial of slope $\leq h$. Then the following act invertibly: $Q^*(t)$ on $\mathbb{D}$, $Q^*(U)$ on $C^*(\mathbb{D})$ and $Q^*(h_t)$ on $H^*(C^*(\mathbb{D}))$.*

Proof. Write $Q(T) = a_0 + a_1 T + \cdots + a_d T^d$ where $a_0 = 1$ and $a_d \neq 0$. Since it has slope $\leq h$, we have that $\operatorname{ord} a_d - \operatorname{ord} a_i \leq (d-i)h$ for every $i = 0, \ldots, d-1$. Therefore for each i,

$$|\frac{a_i}{a_d}| < p^{(d-i)h}.$$

Since $\mathbb{D}$ has slope $> h$, we see that U is a continuous operator on $C^*(\mathbb{D})$ with norm satisfying

$$\|U\| < p^{-h}.$$

Therefore for each i,

$$\|\frac{a_i}{a_d} U^{d-i}\| < 1.$$

Write $\frac{1}{a_d} Q^*(T) = 1 - P(T)$, so that $P(T) = -\frac{1}{a_d} T^d - \frac{a_1}{a_d} T^{d-1} - \cdots - \frac{a_{d-1}}{a_d} T$. It follows that

$$\|P(U)\| < 1.$$

From this we see at once that the action of $Q^*(U)$ on $C^*(\mathbb{D})$ is invertible with inverse given explicitly by the convergent series of operators

$$Q(U)^{-1} = \frac{1}{a_d}(1 + P(U) + P(U)^2 + \cdots + P(U)^n + \cdots).$$

This result descends to h_t acting on cohomology. The assertion for t on $\mathbb{D}$ itself is proved similarly.

Now let R in addition be a noetherian K-Banach algebra. Let M be an ON-able R-Banach module and λ a completely continuous R-endomorphism of M. Then λ has a characteristic power series $P_\lambda(T)$ which is morally speaking the determinant of $1 - \lambda T$. It is an entire power series with coefficients in R.

Now let $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{F}$ be ON-able R-algebras, and $a : \mathbb{D} \to \mathbb{E}$ and $b_j : \mathbb{E} \to \mathbb{F}$ R-linear maps as at the end of Section 2.6. Since $\widetilde{C}^*(X)$ is isomorphic to a finite direct sum of copies of X (whatever X may be), it is an ON-able R-algebras if X is.

Proposition 2.7.6 *If $a : \mathbb{D} \to \mathbb{E}$ is completely continuous and each $b_j : \mathbb{E} \to \mathbb{F}$ has norm ≤ 1 then $H_\sigma : \widetilde{C}^*(\mathbb{D}) \to \widetilde{C}^*(\mathbb{F})$ is a completely continuous map.*

Proof. From formulas §2.6(7-8), we get that for any $y \in F_*^{[i]}$,

$$(\Psi | H_\sigma)(y) = \{(\sum_j \Psi^{[j]} \circ g^{[j]})((f^{[i]}(y))\sigma_j{}^{-1})\}|\sigma_j.$$

Now for each j, $\sigma_j = \sigma \kappa_j$ for some $\kappa_j \in K_{\mathbb{A}_f}$. The expression in the curly braces is in $\mathbb{D}$ and for any r, $\mathbb{D}|\sigma$ modulo p^r is finitely generated over $R/p^r R$. Since there are only a finite number of $F_*^{[i]}$, each finitely generated over $\mathbb{Z}$, the result follows.

If $\mathbb{D} = \mathbb{F}$, we thus can obtain for each degree $*$ the characteristic power series $P_{H_\sigma}(T)$ of H_σ on $\widetilde{C}^*(\mathbb{D})$.

3 Analytic objects

In this chapter we will define the basic analytic structures we consider. Then we construct the main analytic objects we use in our study of the arithmetic cohomology. These include the big cell, s-structures on it, and highest weight modules of distributions. We construct a universal highest weight module. At the end of the chapter we discuss locally algebraic highest weight modules and prove a comparison result between them and their corresponding Verma-type modules.

In this chapter and subsequent ones, we assume that $\mathbf{G}$ is split at p. We will use the notation of section 2.5, except that we write B for the minimal parabolic subgroup of $G(\mathbb{Q}_p)$ rather than P. Also, we find it convenient to write N in place of N^+ and N^{opp} in place of N^-.

Thus $M = \mathbf{T}(\mathbb{Q}_p)$ and $\mathbf{T}(\mathbb{Q}_p)$ is a maximal split torus of $G(\mathbb{Q}_p)$. Also $M^0 = T = \mathbf{T}(\mathbb{Z}_p)$ is the maximal compact subgroup of $\mathbf{T}(\mathbb{Q}_p)$. Then $I = N^{\text{opp}}TN$ is the Iwahori subgroup. We have a subgroup Λ and a subsemigroup Λ^+ of $\mathbf{T}(\mathbb{Q}_p)$ such that (I, Σ, Λ) is a graded Hecke pair.

3.1 Groups and semigroups.

In section 2.5 we defined the semigroup Λ^+. It is easy to see that for any $t \in \Lambda^+$ we have $tN^{\text{opp}}t^{-1} \subseteq N^{\text{opp}}$ and $t^{-1}Bt \subseteq B$.

Let Δ denote the basis for the positive roots with respect to the pair $(\mathbf{T}(\mathbb{Q}_p), B)$. Our conventions are such that if $\delta \in \Delta$ then $\text{ord}_p(\delta(t)) \leq 0$ for any $t \in \Lambda^+$. Then $t \in \mathbf{T}(\mathbb{Q}_p)$ is in $T\Lambda^+$ if and only if $\alpha(t) \leq 0$ for every positive root α. Recall that an element $t \in \Lambda^+$ is said to be *strictly positive* if $\alpha(t) < 0$ for every positive root α.

We order Λ^+ by divisibility: $s_1 \leq s_2$ if and only if there exist $t \in \Lambda^+$ such that $s_1 t = s_2$. This is the same as the ordering guaranteed by property (a) of a graded Hecke pair, Definition 2.5.2. We say $s_j \to \infty$ if $\alpha(s_j) \to -\infty$ for every positive root α.

We have that

$$\Sigma = I\Lambda^+ I = N^{\text{opp}}\Lambda^+ B$$

is a subsemigroup of $\mathbf{G}(\mathbb{Q}_p)$. Every element $\sigma \in \Sigma$ can be expressed uniquely in the form $\sigma = vt\beta$ with $v \in N^{\text{opp}}$, $t \in \Lambda^+$, $\beta \in B$, and the bijection $\Sigma \to N^{\text{opp}} \times \Lambda^+ \times B$ is a homeomorphism. Moreover, the map

$$\delta : \Sigma \longrightarrow \Lambda^+ \quad \text{defined by} \quad \sigma = vt\beta \longmapsto \delta(\sigma) = t$$

is a homomorphism of semigroups.

For any element $s \in \Lambda^+$ and any semigroup $\mathcal{S} \subseteq \mathbf{G}(\mathbb{Q}_p)$ we let

$$\mathcal{S}^s := \mathcal{S} \cap s^{-1}\mathcal{S}s.$$

With this notation, we have

$$\begin{array}{rcl} N^s & = & s^{-1}Ns, \\ B^s & = & s^{-1}Bs, \\ I^s & = & N^{\text{opp}}TN^s = N^{\text{opp}}B^s, \\ \text{and } \Sigma^s & = & I^s\Lambda^+ I^s = N^{\text{opp}}\Lambda^+ B^s. \end{array} \tag{9}$$

We shall refer to the set of these and the other objects in this section that depend on s as an "s-structure" on $\mathbf{G}$.

Proposition 3.1.1 *For any open subgroup T' of T, $N^s N^{opp} \subset N^{opp}T'N^s$ for $s >> 1$.*

Proof. First we show that given $n \in N^s, w \in N^{\mathrm{opp}}$ there exist $w_1 \in N^{\mathrm{opp}}, t \in T, n_1 \in N^s$ such that $nw = w_1tn_1$. Since $N^sN^{\mathrm{opp}} \subset I$, nw can certainly be written in that form, except that all we know a priori is that $n_1 \in N$. Now $n = s^{-1}ms$ for some $m \in N$, so setting $w_1 = sws^{-1} \in N^{\mathrm{opp}}$,

$$nw = s^{-1}msw = s^{-1}mw_1s = s^{-1}w_2tm_1s = s^{-1}w_2sts^{-1}m_1s$$

where $mw_1 = w_2tm_1$ for some $w_2 \in N^{\mathrm{opp}}, t \in T, m_1 \in N$. Since nw and $ts^{-1}m_1s$ are in I, so is $s^{-1}w_2s$ in $I \cap \mathbf{B}^{\mathrm{opp}}(\mathbb{Q}_\mathbf{p}) = \mathbf{N}^{\mathrm{opp}}$ and so $n_1 = s^{-1}m_1s$ is in N^s.

As $s \to \infty$, the N^s shrink down to the identity. The map sending $nw = w_1tn_1$ to t is continuous. The result now follows from the compactness of N^{opp}.

Definition 3.1.2 Fix once and for all a sequence T_s of open subgroups of T which tend to the identity as $s \to \infty$ and such that $T_s \subset T_{s'}$ if $s' \leq s$. For an arbitrary $s \in \Lambda^+$ we define $T(s)$ to be the (open) subgroup of T generated by T_s and all $t \in T$ such that that $nw = w_1tn_1$ for $n \in N^s, w \in N^{\mathrm{opp}}, w_1 \in N^{\mathrm{opp}}, n_1 \in N^s$.

Note that $N^sN^{\mathrm{opp}} \subset N^{\mathrm{opp}}T(s)N^s$, and if $s' \leq s$, then $T(s) \subset T(s')$ and $N^s \subset N^{s'}$.

Remark. If $\mathbf{G}$ is semisimple, $T(s)$ can simply be defined as the smallest subgroup of T such that $N^sN^{\mathrm{opp}} \subset N^{\mathrm{opp}}T(s)N^s$. For in this case, looking at individual root groups and their opposites, one can show that the group generated by all $t \in T$ such that that $nw = w_1tn_1$ is open and the T_s can be dispensed with.

We set (for $r \leq s$)

$$\begin{array}{rcl} I(r,s) & := & N^{\mathrm{opp}} \cdot T(r) \cdot N^s, \\ \Sigma(r,s) & := & I(r,s)\Lambda^+I(r,s) = N^{\mathrm{opp}} \cdot \Lambda^+T(r) \cdot N^s, \\ I(s) & := & I(s,s), \\ \Sigma(s) & := & \Sigma(s,s). \end{array}$$

As $s \to \infty$, the $T(s)$ form a fundamental neighborhood system of open subgroups of 1 of T. A straightforward calculation shows that also $I(r,s)$ is an open subgroup of I and that $\Sigma(r,s)$ is a subsemigroup of Σ.

Whenever $s, s' \in \Lambda^+$ with $s' \leq s$ we have inclusions

$$I^{s'} \subseteq I^s, \quad I(s') \subseteq I(s), \quad \Sigma^{s'} \subseteq \Sigma^s, \quad \Sigma(s') \subseteq \Sigma(s).$$

Moreover, the families $\{I^s\}_{s \in T^+}$, $\{I(s)\}_{s \in T^+}$, $\{\Sigma^s\}_{s \in T^+}$, and $\{\Sigma(s)\}_{s \in T^+}$ form fundamental neighborhood systems about B^{opp}, N^{opp}, $B^{\mathrm{opp}}\Lambda^+$ and $N^{\mathrm{opp}}\Lambda^+$ respectively.

For future reference, we record the following simple proposition, whose proof we leave to the reader.

Proposition 3.1.3 *The canonical group homomorphism $T \longrightarrow T/T(s)$ extends uniquely to a multiplicative map*

$$\Sigma^s \longrightarrow T/T(s)$$

that is trivial on $\Sigma(s)$.

3.2 Locally analytic representations of p-adic Lie groups.

Recall (see 3.1 in [Se]) that a p-adic manifold is a topological space W endowed with a full p-adic analytic atlas, i.e. a maximal compatible family of *charts* (f, U_f), where $f : U_f \longrightarrow \mathbb{Q}_p^d$ is an open function inducing a homeomorphism $U_f \xrightarrow{\sim} f(U_f)$. Morphisms between manifolds are "locally analytic" functions (these are called "analytic functions" in [Se]).

More generally, if V is a locally convex $\mathbb{Q}_p$-vector space and W is a p-adic manifold, then a function $\varphi : W \longrightarrow V$ is said to be *locally analytic* if for every $\mathbb{Q}_p$-Banach space $\overline{V}$ and every continuous linear map $\eta : V \longrightarrow \overline{V}$ the composition

$$\eta \circ \varphi : W \longrightarrow V \longrightarrow \overline{V}$$

is locally analytic, i.e. is defined locally on W by convergent $\overline{V}$-valued power series.

Let H be a p-adic Lie group, i.e. a p-adic manifold endowed with a locally analytic group structure. Let V be a complete locally convex $\mathbb{Q}_p$-vector space and endow $\mathrm{End}_{\mathbb{Q}_p}(V)$ with the weak topology. A continuous representation of H on V is a continuous homomorphism

$$\psi : H \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V).$$

Definition 3.2.1 We say ψ is locally analytic if, for every $v \in V$, the function $\rho_v : H \longrightarrow V$ defined by $\rho_v(\gamma) = v|\gamma$ is a locally analytic function on H.

Every open subset of the $\mathbb{Q}_p$-points of a linear algebraic group is endowed with a canonical p-adic analytic structure. Thus T, N, N^{opp}, and I are all p-adic Lie groups. Moreover, the map

$$N^{\mathrm{opp}} \times T \times N \longrightarrow I, \qquad (v, t, u) \longmapsto vtu \tag{10}$$

is an isomorphism of p-adic manifolds (but not of groups).

Theorem 3.2.2 *Let H be one of the Lie groups T, N, N^{opp}, or I. Then every continuous representation of H on a complete locally convex vector space V is locally analytic.*

The proof is based on the following well-known lemma.

Lemma 3.2.3 *Let R be a complete locally convex $\mathbb{Q}_p$- algebra and let $\psi : \mathbb{Z}_p^d \longrightarrow R^\times$ be a continuous group homomorphism (i.e. $\psi(x + y) = \psi(x) \cdot \psi(y)$ for every $x, y \in \mathbb{Z}_p^d$). Then ψ is locally analytic.*

Proof: It suffices to prove this when R is a $\mathbb{Q}_p$-Banach algebra. In that case, we may choose $m \in \mathbf{N}$ sufficiently large so that $\psi(p^m\mathbb{Z}_p^d) \subseteq 1 + pR^0$ where R^0 is the closed unit ball in R. Then for each $a \in \mathbb{Z}_p^d$ the function $f_{a,m} : a + p^m\mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ defined by $x \longmapsto (x - a)/p^m$ is a local chart at a. Moreover, letting $\lambda := \log(\psi(p^m)) \in pR^0$, the series $\exp(\lambda x)$ converges for every $x \in \mathbb{Z}_p$ and $\psi \circ f_{a,m}^{-1}$ is given on the neighborhood $a + p^m\mathbb{Z}_p^d$ by the convergent power series

$$(\psi \circ f_{a,m}^{-1})(a + p^m x) = \psi(a) \cdot \prod_{i=1}^{d} \exp(\lambda x_i).$$

This completes the proof of the lemma.

Proof of the Theorem: Let $R := \mathrm{End}_{\mathbb{Q}_p}(V)$ endowed with the weak topology. We will show that every continuous group homomorphism

$$\psi : H \longrightarrow R^{\times} \tag{11}$$

is locally analytic. The theorem is an immediate consequence of this assertion.

When $H = T$, (11) follows from the lemma together with the fact that T has an open subgroup isomorphic to $\mathbb{Z}_p^n$.

Now suppose $\psi : N \longrightarrow R^{\times}$ is a continuous group homomorphism. We can write N as a product of root subgroups:

$$\begin{array}{rcc} \eta : N_n \times N_{n-1} \times \cdots \times N_1 & \longrightarrow & N \\ (u_n, u_{n-1}, \ldots, u_1) & \longmapsto & u_n \cdot u_{n-1} \cdots u_1 \end{array} \tag{12}$$

and η is an isomorphism of p-adic manifolds (but not of groups). Each $N_j \cong \mathbb{Z}_p^{d_j}$ as p-adic Lie groups, for some $d_j > 0$.

For each j we let $\psi_j : N \longrightarrow R^{\times}$ be the composition of ψ with projection to the factor N_j in the decomposition (12). Since $\psi_j|_{N_j} : N_j \longrightarrow R^{\times}$ is a continuous group homomorphism and since $N_j \cong \mathbb{Z}_p^{d_j}$, we know from the lemma that $\psi_j|_{N_j}$ is locally analytic and therefore also ψ_j is locally analytic. But again from the decomposition (12) we see that for every $u \in N$ we have $\psi(u) = \psi_n(u) \cdot \psi_{n-1}(u) \cdot \ldots \cdot \psi_1(u)$. Thus ψ is the product of locally analytic functions on N and is therefore locally analytic. This proves the theorem when $H = N$. The case $H = N^{\mathrm{opp}}$ is proved similarly.

The case $H = I$ is proved in exactly the same way but using the decomposition (10) and the fact just proved that any continuous group homomorphism from either N^{opp}, T, or N to $R^{\times}$ is locally analytic. This completes the proof of the theorem.

3.3 Strict p-adic manifolds.

For a $\mathbb{Q}_p$-Banach space V, we let $V\langle X_1, \ldots, X_d\rangle := V \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p\langle X_1, \ldots, X_d\rangle$ where $\mathbb{Q}_p\langle X_1, \ldots, X_d\rangle$ is the $\mathbb{Q}_p$-Tate algebra of dimension d. Thus we have

$$V\langle X_1, \ldots, X_d\rangle := \left\{ \sum_{i_1, \ldots, i_d = 0}^{\infty} v_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d} \;\middle|\; v_I \in V, \text{ and } v_I \to 0 \text{ as } I \to \infty \right\}.$$

The norm of an element of $V\langle X_1, \ldots, X_d\rangle$ is the sup of the norms of its coefficients.

Every power series $F \in V\langle X_1, \ldots, X_d\rangle$ converges on $\mathbb{Z}_p^d$ to a function $\mathbb{Z}_p^d \longrightarrow V$. We say that a function $f : \mathbb{Z}_p^d \longrightarrow V$ is *strictly analytic* if there is a power series $F_f \in V\langle X_1, \ldots, X_d\rangle$ representing f. In particular, a function $f : \mathbb{Z}_p^d \longrightarrow \mathbb{Q}_p^d$ is strictly analytic if and only if each coordinate function of f is represented by an element of the Tate algebra $\mathbb{Q}_p\langle X_1, \ldots, X_d\rangle$. Such a function extends uniquely to a rigid analytic function

$$F_f : \mathcal{O}_{\mathbb{C}_p}^d \longrightarrow \mathbb{C}_p^d$$

where $\mathcal{O}_{\mathbb{C}_p}$ is the ring of integers in $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$.

Definition 3.3.1 Let $f : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ be a function.

(a) We say f is a *strict isomorphism* (of manifolds) if f is a homeomorphism and both f and f^{-1} are strictly analytic as functions $\mathbb{Z}_p^d \longrightarrow \mathbb{Q}_p^d$.

(b) We say f is a *strict immersion* if $F_f : \mathcal{O}_{\mathbb{C}_p}^d \longrightarrow \mathbb{C}_p^d$ is injective.

It is not hard to see that if $f : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ is a strict immersion then the coefficients of F_f are in $\mathbb{Z}_p$ and the derivative $D_f : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ of f at the origin is injective. Moreover, if $L_f : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ is the first Taylor polynomial of f, i.e. $L_f(x) := f(0) + D_f(x)$, then there is a strict isomorphism $g : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ such that $f = L_f \circ g$. It follows from this that the image of a strict immersion f is the same as the image of its first Taylor polynomial L_f. In particular, the image is a compact open subset of $\mathbb{Z}_p^d$. Indeed, the image is an open affine sublattice of $\mathbb{Z}_p^d$.

Definition 3.3.2 A strict immersion $f : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d$ is said to be a *contraction* if $D_f \equiv 0 \pmod{p}$.

Now let W be a p-adic manifold. We say that a chart f on W is *strict* if $f(U_f) = \mathbb{Z}_p^d$. We define the notion of *strict equivalence* of strict charts by

$$f_1 \sim f_2 \iff \begin{cases} U_{f_1} = U_{f_2} \quad \text{and} \\ \\ f_2 \circ f_1^{-1} : \mathbb{Z}_p^d \longrightarrow \mathbb{Z}_p^d \quad \text{is a strict isomorphism.} \end{cases}$$

An atlas $\mathcal{F}$ on W is said to be *strict* if (1) every chart in $\mathcal{F}$ is strict, and (2) the family $\{U_f\}_{f \in \mathcal{F}}$ is a covering of W by *disjoint open subsets*. We define *strict equivalence* of strict atlases by

$$\mathcal{F}_1 \sim \mathcal{F}_2 \iff \forall f_1 \in \mathcal{F}_1,\ \exists f_2 \in \mathcal{F}_2 \text{ such that } f_1 \sim f_2.$$

Definition 3.3.3 A strict p-adic manifold is a manifold W together with a strict equivalence class of strict atlases on W.

Given a p-adic manifold W and a strict atlas $\mathcal{F}$ on W, we will often write $W[\mathcal{F}]$ to denote W endowed with the strict analytic structure represented by $\mathcal{F}$. The collection of disjoint open sets $\{\, U_f \mid f \in \mathcal{F} \,\}$ depends only on the strict analytic structure. We will call these open sets the "cells" of $W[\mathcal{F}]$.

Definition 3.3.4 Let $W_1[\mathcal{F}_1]$ and $W_2[\mathcal{F}_2]$ be strict p-adic manifolds of dimensions d_1 and d_2 respectively. A function $\varphi : W_1[\mathcal{F}_1] \longrightarrow W_2[\mathcal{F}_2]$ is said to be *strictly analytic* (respectively, a *strict immersion*; respectively, a *contraction*) if for every $f_1 \in \mathcal{F}_1$ there is an $f_2 \in \mathcal{F}_2$ such that

(a) $\varphi(U_{f_1}) \subseteq U_{f_2}$, and

(b) $f_2 \circ \varphi \circ f_1^{-1} : \mathbb{Z}_p^{d_1} \longrightarrow \mathbb{Z}_p^{d_2}$ is strictly analytic (respectively, a strict immersion; respectively, a contraction)

If $\mathcal{F}'$ and $\mathcal{F}$ are two strict atlases on the same p-adic manifold W, then we say $\mathcal{F}'$ is a *refinement* of $\mathcal{F}$ and we write

$$\mathcal{F}' \leq \mathcal{F}$$

if the identity map on W induces a strict immersion $W[\mathcal{F}'] \longrightarrow W[\mathcal{F}]$. If the identity map is a contraction then we say $\mathcal{F}'$ is a contracting refinement of $\mathcal{F}$ and write

$$\mathcal{F}' << \mathcal{F}.$$

Definition 3.3.5 Let $W = W[\mathcal{F}]$ be a strict p-adic manifold and let $\varphi : W \longrightarrow V$ be a function from W to a $\mathbb{Q}_p$-Banach space V. Then φ is said to be *strictly analytic* if for every $f \in \mathcal{F}$, the composition $\varphi \circ f^{-1} : \mathbb{Z}_p^d \longrightarrow V$ is strictly analytic.

If $W = W[\mathcal{F}]$ is a strict p-adic manifold and V is a $\mathbb{Q}_p$-Banach space, then we let

$$A(W, V) := \big\{ \varphi : W \longrightarrow V \;\big|\; \varphi \text{ is strictly analytic} \big\} \tag{13}$$

and we endow this space with a norm $\|\cdot\|_W$ defined as follows. For each $f \in \mathcal{F}$ and $\varphi \in A(W, V)$ the function $\varphi \circ f^{-1}$ is represented by a power series $\Phi_f \in V\langle X_1, \cdots, X_d\rangle$. We define $\|\varphi\|_f := \|\Phi_f\|$ and

$$\|\varphi\|_W := \sup_{f \in \mathcal{F}} \|\varphi\|_f.$$

One easily checks that $\|\cdot\|_W$ does not depend on the choice of $\mathcal{F}$ representing the strict analytic structure and that $A(W, V)$ is complete with respect to $\|\cdot\|_W$. We also have the following simple proposition.

Proposition 3.3.6 *Let $\varphi : W_1 \longrightarrow W_2$ be a strictly analytic map of strict p-adic manifolds. Let V be a $\mathbb{Q}_p$-Banach space. Then pullback induces a continuous linear function*

$$\varphi^* : A(W_2, V) \longrightarrow A(W_1, V).$$

If, moreover, φ is a contraction, then φ^* is a completely continuous linear map of $\mathbb{Q}_p$-Banach spaces.

3.4 Locally analytic distributions on p-adic manifolds.

Let $W = W[\mathcal{F}_W]$ be a *compact* strict p-adic manifold of dimension d and let R be a $\mathbb{Q}_p$-Banach algebra. (If $R = \mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$ easily supplied from the context, we will suppress the R's from the notation.)

As in §3.3(13) we define the Banach algebra

$$A(W, R) := \{\varphi : W \longrightarrow R \,|\, \varphi \text{ is strictly analytic}\}.$$

We also define

$$\mathcal{A}(W, R) := \big\{ \varphi : W \longrightarrow R \;\big|\; \varphi \text{ is locally analytic} \big\}$$

and note that we have a canonical isomorphism

$$\mathcal{A}(W, R) = \varinjlim_{\mathcal{F}} A(W[\mathcal{F}], R)$$

where $\mathcal{F}$ runs over the directed set of all strict atlases on W refining $\mathcal{F}_W$. We endow $\mathcal{A}(W, R)$ with the locally convex final topology with respect to the inclusions $A(W[\mathcal{F}], R) \hookrightarrow \mathcal{A}(W, R)$. This is the finest locally convex topology for which all these inclusions are continuous (see p. 22 [Sch]).

Dually, a continuous R-linear functional $\mu : A(W, R) \longrightarrow R$ will be called an R-valued strictly analytic distribution on W. We denote the space of all such distributions by

$$\mathbb{D}(W, R) := \big\{ \mu : A(W, R) \longrightarrow R \;\big|\; \mu \text{ is } R\text{-linear and strictly analytic} \big\}$$

and endow $\mathbb{D}(W,R)$ with the norm $\|\cdot\|_W$ dual to the norm on $A(W,R)$. Clearly, $\mathbb{D}(W,R)$ is complete with respect to $\|\cdot\|_W$. For any strictly analytic map $\varphi : W_1 \longrightarrow W_2$ of strict manifolds, we have a canonical R-linear continuous map

$$\varphi_* : \mathbb{D}(W_1[\mathcal{F}_1],R)\longrightarrow\mathbb{D}(W_2[\mathcal{F}_2],R).$$

The map φ_* is completely continuous if and only if φ is a contraction.

Finally, we define the space of R-valued *locally analytic distributions* on W to be the space

$$\mathcal{D}(W,R) := \left\{ \mu : \mathcal{A}(W,R)\longrightarrow R \;\middle|\; \mu \text{ is } R\text{-linear and continuous}\right\}$$

and we endow $\mathcal{D}(W,R)$ with the coarsest topology for which the maps $\mathcal{D}(W,R)\longrightarrow\mathbb{D}(W[\mathcal{F}],R)$ are continuous (on p. 20 of [Sch] this is called the initial topology). In particular, the canonical map

$$\mathcal{D}(W,R)\longrightarrow \varprojlim_{\mathcal{F}} \mathbb{D}(W[\mathcal{F}],R)$$

is an isomorphism of locally convex R-modules.

We summarize this discussion with the following theorem.

Theorem 3.4.1 *The space $\mathcal{A}(W,R)$ is isomorphic to a compact inductive limit of Banach R-modules and $\mathcal{D}(W,R)$ is a compact projective limit of Banach R-modules. In particular, both of these spaces are complete locally convex R-modules. Moreover, $\mathcal{D}(W,R)$ is Frechet.*

Proof: The proof is immediate from the above discussion together with the simple observation that every strict atlas $\mathcal{F}$ admits a sequence of strict refinements $\{\mathcal{F}^{(n)}\}_n$ that is cofinal in the analytic structure and satisfies $\mathcal{F}^{(n+1)} << \mathcal{F}^{(n)}$ for every n.

Let $\mathcal{B}^d \subseteq \mathbb{C}_p^d$ be the open unit polydisk centered at 1 in $\mathbb{C}_p^d$, endowed with the $\mathbb{Q}_p$-rigid analytic structure associated to convergent power series at 1 with coefficients in $\mathbb{Q}_p$. To each cell U of W we associate the $\mathbb{Q}_p$-rigid analytic space $\mathcal{X}_U := \mathcal{B}^d$ and define the $\mathbb{Q}_p$-rigid analytic space

$$\mathcal{X}_W := \coprod_U \mathcal{X}_U.$$

Here U runs over the cells of the strict manifold W. We let $A(\mathcal{X}_W)$ be the locally convex space of $\mathbb{Q}_p$-rigid analytic functions on $\mathcal{X}_W$.

Theorem 3.4.2 *There is a (non-canonical) isomorphism*

$$\mathcal{D}(W) \xrightarrow{\sim} A(\mathcal{X}_W)$$

of locally convex vector spaces.

Proof: For $t \in \mathcal{B}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{Z}_p^d$ we define

$$\psi(t,x) := t_1^{x_1} \cdot \ldots \cdot t_d^{x_d} \in \mathcal{B}. \tag{14}$$

For fixed $x \in \mathbb{Z}_p^d$, we have $\psi(\cdot,x) \in A(\mathcal{B}^d)$. On the other hand, for fixed $t \in \mathcal{B}^d$ we have $\psi(t,\cdot) \in \mathcal{A}(\mathbb{Z}_p^d,\mathbb{C}_p)$. Thus for $\mu \in \mathcal{D}(\mathbb{Z}_p^d)$ we may define $M_\mu : \mathcal{B}^d \longrightarrow \mathbb{C}_p$ by

$$M_\mu(t) = \int_{\mathbb{Z}_p^d} \psi(t,x)\; d\mu(x).$$

Theorem 3.4.2 is now a consequence of the following well-known theorem of Amice and Velu.

Theorem 3.4.3 *For all $\mu \in \mathcal{D}(\mathbb{Z}_p^d)$ the function M_μ is rigid analytic, i.e. $M_\mu \in A(\mathcal{B}^d)$. Moreover, the map*

$$M : \mathcal{D}(\mathbb{Z}_p^d)\longrightarrow A(\mathcal{B}^d) \quad \text{defined by} \quad \mu \longmapsto M_\mu$$

is an isomorphism of locally convex $\mathbb{Q}_p$-vector spaces.

3.5 The weight space.

The Lie group T is canonically isomorphic to the product of a finite group Δ_T and an open subgroup U that is (non-canonically) isomorphic to $\mathbb{Z}_p^n$:

$$T = \Delta_T \times U.$$

This gives us a canonical strict analytic structure on T whose cells are in natural one-one correspondence with the elements of Δ_T. Let $\Delta_T^* := \mathrm{Hom}(\Delta_T, \mathbb{Z}_p^\times)$ be the character group of Δ_T and fix an isomorphism $\delta : \Delta_T^* \longrightarrow \Delta_T$. Then we have a rigid isomorphism

$$\mathcal{X}_T \cong \Delta_T^* \times \mathcal{B}^n.$$

Proposition 3.5.1 *For complete subfields $K \subseteq \mathbb{C}_p$, there is a functorial (in K) group isomorphism*

$$\mathcal{X}_T(K) \xrightarrow{\sim} \mathit{Hom}_{cont}(T, K^\times).$$

Proof: Fix a continuous (hence strictly analytic) group isomorphism $\phi : U \longrightarrow \mathbb{Z}_p^n$. The map $\psi : \mathcal{B}^n \times \mathbb{Z}_p^n \longrightarrow \mathbb{C}_p^\times$ defined in §3.4(14) induces a natural isomorphism of groups

$$\mathcal{B}^n(K) \xrightarrow{\sim} \mathrm{Hom}_{cont}(\mathbb{Z}_p^n, K^\times) \quad \text{defined by} \quad t \longmapsto \psi(t, \cdot).$$

Pulling back by ϕ we thus have natural isomorphisms

$$\mathcal{X}_T(K) \cong \Delta_T^* \times \mathrm{Hom}_{cont}(U, K^\times) = \mathrm{Hom}_{cont}(T, K^\times).$$

This completes the proof.

More generally, we make the following definitions.

Definition 3.5.2 A *weight* on T is a pair $k := (k, R)$ consisting of a complete locally convex $\mathbb{Q}_p$-algebra R and a continuous group isomorphism $k : T \longrightarrow R^\times$. An R-valued weight is called an R-weight. If $k_i : T \longrightarrow R_i^\times$, $i = 1, 2$, are two weights then a morphism $\psi : k_1 \longrightarrow k_2$ is a continuous $\mathbb{Q}_p$-algebra homomorphism $\psi : R_1 \longrightarrow R_2$ for which $k_1 = \psi \circ k_2$.

We define the *group of R-weights* to be the group

$$\mathcal{X}_T(R) := \mathrm{Hom}_{cont}(T, R^\times).$$

We will write the group law of $\mathcal{X}_T(R)$ additively and for any $k \in \mathcal{X}_T(R)$, $t \in T$, we let $t^k \in R^\times$ denote the value of k on t. With these conventions, we have $t^{k_1+k_2} = t^{k_1} \cdot t^{k_2}$ for any $k_1, k_2 \in \mathcal{X}_T(R)$ and any $t \in T$.

For an important example, we take R to be $\mathcal{D}(T)$ endowed with the $\mathbb{Q}_p$-algebra structure given by *convolution product*. Explicitly, if $\mu, \nu \in \mathcal{D}(T)$ then the convolution $\mu * \nu$ is the distribution whose value on a locally analytic function $\varphi \in \mathcal{A}(T)$ is given by the "integration formula"

$$(\mu * \nu)(\varphi) := \int_T \left(\int_T \varphi(st)\, d\mu(s) \right) d\mu(t).$$

Moreover, for each $t \in T$, we let $[t]$ be the Dirac distribution concentrated at t. The map $[\cdot] : T \longrightarrow \mathcal{D}(T)^\times$ is a continuous group homomorphism. In theorem 3.5.4 we show that $[\cdot]$ is universal in the category of weights on T.

From theorem 3.4.3 we see that if $\mu \in \mathcal{D}(T)$, then the map $\eta_\mu : \mathcal{X}_T(\mathbb{C}_p) \longrightarrow \mathbb{C}_p$ defined by

$$\eta_\mu(k) := \int_T t^k \, d\mu(t)$$

is $\mathbb{Q}_p$-rigid analytic. Moreover, the canonical map

$$\mathcal{D}(T) \xrightarrow{\sim} A(\mathcal{X}_T)$$

is an isomorphism of locally convex $\mathbb{Q}_p$-algebras.

Now let R be a complete locally convex $\mathbb{Q}_p$-algebra and for each continuous $\mathbb{Q}_p$-linear map $\varphi : \mathcal{D}(T) \longrightarrow R$ define $F_\varphi : T \longrightarrow R$ by $F_\varphi(t) := \varphi([t])$. Since $[\cdot] : T \longrightarrow \mathcal{D}(T)$ is locally analytic and $\varphi : \mathcal{D}(T) \longrightarrow R$ is continuous, we conclude that $F_\varphi : T \longrightarrow R$ is locally analytic, i.e. $F_\varphi \in \mathcal{A}_R(T)$.

Lemma 3.5.3 *The map $F : Hom_{cont}(\mathcal{D}(T), R) \longrightarrow \mathcal{A}_R(T)$ defined by $\varphi \longmapsto F_\varphi$ is an isomorphism of locally convex $\mathbb{Q}_p$-vector spaces.*

Proof: Since $\mathcal{A}_R(T)$ is a compact inductive limit of $\mathbb{Q}_p$-Banach spaces, we know from [Sch] that $\mathcal{A}_R(T)$ is reflexive. Thus the canonical map $\Phi : \mathcal{A}_R(T) \longrightarrow \mathrm{Hom}_{cont}(\mathcal{D}(T), R)$ defined by $f \longmapsto (\Phi_f : \mu \longmapsto \mu(f))$ is an isomorphism. But we clearly have $F_{\Phi_f} = f$ for every $f \in \mathcal{A}_R(T)$, so F is the inverse map to Φ and is therefore an isomorphism. This proves the lemma.

Theorem 3.5.4 *The character $[\cdot]$ satisfies the following universal property: for every complete locally convex $\mathbb{Q}_p$-algebra R and every R-weight $k \in \mathcal{X}_T(R)$ there is a unique continuous $\mathbb{Q}_p$-algebra morphism $\varphi : \mathcal{D}(T) \xrightarrow{\varphi} R$ such that $\varphi([t]) = t^k$ for every $t \in T$.*

Proof: Let R be a complete locally convex $\mathbb{Q}_p$-algebra. Then R is isomorphic to a projective limit

$$R = \varprojlim_\nu R_\nu$$

where $\{R_\nu\}_\nu$ is a projective system of p-adic Banach algebras. For each ν, we let $k_\nu \in \mathcal{X}_T(R_\nu)$ be the character obtained by composing k with projection to R_ν. By the results of the last section we know that each k_ν is locally analytic, i.e. that the function $T \longrightarrow R_\nu^\times$, $t \longrightarrow t^{k_\nu}$ is locally analytic on T. We define $\varphi_\nu : \mathcal{D}(T) \longrightarrow R_\nu$ by

$$\varphi_\nu(\mu) := \int_T t^{k_\nu} d\mu(t)$$

for $\mu \in \mathcal{D}(T)$. We see at once that φ_ν is a continuous morphism of $\mathbb{Q}_p$-algebras. Moreover, the system $\varphi(\mu) := \{\varphi_\nu(\mu)\}_\nu$ is coherent for the given inductive system and therefore defines an element of $R = \varprojlim_\nu R_\nu$. Thus we have defined a continuous $\mathbb{Q}_p$-algebra morphism $\mathcal{D}(T) \longrightarrow R$. From the definitions we see that φ has the desired properties, proving the existence statement of the theorem.

Uniqueness follows from the lemma. Indeed, if $\varphi, \psi : \mathcal{D}(T) \longrightarrow R$ are two continuous $\mathbb{Q}_p$-algebra morphisms for which $\varphi([t]) = \psi([t])$ for every $t \in T$, then, in the notation of the lemma, we have $F_\varphi = F_\psi$. But from the lemma we know F is injective. Hence $\varphi = \psi$ and the theorem is proved.

We conclude this section with a classification of the K-weights for any finite extension K of $\mathbb{Q}_p$. Let χ_i, $i = 1, \ldots, n$, be the dominant weights of $\mathbf{G}/\mathbb{Q}_p$. (See section 3.8 below for a review of this concept.) In standard terminology, these are the characters that are *dominant integral for the pair* (B^{opp}, T). Every algebraic character ψ of T can be expressed in exactly one way in the form $\psi = \prod_{i=1}^n \chi_i^{k_i}$ with $k_1, \ldots, k_n \in \mathbb{Z}$.

Definition 3.5.5 Let $k \in \mathcal{X}_T(K)$ and $s \in T^+$.

(a) We say k has level s if k is strictly analytic on $T(s)$.

(b) We say k is *locally algebraic of level* s if there is an algebraic character ψ of T for which $\psi(t) = t^k$ for every $t \in T(s)$. In this case we call ψ the algebraic character associated to k.

(c) We say k is *arithmetic of level* s if k is locally algebraic and the associated algebraic character is dominant.

3.6 Strict analytic structures on the big cell.

In this section we use a slightly different notation to conform with usual conventions. We let bold capital letters denote varieties defined over $\mathbb{Q}_p$. If we use the letter without an argument, it stands for the $\mathbb{C}_p$-points of the variety.

The algebraic "big cell" $\widetilde{\mathbf{Y}} \subseteq \mathbf{G}$ is the Zariski open subset

$$\widetilde{\mathbf{Y}} := \mathbf{N}^{\mathrm{opp}}\mathbf{T}\mathbf{N} \subseteq \mathbf{G}.$$

Recall that Λ was defined in section 1.5. Then Λ is a discrete subgroup of $\mathbf{G}(\mathbb{Q}_p)$ and therefore $\Lambda\backslash\mathbf{G}(\mathbb{Q}_p)$ is locally isomorphic to $\mathbf{G}(\mathbb{Q}_p)$. We define

$$Y := \mathbf{N}^{\mathrm{opp}}(\mathbb{Q}_p)\Lambda\backslash\widetilde{\mathbf{Y}}(\mathbb{Q}_p)$$

and for $r \leq s \in \Lambda^+$ we define the nested sequence of open subsets of $\mathbf{N}^{\mathrm{opp}}(\mathbb{Q}_p)\Lambda\backslash\mathbf{G}(\mathbb{Q}_p)$

$$X(r,s) \subseteq X^s \subseteq X \subseteq Y \tag{15}$$

to be the image of the sequence $\Sigma(r,s) \subseteq \Sigma^s \subseteq \Sigma \subseteq \widetilde{\mathbf{Y}}(\mathbb{Q}_p)$. See Section 3.1 for the notation. Equivalently, this is the image of the sequence $I(r,s) \subseteq I^s \subseteq I \subseteq \widetilde{\mathbf{Y}}(\mathbb{Q}_p)$. We write $X(s) = X(s,s)$ and $\Sigma(s) = \Sigma(s,s)$.

From the decomposition $\Sigma^s = N^{\mathrm{opp}}\Lambda^+ \cdot B^s$ we see that the map $B^s \longrightarrow X^s$ is a bijection. We endow X^s with the structure of p-adic manifold induced from the natural structure on B^s. Both the right action of Σ^s and the left action of T on X^s preserve this analytic structure. The stabilizer in I of $X(r,s)$ is $I(r,s)$ and X is the disjoint union of the "(r,s)-cells" $X(r,s)\gamma$:

$$X = \coprod_{\gamma \in I(r,s)\backslash I} X(r,s)\gamma.$$

We endow X with a strict analytic structure $\mathcal{F}_{r,s}$ as follows. First, choose a strict chart

$$\phi_{r,s} : X(r,s) \longrightarrow \mathbb{Z}_p^d,$$

(where d is the dimension of $\widetilde{\mathbf{Y}}$) by using the fact that the map $T(r) \times N^s \longrightarrow X(r,s)$ defined by $(t,u) \longmapsto tu \pmod{\mathbf{N}^{\mathrm{opp}}(\mathbb{Q}_p)\Lambda}$ is an isomorphism of p-adic manifolds. The definitions of $T(r)$ and N^s make clear that there are strict charts $T(r) \xrightarrow{\sim} \mathbb{Z}_p^n$ and $N^s \xrightarrow{\sim} \mathbb{Z}_p^{d-n}$. The product of these charts gives us the desired chart $\phi_{r,s}$ on $X(r,s)$. Now choose a set of representatives $\{\gamma\}$ of $I(r,s)\backslash I$ and right-translate $\phi_{r,s}$ by each γ to obtain a strict atlas

$$\mathcal{F}_{r,s} := \left\{ \phi_{r,s} \circ \gamma^{-1} \;\middle|\; \gamma \in I(r,s)\backslash I \right\}$$

on X. We let

$$X[r,s] := X[\mathcal{F}_{r,s}] \tag{16}$$

denote X endowed with the strict analytic structure represented by $\mathcal{F}_{r,s}$. As usual, we set $\mathcal{F}_s = \mathcal{F}_{s,s}$, $X[s] = X[s,s]$ etc.

Definition 3.6.1 A strictly analytic function on $X[s]$ (resp. $X[r,s]$) will be called a *locally analytic function of level s (resp. (r,s))* on X.

If an open set $U \subseteq X$ is a union of (r,s)-cells then we say U is compatible with the analytic structure $X[r,s]$. In that case we use the corresponding charts in $\mathcal{F}_{r,s}$ to endow U with a strict analytic structure which we denote

$$U[r,s] \subseteq X[r,s].$$

We leave the proof of the following simple proposition to the reader.

Proposition 3.6.2 *Let $s \in \Lambda^+$ and let U be an open subset of X that is compatible with $X[r,s]$.*

(a) For each $t \in \Lambda^+$, the identity map induces a strictly analytic map

$$U[rt, st] \longrightarrow U[r,s]$$

and this map is a contraction if and only if t is strictly positive. Moreover, we have a canonical isomorphism of locally convex vector spaces

$$\mathcal{D}(U) \xrightarrow{\sim} \varprojlim_{t\in\Lambda^+} \mathbb{D}(U[rt,st]),$$

the projective limit taken with respect to the maps $\mathbb{D}(U[rt_1, st_1]) \longleftarrow \mathbb{D}(U[rt_2, st_2])$ for $t_1 \leq t_2$ in Λ^+.

(b) Let $\sigma \in \Sigma$ with $\delta(\sigma) = t \in \Lambda^+$ and let $U' := U\sigma$. Then U' is compatible with $X[t, st]$ and right translation by σ induces a strictly analytic isomorphism

$$\sigma : U[s] \xrightarrow{\sim} U'[t, st].$$

In particular, σ induces isomorphisms

$$\mathbb{D}(U[s]) \xrightarrow{\sim} \mathbb{D}(U'[t,st]) \qquad \text{and} \qquad \mathcal{D}(t,U) \xrightarrow{\sim} \mathcal{D}(t,U')$$

where $\mathcal{D}(t,U)$ denotes the space of distributions on U which is strictly analytic of level t in the T variable and locally analytic in the N variable.

Now let Ω be a connected admissible affinoid open subset of the weight space $\mathcal{X}_T$. Let $A(\Omega) = A(\Omega, \mathbb{Q}_p)$ denote the Banach algebra of $\mathbb{Q}_p$-valued strictly analytic functions on Ω, and $\mathcal{D} = \mathcal{D}(X, \mathbb{Q}_p)$ the module of distribtuions (see Section 3.4).

Definition 3.6.3 We define the $A(\Omega) - \Sigma$-module $\mathcal{D}_\Omega := \mathcal{D}\hat{\otimes}_{\mathcal{D}(T)} A(\Omega)$ where the tensor product is taken according to the natural map $\mathcal{D}(T) \approx A(\mathcal{X}_T) \to A(\Omega)$ given by the Amice-Velu isomorphism and restriction of functions.

We want to show that $\mathcal{D}_\Omega$ is the inverse limit of Banach modules over $A(\Omega)$. First we need a simple lemma.

Lemma 3.6.4 *Let $T[s]$ be the structure of strict analytic manifold on T obtained by translating the strict chart $T(s) \to \mathbb{Z}_p^n$ used in the discussion preceding (16).*

Let Ω be an affinoid open subset of the weight space $\mathcal{X}_T$. Then there exists $s(\Omega) \in \Lambda^+$ such that for all $s \geq s(\Omega)$, the Amice-Velu isomorphism (Theorem 3.4.3) induces a map $\mathbb{D}(T[s]) \to A(\Omega)$.

Proof: Use of the Maximum Modulus Principle shows that one m in the proof of Lemma 3.2.3 can be chosen to work for all characters in Ω. That proof then shows that if $s >> 1$, $t \mapsto t^k$ is strictly analytic on any translate of $T(s)$ for all $k \in \Omega$.

Note that $\mathcal{D}(T)$ is the projective limit of the $\mathbb{D}(T[s])$, with surjective transition maps.

The Amice-Velu isomorphism is given by evaluating a distribution on t^k. Given $\mu \in \mathbb{D}(T[s])$, lift it to $\mu' \in \mathcal{D}(T)$. Since $t \mapsto t^k$ is in $A(T[s])$, for $k \in \Omega$

$$\int t^k d\mu(t)$$

makes sense independently of the choice of μ' and is the restriction to Ω of the function in $A(\mathcal{X}_T)$ given by

$$\int t^k d\mu'(t).$$

This completes the proof of the lemma.

Definition 3.6.5 For each Ω as above we fix once and for all $s(\Omega) \in \Lambda^+$ satisfying the conclusion of Lemma 3.6.4. Then for any $s \geq r \geq s(\Omega)$, we define the $A(\Omega) - I$-module $\mathbb{D}_\Omega[r,s] := \mathbb{D}(X[r,s])\hat{\otimes}_{\mathbb{D}(T[r])}A(\Omega)$ where the tensor product is taken according to the convolution action of $\mathbb{D}(T[r])$ on $\mathbb{D}(X[r,s])$ and the map $\mathbb{D}(T[r]) \to A(\Omega)$ given by Lemma 3.6.4. Then also $\mathbb{D}_\Omega[r,s] = \mathbb{D}(X[r,s])\hat{\otimes}_{\mathcal{D}(T)}A(\Omega)$, taking the tensor product with respect to the natural surjective map $\mathcal{D}(T) \to \mathbb{D}(T[r])$.

More generally, if U is an open subset of X compatible with $X[r,s]$ and T-stable on the left, we define $\mathbb{D}_\Omega(U[r,s]) := \mathbb{D}(U[r,s])\hat{\otimes}_{\mathbb{D}(T[r])}A(\Omega)$. As usual, we set $\mathbb{D}_\Omega(U[s]) = \mathbb{D}_\Omega(U[s,s])$ and in particular, $\mathbb{D}_\Omega([s]) = \mathbb{D}_\Omega([s,s])$.

It follows from Proposition 3.6.2 that for such U, and $s \geq s(\Omega)$,

$$\mathcal{D}(U)_\Omega := \mathcal{D}(U)\hat{\otimes}_{\mathcal{D}(T)}A(\Omega) \xrightarrow{\sim} \varprojlim_{t\in\Lambda^+} \mathbb{D}_\Omega(U[st]).$$

Moreover, we have the following important fact:

Lemma 3.6.6 *For $s \geq r \geq s(\Omega)$, $\mathbb{D}_\Omega(U[r,s])$ is independent of r. That is, for each r it can be identified with $\mathbb{D}_\Omega(U[s])$.*

Proof: This follows from associativity of tensor products. Write $U \approx M \times T$ where M is an open subset of N compatible with $N[s]$. Then

$$\mathbb{D}_\Omega(U[r,s]) = \mathbb{D}(U[r,s])\hat{\otimes}_{\mathbb{D}(T[r])}A(\Omega) \approx \mathbb{D}(M[s])\hat{\otimes}_K\mathbb{D}(T[r])\hat{\otimes}_{\mathbb{D}(T[r])}A(\Omega) \approx \mathbb{D}(M[s])\hat{\otimes}_K A(\Omega).$$

Note that in particular, for any $r \leq s$, $\mathbb{D}_\Omega([r,s]) = \mathbb{D}_\Omega([s])$.

Let Ind denote the usual induction functor for right modules over a group. Then we have the following proposition.

Proposition 3.6.7 *(a)* $\mathbb{D}_\Omega(X^s)$ *is an* $A(\Omega) - I^s$ *Banach module.*

(b) $\mathrm{Ind}_{I^s}^I \mathbb{D}_\Omega(X^s)$ *is naturally isomorphic to* $\mathbb{D}_\Omega[s]$ *as* $A(\Omega) - I$ *Banach module.*

(c) We have the natural isomorphism

$$\mathcal{D}_\Omega \xrightarrow{\sim} \varprojlim_{s \geq s(\Omega)} \mathbb{D}_\Omega[s].$$

(d) For each $s \geq s(\Omega)$*,* $\mathbb{D}_\Omega(X^s)$ *is ON-able as an* $A(\Omega)$*-Banach module. The elements of* I^s *act on it as operators of norm 1 while any strictly positive* $t \in \Lambda^+$ *induces a completely continuous map* $\mathbb{D}_\Omega(X^s) \to \mathbb{D}_\Omega(X^{st})$*.*

(e) For each $s \geq s(\Omega)$*,* $\mathbb{D}_\Omega[s]$ *is ON-able as an* $A(\Omega)$*-Banach module. The elements of* I *act on it as operators of norm 1 while any strictly positive* $t \in \Lambda^+$ *induces a completely continuous map* $\mathbb{D}_\Omega[s] \to \mathbb{D}_\Omega(X^t[st])$*.*

Proof: Points (a), (b) and (c) are obvious and (e) follows easily from the preceding points. For (d), first, to see the ON-ability, note that N^s is isomorphic as p-adic manifold to a direct product of one-dimensional root groups. Choosing a coordinate on each root group, we obtain a strict chart on N^s such that the monomials in those coordinates provide an ON basis for the K-Banach space $A(N^s)$.

With respect to that strict analytic structure on N^s we have that $\mathbb{D}(N^s)$ is a K-Banach space, and hence ON-able by Serre's theory. Now if V is an ON-able Banach A-module and B is an A-Banach algebra, then $V \hat{\otimes}_A B$ is an ON-able Banach B-module. Apply this to

$$\mathbb{D}_\Omega(X^s) \approx \mathbb{D}(N^s) \hat{\otimes}_K A(\Omega).$$

Since I^s is a group, permuting the $\mathbb{C}_p$-points of X^s, and the norm on the strictly analytic functions is the sup norm, its elements act with norm 1. Since t is strictly positive, it acts as multiplication by a positive power of p on each root group in N^s and the completely continuous nature of its action is obvious.

Remark: The displayed isomorphisms in the proof are compatible with taking the projective limit in s and induce the isomorphism $\mathcal{D}_\Omega = \mathcal{D}_\Omega(X) \approx \mathcal{D}(N) \hat{\otimes}_K A(\Omega)$. This last is compatible with the isomorphism of $\mathcal{D}(T)$-modules $\mathcal{D} = \mathcal{D}(X) \approx \mathcal{D}(N) \hat{\otimes}_K \mathcal{D}(T)$.

3.7 Universal highest weight modules.

Let $s \in \Lambda^+$ and let (I^s, Σ^s) be the Hecke pair of level s defined in §3.1(9).

Definition 3.7.1 Let R be a complete locally convex $\mathbb{Q}_p$-algebra and fix an R-weight $k \in \mathcal{X}_T(R)$. Let V be a locally convex R-module endowed with a continuous right action of Σ^s.

(a) An element $v \in V$ is called a *weight vector of weight* k (with respect to T) if $v|t = t^k v$ for all $t \in T$ and $v|s = v$ for all $s \in \Lambda^+$.

(b) If, moreover, $v|u = v$ for every $u \in N^{\mathrm{opp}}$, then we say v is a *highest* weight vector of weight k.

(c) The pair (V, v) is called a *highest weight module* of weight k and level s if v is a highest weight vector of weight k and also v generates V topologically as an $R[I^s]$-module.

Every highest weight module inherits a canonical structure of $\mathcal{D}(T)$-module. Indeed, if (V, v) is a highest weight module of weight $k \in \mathcal{X}_T(R)$, then we may view V as a $\mathcal{D}(T)$-module via the algebra morphism $\varphi_k : \mathcal{D}(T) \longrightarrow R$ associated to k by the universality of the character $[\cdot] : T \longrightarrow \mathcal{D}(T)^\times$.

Both the right action of Σ^s and the left action of T on X^s preserve the analytic structure. Thus on $\mathcal{D}(X^s)$ we have a continuous left action of T as well as a continuous right action of Σ^s. The left action of T induces a left convolution action of $\mathcal{D}(T)$ on $\mathcal{D}(X^s)$ which is concretely given as follows: for $\lambda \in \mathcal{D}(T)$ and $\mu \in \mathcal{D}(X^s)$ we define the convolution $\lambda * \mu \in \mathcal{D}(X^s)$ by the integration formula:

$$(\lambda * \mu)(f) = \int_T \left(\int_X f(tx)\, d\mu(x) \right) d\lambda(t)$$

for every $f \in \mathcal{A}(X^s)$. With these definitions $\mathcal{D}(X^s)$ becomes a $\mathcal{D}(T)$-module, endowed with a continuous right action of Σ^s (commuting with the $\mathcal{D}(T)$-structure).

Theorem 3.7.2 *Let $\delta_1 \in \mathcal{D}(X^s)$ be the Dirac distribution at the origin of X^s. Then $(\mathcal{D}(X^s), \delta_1)$ is a highest weight module whose highest weight is the universal weight $[\cdot] : T \longrightarrow \mathcal{D}(T)$. Moreover, $(\mathcal{D}(X^s), \delta_1)$ satisfies the following universal property: for every complete locally convex $\mathbb{Q}_p$-algebra R and every highest weight module (V, v) of level s and weight $k \in \mathcal{X}_T(R)$ over R, there is a unique $\mathcal{D}(T)[\Sigma^s]$-equivariant map*

$$\psi : \mathcal{D}(X^s) \longrightarrow V$$

sending δ_1 to v. If, moreover, R is a Banach algebra and k has level st for some $t \in \Lambda^+$, then ψ factors through $\mathcal{D}(X^s) \longrightarrow \mathbb{D}(X^s[st])$.

Proof: Consider the map $\tilde{J} : I^s \longrightarrow V$ defined by $\tilde{J}(\gamma) := v|\gamma$, for $\gamma \in I^s$. By Theorem 3.2.2, we know that $\tilde{J}$ is locally analytic. Moreover, since v is a highest weight vector, we have $\tilde{J}(u\gamma) = \tilde{J}(\gamma)$ whenever $u \in N^{\mathrm{opp}}\Lambda^+$. Thus $\tilde{J}$ descends to a locally analytic function

$$J : X^s \longrightarrow V.$$

We define $\psi : \mathcal{D}(X^s) \longrightarrow V$ by

$$\psi(\mu) = \int_{X^s} J(x)\, d\mu(x)$$

for every $\mu \in \mathcal{D}(X^s)$. A simple verification shows that ψ has the desired properties. Uniqueness of ψ follows from the fact that $\mathcal{D}(X^s)$ is topologically generated by δ_1 as a $\mathcal{D}(T)[I^s]$-module.

More generally, let $k \in \mathcal{X}_T(R)$ be a fixed R-weight for some complete locally convex $\mathbb{Q}_p$-algebra R. We define the locally analytic induced modules

$$\mathcal{A}_k^s := \mathcal{A}_k(X^s) := \left\{ f \in \mathcal{A}(X^s, R) \;\middle|\; f(tx) = t^k f(x), \quad \forall t \in T,\, x \in X^s \right\}. \tag{17}$$

Note that $\mathcal{A}_k^s$ is naturally a left Σ^s-module since Σ^s acts on X^s on the right by right translation.

If $t \in \Lambda^+$ and k has level st, then we also define

$$A_k^s[st] := A_k(X^s[st], R) := \left\{ f \in \mathcal{A}_k^s \;\middle|\; f \text{ is locally analytic of level } st \right\}. \tag{18}$$

(See Definition 3.6.1 for "locally analytic of level st".) These are closed spaces of $\mathcal{A}(X^s, R)$ and $A(X[st], R)$, respectively. We endow each with the induced locally convex topology. Dually, we also define

$$\mathcal{D}_k^s := \mathrm{Hom}_R(\mathcal{A}_k^s, R) \qquad \text{and} \qquad \mathbb{D}_k^s[st] := \mathrm{Hom}_R(A_k^s[st], R)$$

endowing each with the strong topology. Note that $\mathcal{D}_k^s$ is naturally a right Σ^s-module.

As before (see Proposition 3.6.2) we have isomorphisms

$$\mathcal{A}_k^s \cong \varinjlim_t A_k^s[st] \qquad \text{and} \qquad \mathcal{D}_k^s \cong \varprojlim_t \mathbb{D}_k^s[st]$$

and the semigroup Σ^s acts continuously on these inductive and projective systems, with each $\sigma \in \Sigma^s$ changing the level of analyticity as in 3.6.2(b).

Let $\delta_1 \in \mathcal{D}_k^s$ be the Dirac distribution supported at $1 \in X^s$. The proof of Theorem 3.7.2 is easily modified into a proof of the following theorem.

Theorem 3.7.3 *Let R be a complete locally convex $\mathbb{Q}_p$-algebra and k be an R-weight. Then the pair $(\mathcal{D}_k(X^s), \delta_1)$ is a highest weight module over R of weight k and level s. Moreover, $(\mathcal{D}_k(X^s), \delta_1)$ satisfies the following universal property: for any complete locally convex algebra R_0 and any highest weight module (V, v) of level s and weight $k_0 \in \mathcal{X}_T(R_0)$, if k_0 is a specialization of k then there is a unique continuous $R[\Sigma^s]$-morphism $\psi : \mathcal{D}_k(X^s) \longrightarrow V$ such that $\psi(\delta_1) = v$.*

Now let K be a finite extension of $\mathbb{Q}_p$. If M is any $\mathbb{Q}_p$-module, let M_K denote $M \otimes_{\mathbb{Q}_p} K$. If Y is a p-adic manifold, and $\mathcal{D}(Y)$ stands for $\mathcal{D}(Y, \mathbb{Q}_p)$, note that $\mathcal{D}(Y, K)$ is naturally isomorphic to $\mathcal{D}(Y)_K$. Let $k \in \mathcal{X}_T(K)$ be a K-weight. Then theorem 3.7.2 gives us a canonical continuous $\mathcal{D}(T)[\Sigma^s]$-morphism $\mathcal{D}(X^s)_K \longrightarrow \mathcal{D}_k(X^s)$. This map has a canonical extension to a $\mathcal{D}(T)_K$-morphism

$$\eta_k : \mathcal{D}(X^s)_K \longrightarrow \mathcal{D}_k(X^s).$$

Let $I_k \subseteq \mathcal{D}(T)_K$ be the kernel of specialization to k. Then clearly $I_k \mathcal{D}(X^s)_K \subseteq \ker(\eta_k)$. In fact, we have the following result.

Theorem 3.7.4 *The sequence*

$$0 \longrightarrow I_k \mathcal{D}(X^s, K) \longrightarrow \mathcal{D}(X^s, K) \xrightarrow{\eta_k} \mathcal{D}_k(X^s) \longrightarrow 0$$

is an exact sequence of $\mathcal{D}(T, K)[\Sigma^s]$-modules.

Proof: The natural map $T \times N^s \longrightarrow X^s$ is an isomorphism of p-adic manifolds. Let $\mathcal{X}_{N^s}$ be the rigid analytic space associated to a strict chart on N^s. Then $\mathcal{D}(X^s, K) \cong A(\mathcal{X}_T \times \mathcal{X}_{N^s})_K$ and the $\mathcal{D}(T, K)$-structure is induced by the isomorphism $\mathcal{D}(T, K) \cong A(\mathcal{X}_T)_K$. Thus the above sequence is isomorphic to the sequence $0 \longrightarrow I_k A(\mathcal{X}_T \times \mathcal{X}_{N^s})_K \longrightarrow A(\mathcal{X}_T \times \mathcal{X}_{N^s})_K \xrightarrow{\eta_k} A(\mathcal{X}_{N^s})_K \longrightarrow 0$, which is clearly exact.

3.8 Characters of algebraic groups.

In this and the next two subsections we suspend our usual notation. In particular we use Λ to denote the weight lattice of an algebraic group.

Let $\mathbf{G}$ be a connected reductive algebraic group defined over a field K of characteristic 0. We assume $\mathbf{G}$ is split over K and fix $\mathbf{T}$ to be a K-split maximal torus in $\mathbf{G}$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ and $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$. Also let $\mathbf{B}^{\mathrm{opp}}$ be the Borel subgroup opposite to $\mathbf{B}$ with respect to $\mathbf{T}$, and let $\mathbf{N}^{\mathrm{opp}}$ be the unipotent radical of $\mathbf{B}^{\mathrm{opp}}$.

The Cartan decomposition of Lie algebras, $\mathfrak{g} = \mathfrak{n}^{\mathrm{opp}} \oplus \mathfrak{t} \oplus \mathfrak{n}$, is preserved by the right adjoint action of $\mathbf{T}$, i.e. the action derived from right conjugation, $(\gamma, t) \mapsto t^{-1}\gamma t$. Let Φ be the set of roots of $\mathbf{T}$, and choose the ordering of Φ to be the one in which the roots occurring in $\mathfrak{n}^{\mathrm{opp}}$ are the positive ones. Let Δ be the basis of Φ determined by this ordering.

In what follows we will use some non-standard but useful terminology. We let $\Lambda := \mathrm{Hom}_K(\mathbf{T}, \mathbf{G}_m)$ be the character group of $\mathbf{T}$ and $\Lambda_r \subseteq \Lambda$ be the "root lattice", defined as the sublattice generated by Δ. Note that in general Λ_r does not have finite index in Λ. We have the Cartan "pairing"

$$\langle\ ,\ \rangle : \Lambda \times \Lambda_r \longrightarrow \mathbb{Z}$$

which is linear in the first variable, but not the second. The ordering on Φ extends to a partial ordering on Λ defined by

$$\psi' \geq \psi \iff \psi'\psi^{-1} = \prod_{\alpha\in\Delta} \alpha^{n_\alpha} \text{ with all } n_\alpha \text{ integers and } \geq 0.$$

We say a character $\psi \in \Lambda$ is *positive* if $\psi \geq 1$. (This is not the usual notion of "positive" for weights.) Dually, we say $\lambda \in \Lambda$ is *dominant* if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$ and let

$$\Lambda^+ := \left\{ \lambda \in \Lambda \;\middle|\; \lambda \text{ is dominant} \right\}.$$

Note that if Δ has more than one element and Φ is irreducible, then the elements of Δ are positive but not dominant. In the other direction, a dominant character is positive if and only if it lies in the root lattice.

We define the "big cell" $\widetilde{\mathbf{Y}} \subseteq \mathbf{G}$ by

$$\widetilde{\mathbf{Y}} := \mathbf{N}^{\mathrm{opp}}\mathbf{T}\mathbf{N}.$$

Key Fact 3.8.1 *$\widetilde{\mathbf{Y}}$ is an affine open subset of $\mathbf{G}$ isomorphic to $\mathbf{N}^{opp} \times \mathbf{T} \times \mathbf{N}$ in the obvious way.*

Let $K[\widetilde{\mathbf{Y}}]$ be the affine coordinate ring of $\widetilde{\mathbf{Y}}$. Any $\psi \in \Lambda$ extends uniquely to an algebraic character $\psi : \mathbf{B}^{\mathrm{opp}} \longrightarrow \mathbf{G}_m$ that is trivial on $\mathbf{N}^{\mathrm{opp}}$. This then extends uniquely to a function $\widetilde{\psi} \in K[\widetilde{\mathbf{Y}}]$ that is both left translation invariant under $\mathbf{N}^{\mathrm{opp}}$ and right translation invariant under $\mathbf{N}$.

The following proposition is standard. It follows, for example, from [H2], exercise 4, p. 195.

Proposition 3.8.2 *A character $\psi \in \Lambda$ is dominant if and only if $\widetilde{\psi}$ extends to a regular function on $\mathbf{G}$.*

3.9 Algebraic induced modules.

For $\psi \in \Lambda$ we define the algebraic induced module

$$L_\psi(\mathbf{G}) := \left\{ F \in K(G) \;\middle|\; F(\beta x) = \psi(\beta)F(x) \text{ for all } \beta \in \mathbf{B}^{\mathrm{opp}},\, x \in \mathbf{G} \right\}$$

endowed with the (left) action of $\mathbf{G}$ given by right translation: $(\gamma F)(x) = F(x\gamma)$. By differentiating this action, we also obtain an action of the Lie algebra $\mathfrak{g}$.

For any Zariski open subset $U \subseteq \mathbf{G}$ we let

$$L_\psi[U] := \left\{ F \in L_\psi(\mathbf{G}) \;\middle|\; F \text{ is regular on } U \right\}$$

and note that $L_\psi[U]$ is a $\mathfrak{g}$-invariant subspace of $L_\psi(\mathbf{G})$. In particular we have the inclusion

$$L_\psi[\mathbf{G}] \subseteq L_\psi[\widetilde{\mathbf{Y}}].$$

We emphasize that $L_\psi[\mathbf{G}]$ is a $\mathbf{G}$-module, but $L_\psi[\widetilde{\mathbf{Y}}]$ is only a $\mathbf{B}$-module. But both are $\mathfrak{g}$-modules and the above inclusion is $\mathfrak{g}$-equivariant.

The following theorem is standard.

Theorem 3.9.1 *Let $\psi \in \Lambda$. Then $L_\psi[\mathbf{G}]$ is finite dimensional and*

$$L_\psi[\mathbf{G}] \neq 0 \Longleftrightarrow \psi \in \Lambda^+.$$

Moreover, $L_\psi[\mathbf{G}]$ is the unique finite dimensional irreducible representation of $\mathbf{G}$ of highest weight ψ.

For each $\psi \in \Lambda$ we define

$$\psi^* := \prod_{\alpha\in\Delta} \alpha^{\langle\psi,\alpha\rangle}.$$

Then the map $\psi \mapsto \psi^*$ gives us a homomorphism $\Lambda \longrightarrow \Lambda_r$ which sends dominant weights to positive roots. This map is surjective if and only if $\mathbf{G}$ is simply connected (see [H2] p. 189).

Theorem 3.9.2 *Let $\psi \in \Lambda^+$ and suppose $F \in L_\psi[\widetilde{\mathbf{Y}}]$ is a weight vector of weight $\chi \in \Lambda$ where χ satisfies the inequality*

$$\chi\psi^* \geq \psi.$$

Then $F \in L_\psi[\mathbf{G}]$, i.e. F extends to a regular function on $\mathbf{G}$.

The proof is sketched in the next section.

3.10 Proof of Theorem 3.9.2.

The strategy is to reduce the theorem to standard results about Verma modules.

For each $\chi \in \Lambda$ we let $L_\psi[\widetilde{\mathbf{Y}}](\chi)$ be the weight χ subspace. Since any function $f \in L_\psi[\widetilde{\mathbf{Y}}]$ is determined by $f|\mathbf{N}$, we have the following simple proposition.

Proposition 3.10.1 $L_\psi[\widetilde{\mathbf{Y}}] = \bigoplus_{\chi\leq\psi} L_\psi[\widetilde{\mathbf{Y}}](\chi)$.

We say that a linear functional $\mu : L_\psi[\widetilde{\mathbf{Y}}] \longrightarrow K$ is *admissible* if μ vanishes on $L_\psi[\widetilde{\mathbf{Y}}](\chi)$ for all but finitely many $\chi \in \Lambda$. We then define the dual of $L_\psi[\widetilde{\mathbf{Y}}]$ to be the space

$$L_\psi[\widetilde{\mathbf{Y}}]^* := \left\{ \mu \in \mathrm{Hom}_K\left(L_\psi[\widetilde{\mathbf{Y}}], K\right) \;\middle|\; \mu \text{ is admissible} \right\}.$$

Now let $\mathfrak{U}$ be the universal enveloping algebra of $\mathfrak{g}$ and extend the action of $\mathfrak{g}$ on $L_\psi[\widetilde{\mathbf{Y}}]$ to a left action of $\mathfrak{U}$. By duality, we obtain a right action of $\mathfrak{U}$ on $L_\psi[\widetilde{\mathbf{Y}}]^*$. Define the pairing

$$\begin{aligned} [\ ,\ ] : \mathfrak{U} \times L_\psi[\widetilde{\mathbf{Y}}] &\longrightarrow K \\ u, F &\longmapsto [u, F] := (uF)(1) \end{aligned}$$

and note that $[uv, F] = [u, vF]$ for any $u, v \in \mathfrak{U}$. Thus the pairing $[\ ,\ ]$ induces a $\mathfrak{U}$-equivariant map

$$\mathfrak{U} \longrightarrow L_\psi[\widetilde{\mathbf{Y}}]^*.$$

One easily verifies that this map factors through the canonical map $\mathfrak{U} \longrightarrow Z(\psi)$ where $Z(\psi)$ is the universal cyclic (right) $\mathfrak{U}$-module with highest weight ψ.

Thus we obtain a $\mathfrak{U}$-equivariant map

$$\eta : Z(\psi) \longrightarrow L_\psi[\widetilde{\mathbf{Y}}]^*.$$

Theorem 3.10.2 *η is an isomorphism.*

The existence of η and the fact that it is an isomorphism both follow from the way $L_\psi[\widetilde{\mathbf{Y}}]^*$ is generated by the maximal vector δ_1 (evaluation at 1) under $\mathfrak{U}$. Indeed, $L_\psi[\widetilde{\mathbf{Y}}]^*$ is a standard cyclic module whose maximal vector is killed by $\mathfrak{n}^{\mathrm{opp}}$. (Because we are working with the dual, the roles of $\mathfrak{n}$ and $\mathfrak{n}^{\mathrm{opp}}$ are switched.) See [H1] p. 110.

Corollary 3.10.3 *Let $\psi \in \Lambda^+$. Then the set of maximal weights occurring in $L_\psi[\widetilde{\mathbf{Y}}]/L_\psi[\mathbf{G}]$ is the set*

$$\{\, \psi\alpha^{-\langle\psi,\alpha\rangle-1} \,|\, \alpha \in \Delta \,\}.$$

Proof: Since $\mathbb{K}(\psi) := \eta^{-1}(L_\psi[\mathbf{G}]^\perp) \subseteq Z(\psi)$ is isomorphic to the dual of $L_\psi[\widetilde{\mathbf{Y}}]/L_\psi[\mathbf{G}]$, the maximal weights occurring in one are the same as the maximal weights occurring in the other. But the set of maximal weights occurring in $\mathbb{K}(\psi)$ is well-known to be the set given in the statement of the corollary (see [H1] p. 115).

Proof of Theorem 3.9.2: Suppose F is not regular on G. Then F maps to a non-zero weight vector in $L_\psi[\widetilde{\mathbf{Y}}]/L_\psi[\mathbf{G}]$ and therefore χ occurs as a weight in this space. But according to Corollary 3.10.3 there must then be an $\alpha \in \Delta$ such that

$$\chi \leq \psi\alpha^{-\langle\psi,\alpha\rangle-1}.$$

But by hypothesis we have $\psi\chi^{-1} \leq \psi^*$ so we have

$$\alpha^{\langle\psi,\alpha\rangle+1} \leq \psi\chi^{-1} \leq \psi^*.$$

This is a contradiction and the theorem is proved.

3.11 Locally algebraic highest weight modules.

In this section we revert to the notation of Section 3.7 and previous sections.

Again let K be a finite extension of $\mathbb{Q}_p$ and let $k \in \mathcal{X}_T(K)$ be a K-weight that is arithmetic of level s (definition 3.5.5)). We let $\psi := \psi_k$ be the dominant character of $\mathbf{T}$ associated to k and let $\epsilon := \epsilon_k : T \longrightarrow K^\times$ be the finite order character for which $k = \psi + \epsilon$. Since k has level s, we have ϵ is trivial on $T(s)$. Using proposition 3.1.3 we extend ϵ to a homomorphism $\epsilon : \Sigma^s \longrightarrow K^\times$ that is trivial on $\Sigma(s)$ and define

$$L^{alg}{}_k := L_\psi[\mathbf{G}](\epsilon)$$

to be the Σ^s-module obtained by twisting the action of Σ^s on $L_\psi[\mathbf{G}] \otimes_{\mathbb{Q}_p} K$ by ϵ: $(\sigma F)(x) := \epsilon(\sigma)F(x\sigma)$. By definition, we have $L^{alg}{}_k \cong L_\psi[\mathbf{G}]$ as $\Sigma(s)$-modules. We also define

$$V^{alg}{}_k := \mathrm{Hom}_K(L^{alg}{}_k, K)$$

to be the space of K-linear functionals on ${L^{alg}}_k$, endowed with the "dual (right) action" of $\mathbf{G}$ defined by: $(\ell|\gamma)(F) := \ell(\gamma F)$, for $\ell \in {V^{alg}}_k$, $F \in {L^{alg}}_k$, and $\gamma \in \mathbf{G}$.

A function $f : \mathbf{G}(\mathbb{Q}_p) \longrightarrow K$ is said to be algebraic on an open subset $U \subseteq \mathbf{G}(\mathbb{Q}_p)$ if there is a K-regular function $F \in K[\mathbf{G}]$ such that $f(x) = F(x)$ for all $x \in U$. We say f is locally algebraic if every point of $\mathbf{G}(\mathbb{Q}_p)$ has a neighborhood on which f is algebraic.

Suppose $k \in \mathcal{X}_T^+(K)$ is arithmetic of level s and consider the space $\mathcal{A}_k^s$ defined in §3.7(17). If $f \in \mathcal{A}_k^s$ we let $\tilde{f} : \Sigma^s \longrightarrow K$ be the composition of f with the projection $\Sigma^s \longrightarrow X^s$:

$$\tilde{f} : \Sigma^s \longrightarrow X^s \xrightarrow{f} K.$$

The function $f \in \mathcal{A}_k^s$ is said to be locally algebraic if $\tilde{f}$ is locally algebraic. We define

$$L_k^s := L_k(X^s) := \left\{ f \in \mathcal{A}_k^s \;\middle|\; \tilde{f} \text{ is algebraic on } I(s) \right\}.$$

We note that L_k^s is preserved by the action of Σ^s on $\mathcal{A}_k^s$ and therefore inherits a natural structure as Σ^s-module. We define

$$V_k^s := \mathrm{Hom}_K(L_k^s, K)$$

to be the space of K-linear functionals on L_k^s, endowed with the dual right action of Σ^s. We have a canonical map

$$\xi_k : L_k^s \longrightarrow {L^{alg}}_k$$

defined by sending a locally algebraic function f on X^s to the regular function representing $\tilde{f}$ at the origin. We also have the dual map

$$\xi_k^* : {V^{alg}}_k \longrightarrow V_k^s.$$

Both ξ_k and ξ_k^* are isomorphisms of K-vector spaces. If we define $\tau_\psi : \Sigma \longrightarrow \mathbb{Q}_p^\times$ to be the composition

$$\tau_\psi := \psi \circ \delta : \Sigma \xrightarrow{\delta} \Lambda^+ \xrightarrow{\psi} \mathbb{Q}_p^\times$$

then ξ_k, ξ_k^* satisfy the intertwining relations

$$\xi_k(\sigma f) = \tau_\psi^{-1}(\sigma) \cdot \sigma \xi_k(f) \qquad \text{and} \qquad \xi_k^*(\ell|\sigma) = \tau_\psi(\sigma) \cdot \xi_k^*(\ell)|\sigma \tag{19}$$

for any $\sigma \in \Sigma^s$, $f \in L_k^s$, and $\ell \in {V^{alg}}_k$. In particular, ξ_k and ξ_k^* are isomorphisms of I^s- modules (but not of Σ^s-modules, since Λ^+ acts trivially on the highest weight vectors in L_k^s and V_k^s, but non-trivially on those in ${L^{alg}}_k$ and ${V^{alg}}_k$ if $\psi \neq 1$).

Remark. The Σ^s-modules V_k^s (and L_k^s) have better analytic properties than ${V^{alg}}_k$ (and ${L^{alg}}_k$) as k varies over the weight space. We will see that the standard action of the Hecke operators on the cohomology of V_k^s varies continuously in p-adic families as k varies. However, the action of the Hecke operators at p on the cohomology of ${V^{alg}}_k$ *does not* vary continuously. The relation between these actions is encoded in (19). The factor $\tau_\psi(\sigma)$ is the "part of the Hecke operator" that does not vary continuously as a function of k.

Clearly, L_k^s is a closed Σ^s-submodule of $A_k^s := A_k^s[s]$ (see §3.7(18)). By duality, we have a surjective continuous map $\mathbb{D}_k^s := \mathbb{D}_k^s[s] \longrightarrow V_k^s$ whose kernel we denote $\mathbb{K}_k^s$. Thus we have a canonical exact sequence

$$0 \longrightarrow \mathbb{K}_k^s \longrightarrow \mathbb{D}_k^s \longrightarrow V_k^s \longrightarrow 0 \tag{20}$$

of K-Banach spaces.

Now let ψ be a dominant weight and $t \in \Lambda^+$. For each $\alpha \in \Delta$ define $v_\alpha := -\mathrm{ord}_p(\alpha(t))$ and note that $v_\alpha \geq 0$. We then define

$$m_\psi(t) := \min_{\alpha\in\Delta}\big(v_\alpha(1 + \langle\psi, \alpha\rangle)\big). \tag{21}$$

Theorem 3.11.1 *Let $k \in \mathcal{X}_T^+(K)$ be an arithmetic weight of level s and weight ψ, let $\sigma \in \Sigma^s$ and suppose $t = \delta(\sigma)$. Then the action of σ on $\mathbb{K}_k^s$ induces a continuous linear endomorphism of Banach spaces $\mathbb{K}_k^s \xrightarrow{\sigma} \mathbb{K}_k^s$ and the norm of this operator satisfies the inequality*

$$\|\sigma\|_{\mathbb{K}_k^s} \leq p^{-m_\psi(t)}.$$

Proof: It suffices to prove the theorem in the special case $\sigma = t$. We need to prove the inequality

$$(*) \qquad |(\mu|t)(f)| \leq p^{-m_\psi(t)} \cdot |\mu(f)|$$

for every $\mu \in \mathbb{K}_k^s$ and $f \in A_k^s$. Since L_k^s is a dense subspace of A_k^s it suffices to prove $(*)$ for $f \in L_k^s$. Since $L_\psi[\widetilde{\mathbf{Y}}]$ is spanned by weight vectors, we may even assume $F := \xi_k(f)$ is a weight vector for some character χ with $\chi{\leq}\psi$. This means $tf = \chi\psi^{-1}(t) \cdot f$. Writing $\psi\chi^{-1} = \prod_\alpha \alpha^{n_\alpha}$ with integers $n_\alpha \geq 0$ we have $\chi\psi^{-1}(t) = \prod p^{v_\alpha n_\alpha}$ and therefore

$$(**) \qquad |(\mu|t)(f)| = \Big(\prod p^{-v_\alpha n_\alpha}\Big) \cdot |\mu(f)|.$$

If $\chi\psi^*{\geq}\psi$ then F is regular on $\mathbf{G}$ and therefore $f \in L_k^{alg}$. Since $\mu \in \mathbb{K}_k^s$ we then have $\mu(f) = 0$. Thus $(*)$ is a trivial consequence of $(**)$ in this case. If the inequality $\chi\psi^*{\geq}\psi$ does not hold then we can choose $\alpha \in \Delta$ such that $n_\alpha \geq 1 + \langle\psi, \alpha\rangle$. Hence for this choice of α we have $v_\alpha n_\alpha \geq m_\psi(t)$ and once again $(*)$ is an immediate consequence of $(**)$. So we have proved $(*)$ in all cases and the theorem is proved.

Remarks:

(1) The bound given in the last theorem is *almost* best possible. More precisely, let ϵ be dominant and $\beta \in \Delta$ be chosen arbitrarily such that $\langle\epsilon, \beta\rangle > 0$. Then we have the inequality

$$\|t\|_{\mathbb{K}_k^s} \geq p^{-m_\psi(t)+v_\alpha-v_\beta\langle\epsilon,\beta\rangle}.$$

Indeed, choose $\alpha \in \Delta$ such that $v_\alpha(1+\langle\psi, \alpha\rangle) = m_\psi(t)$ and let $\sigma_\alpha, \sigma_\beta$ be the simple reflections in the Weyl group associated to α and β. Then the rational function

$$F := (\sigma_\alpha\widetilde{\psi}) \cdot (\sigma_\beta\widetilde{\epsilon})/\widetilde{\epsilon}$$

is a weight vector in $L_\psi[\widetilde{\mathbf{Y}}]$ that is not an element of $L_\psi[\mathbf{G}]$. Moreover the weight of F is $\chi = \psi\alpha^{-\langle\psi,\alpha\rangle}\beta^{-\langle\epsilon,\beta\rangle}$. Now let $f \in A_k^s$ for which $\xi_k(f) = F$, and choose $0 \neq \mu \in \mathbb{K}_k^s$ such that $\|\mu\| \cdot \|f\| = |\mu(f)|$. Then we have

$$|(\mu|t)(f)| = |\chi\psi^{-1}(t)| \cdot |\mu(f)| = p^{-m_\psi(t)+v_\alpha-v_\beta\langle\epsilon,\beta\rangle} \cdot \|\mu\| \cdot \|f\|.$$

Thus $\|(\mu|t)\| \geq p^{-m_\psi(t)+v_\alpha-v_\beta\langle\epsilon,\beta\rangle} \cdot \|\mu\|$ and consequently $\|t\| \geq p^{-m_\psi(t)+v_\alpha-v_\beta\langle\epsilon,\beta\rangle}$.

(2) In particular, if $\mathbf{G}$ is simply connected, then the inequality stated in the theorem is, in fact, an equality. Indeed, when $\mathbf{G}$ is simply connected, we may choose α as above, then set $\beta = \alpha$ and choose ϵ such that $\langle\epsilon, \beta\rangle = 1$.

4 Slope $\leq h$ decompositions

In this chapter, we define slope $\leq h$ decompositions for Netwon polygons and Banach modules, and we define slope $\leq h$ factorizations for Fredholm power series. We apply these notions to obtain slope $\leq h$ decompositions of Banach modules of cochains and of cohomology.

We begin with a more general algebraic notion of $\mathcal{S}$-decompositions, which enable us to descend from cochains to cohomology.

4.1 $\mathcal{S}$-decompositions

In this section, we assume R is a commutative noetherian ring, $\mathcal{R}$ is a commutative R-algebra, and $\mathcal{S} \subseteq \mathcal{R}$ is a multiplicative subset. For any $\mathcal{R}$-module H we put

$$H_{\mathcal{S}} := \left\{ h \in H \;\middle|\; \exists \alpha \in \mathcal{S} \text{ such that } \alpha h = 0 \right\} \tag{22}$$

and note that $H_{\mathcal{S}}$ is an $\mathcal{R}$-submodule of H.

Definition 4.1.1 An $\mathcal{S}$-decomposition of H is an $\mathcal{R}$-module decomposition

$$H = H_{\mathcal{S}} \oplus H',$$

with $H_{\mathcal{S}}$ given by (22), satisfying the following two properties:

(a) $H_{\mathcal{S}}$ is finitely generated as R-module; and

(b) H' is an $\mathcal{R}$-submodule of H on which every element of $\mathcal{S}$ acts invertibly (i.e. has a two-sided inverse in $\mathrm{End}_{\mathcal{R}}(H')$).

The facts we need about $\mathcal{S}$-decompositions are summarized in the following proposition.

Proposition 4.1.2 *Let $\mathcal{R}$ be a commutative R-algebra where R is a noetherian ring, and let $\mathcal{S}$ be a multiplicative subset of $\mathcal{R}$.*

(a) Let $A = A_{\mathcal{S}} \oplus A'$ and $B = B_{\mathcal{S}} \oplus B'$ be $\mathcal{S}$-decompositions of the $\mathcal{R}$-modules A and B and let $\psi : A \longrightarrow B$ be an $\mathcal{R}$-morphism. Then $\psi(A_{\mathcal{S}}) \subseteq B_{\mathcal{S}}$ and $\psi(A') \subseteq B'$. In particular, an $\mathcal{R}$-module can have at most one $\mathcal{S}$-decomposition. Moreover, the kernel and image of ψ both have $\mathcal{S}$-decompositions.

(b) Let $\lambda : \mathcal{R} \longrightarrow \mathcal{R}[\mathcal{S}^{-1}]$ be the localization morphism with respect to $\mathcal{S}$. An $\mathcal{R}$-module A has an $\mathcal{S}$-decomposition if and only if (i) $A_{\mathcal{S}}$ is finitely generated over R and (ii) the canonical sequence

$$0 \longrightarrow A_{\mathcal{S}} \xrightarrow{j} A \xrightarrow{1\otimes\lambda} A \otimes_{\mathcal{R}} \mathcal{R}[\mathcal{S}^{-1}] \longrightarrow 0$$

is exact. Here j is the canonical inclusion.

(c) Let

$$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$$

be an exact sequence of $\mathcal{R}$-modules. If A, B, D, E have $\mathcal{S}$-decompositions, then so does C. Moreover, the sequences

$$A' \longrightarrow B' \longrightarrow C' \longrightarrow D' \longrightarrow E'$$

$$A_{\mathcal{S}} \longrightarrow B_{\mathcal{S}} \longrightarrow C_{\mathcal{S}} \longrightarrow D_{\mathcal{S}} \longrightarrow E_{\mathcal{S}}$$

are both exact.

(d) *Let* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\psi} C \longrightarrow 0$ *be an exact sequence of* $\mathcal{R}$*-modules. If two of the three modules* A, B, C *have* $\mathcal{S}$*-decompositions, then so does the third.*

(e) *Let* C^* *be a cochain complex of* $\mathcal{R}$*-modules and suppose each* C^i *has an* $\mathcal{S}$*-decomposition. Then the cohomology of* C^* *has an* $\mathcal{S}$*-decomposition as well.*

Proof. Suppose A and B are as in (a). Then clearly $\psi(A_{\mathcal{S}}) \subseteq B_{\mathcal{S}}$, so we need only prove the second inclusion. Since $A_{\mathcal{S}}$ and $B_{\mathcal{S}}$ are finitely generated R-modules, we may choose $\alpha \in \mathcal{S}$ such that α annihilates both $A_{\mathcal{S}}$ and $B_{\mathcal{S}}$. Let $a' \in A'$ be arbitrary. Choose $a_1 \in A'$ such that $\alpha \cdot a_1 = a'$. Then $\psi(a_1) \in B$ decomposes as $\psi(a_1) = b + b'$ with $b \in B_{\mathcal{S}}$ and $b' \in B'$. Thus $\psi(a') = \psi(\alpha \cdot a_1) = \alpha \cdot b + \alpha \cdot b' = \alpha b' \in B'$. This proves (a).

We now turn to the proof of (b). If $A = A_{\mathcal{S}} \oplus A'$ is an $\mathcal{S}$-decomposition, then clearly $A' \cong A \otimes_{\mathcal{R}} \mathcal{R}[\mathcal{S}^{-1}]$ and the sequence (b) is exact. Conversely, suppose $A_{\mathcal{S}}$ is finitely generated and the sequence (b) is exact. Choose $\alpha \in \mathcal{S}$ such that $\alpha A_{\mathcal{S}} = 0$ and set $A' := \alpha A$. Then from the surjectivity of the arrow on the right, we see that $A' \cong A \otimes_{\mathcal{R}} \mathcal{R}[\mathcal{S}^{-1}]$. Thus every element of $\mathcal{S}$ is invertible on A'. Letting β be the inverse of α on A' we see that the morphism $A \longrightarrow A'$ defined by $a' \longmapsto \beta \cdot \alpha a'$ is an idempotent $e_{\mathcal{S}} \in \mathrm{End}_{\mathcal{R}}(A)$ mapping $A \longrightarrow A'$ surjectively and having $A_{\mathcal{S}}$ as its kernel. This proves (b).

To prove (c) we first note that since localization is an exact functor, the sequence

$$A[\mathcal{S}^{-1}] \longrightarrow B[\mathcal{S}^{-1}] \longrightarrow C[\mathcal{S}^{-1}] \longrightarrow D[\mathcal{S}^{-1}] \longrightarrow E[\mathcal{S}^{-1}]$$

is exact, where we write $M[\mathcal{S}^{-1}]$ for $M \otimes_{\mathcal{R}} \mathcal{R}[\mathcal{S}^{-1}]$ for any $\mathcal{R}$-module M. By (b) and our hypotheses we have $M = M_{\mathcal{S}} \oplus M'$, with $M' = M[\mathcal{S}^{-1}]$, for $M = A, B, D$, or E. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C[\mathcal{S}^{-1}] & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

in which the vertical maps are the localization maps, the horizontal sequences are exact, and all but the middle vertical arrows are surjective. We also note that the vertical surjective arrow $D \longrightarrow D'$ has a canonical section $D' \hookrightarrow D$, coming from the $\mathcal{S}$-decomposition of D. An easy diagram chase now shows that the middle vertical arrow must also be surjective. Another diagram chase shows that the sequence $A_{\mathcal{S}} \longrightarrow B_{\mathcal{S}} \longrightarrow C_{\mathcal{S}} \longrightarrow D_{\mathcal{S}} \longrightarrow E_{\mathcal{S}}$ is exact. Since $B_{\mathcal{S}}$ and $D_{\mathcal{S}}$ are finitely generated over the noetherian ring R, it follows that $C_{\mathcal{S}}$ is finitely generated over R as well. Thus by (b) C has an $\mathcal{S}$-decomposition, and (c) is proved.

Assertion (d) is a special case of (c).

Finally, to prove (e), we note that according to (a), the $\mathcal{S}$-decomposition of C^* is respected by the coboundary morphisms. Hence we obtain an $\mathcal{S}$-decomposition *of complexes*

$$C^* = C^*_{\mathcal{S}} \oplus (C')^*.$$

The cohomology of C^* is the direct sum of the cohomology of $C^*_{\mathcal{S}}$ and C'^* giving us the desired $\mathcal{S}$-decomposition of the cohomology. This completes the proof of the proposition.

4.2 Slope $\leq h$ decompositions of Newton polygons.

For our study of power series over an affinoid algebra it will be convenient to use the language of Newton polygons and sup-convexity. We say that a subset $\mathcal{N}$ of $\mathbb{R}^2$ is "sup-convex" if (1) $\mathcal{N}$ is convex, and (2) $(0,t)+\mathcal{N}\subseteq\mathcal{N}$ for every $t\geq 0$. The sup-convex hull of a subset $\mathcal{S}$ of $\mathbb{R}^2$ is the set $\mathcal{H}_+(\mathcal{S})$ defined as the intersection of all sup-convex sets containing $\mathcal{S}$.

A *Newton polygon* is any subset $\mathcal{N}\subseteq\mathbb{R}^2$ of the form

$$\mathcal{N}=\mathcal{N}_\omega:=\mathcal{H}_+\Big(\big\{(i,\omega(i))\ \big|\ i\in I\,\big\}\Big)$$

where $\omega: I\longrightarrow\mathbb{R}$ is a function on some non-empty set I of non-negative integers. If I is finite we will say $\mathcal{N}$ is a *finite Newton polygon*. In this case, we define the degree of $\mathcal{N}$ to be the maximal element of I.

The *vertices* of a Newton polygon $\mathcal{N}=\mathcal{N}_\omega$ are the points $P=(n,\omega(n))$ for which we have a strict inequality

$$\frac{\omega(n)-\omega(r)}{n-r}<\frac{\omega(s)-\omega(n)}{s-n}$$

whenever $r,s\in I$ satisfy $r<n<s$. Every Newton polygon has at least one vertex, namely, $(i_0,\omega(i_0))$ where i_0 is the smallest element of I. It is possible for a Newton polygon to have precisely one vertex. For example, this is true of each of the Newton polygons $\mathcal{N}_{\omega_i}$, $i=0,1$, where the $\omega_i:\mathbb{Z}^{\geq 0}\longrightarrow\mathbb{R}$ are defined by $\omega_0(0)=\omega_1(0)=0$ and $\omega_i(n)=i$ for $n>0$. We have $\mathcal{N}_{\omega_0}$ is the closed first quadrant and $\mathcal{N}_{\omega_1}$ is $\mathcal{N}_{\omega_0}$ with the positive real axis removed. Both of these Newton polygons have precisely one vertex, namely, the origin.

The *edges* of a Newton polygon $\mathcal{N}$ are the line segments contained in the boundary of $\mathcal{N}$ of the form $\overline{PP'}$ where P and P' are distinct vertices of $\mathcal{N}$. If e is an edge of $\mathcal{N}$, then clearly $\mathcal{H}_+(e)\subseteq\mathcal{N}$. However, it need not be true that $\mathcal{N}$ is the union of the sets $\mathcal{H}_+(e)$ where e ranges over the edges of $\mathcal{N}$, as the two examples of the last paragraph illustrate.

The slope of any edge of $\mathcal{N}$ will be called a *slope* of $\mathcal{N}$.

Remark: The collection of all Newton polygons is a monoid under addition. Indeed, one checks easily that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are Newton polygons, then

$$\mathcal{N}_1+\mathcal{N}_2:=\{x_1+x_2\,|\,x_i\in\mathcal{N}_i\,\}$$

is also a Newton polygon. Moreover the non-negative y-axis plays the role of additive identity.

Definition 4.2.1 If $\mathcal{N}$ is a Newton polygon and $(0,0)$ is a vertex of $\mathcal{N}$ then we will say $\mathcal{N}$ is of *Fredholm type*. If $\mathcal{N}$ is a Newton polygon of Fredholm type and $h\in\mathbb{R}$, then we say $\mathcal{N}$ *has slope* $>h$ if, for every non-zero $P\in\mathcal{N}$,the line through P and the origin has slope $>h$.

Definition 4.2.2 Let $\mathcal{N}$ be a Newton polygon and $h\in\mathbb{R}$. Then a decomposition

$$\mathcal{N}=\mathcal{N}_h+\mathcal{N}_h^*$$

is called a *slope* $\leq h$ *decomposition* of $\mathcal{N}$ if

(1) $\mathcal{N}_h$ is a finite Newton polygon whose largest slope is $\leq h$; and

(2) $\mathcal{N}_h^*$ is a Newton polygon of Fredholm type having slope $>h$.

Remarks:

(1) A Newton polygon admits at most one slope $\leq h$ decomposition for any given $h\in\mathbb{R}$.

(2) If a Newton polygon admits a slope $\leq h$ decomposition then it admits a slope $\leq h'$ decomposition for any $h'<h$.

4.3 Slope $\leq h$ factorizations of power series.

Let $(A, |\cdot|_A)$ be a non-archimedean K-Banach algebra. For simplicity, we suppose A is an integral domain. An element $a \in A^\times$ such that $|ax|_A = |a|_A|x|_A$ for every $x \in A$ will be called a multiplicative unit. It is easy to see that if A is an affinoid algebra, and $a \in A$ is a multiplicative unit, then the norm of a equals the norm of any of its specializations.

We also define $v_A : A \setminus \{0\} \longrightarrow \mathbb{Q}$ by

$$|\alpha|_A = p^{-v_A(\alpha)}.$$

We always normalize so that $v_A(p) = 1$. For a non-zero power series

$$F = \sum_{n \geq 0} a_n T^n \in A[[T]]$$

we let $I_F := \{\, n \geq 0 \,|\, a_n \neq 0 \,\}$ and put

$$\mathcal{S}(F) := \left\{ \big(n, v_A(a_n)\big) \in \mathbb{R}^2 \;\middle|\; n \in I_F \right\}.$$

We then define the *Newton polygon* of F to be the Newton polygon

$$\mathcal{N}(F) := \mathcal{H}_+\big(\mathcal{S}(F)\big).$$

The *vertices, edges,* and *slopes* of a power series F are, by definition, the vertices, edges, and slopes of $\mathcal{N}(F)$, respectively.

Definition 4.3.1 A power series F is called a Fredholm series if $F(0) = 1$. Note that in this case, the Newton polygon $\mathcal{N}(F)$ is of Fredholm type. For $h \in \mathbb{R}$, we say that a Fredholm series F has slope $> h$ if $\mathcal{N}(F)$ has slope $> h$.

For a power series $F = \sum_{n \geq 0} a_n T^n \in A[[T]]$ recall that the *interval of convergence* of F is the set of all non-negative real numbers r for which $\lim_{n\to\infty} |a_n| r^n = 0$. Thus, for $h \in \mathbb{R}$, the real number p^h is in the interval of convergence of F if and only if $v_A(a_n) - nh \to \infty$ as $n \to \infty$.

Definition 4.3.2 *Let $F \in A[[T]]$ be a power series and $h \in \mathbb{R}$. A slope $\leq h$ factorization of F is a factorization*

$$F = Q \cdot S$$

in $A[[T]]$ where Q is a polynomial whose leading coefficient is a multiplicative unit, S is a Fredholm series, and such that

(a) every slope of Q is $\leq h$ (in which case we say "Q has slope $\leq h$");

(b) S has slope $> h$; and

(c) p^h is in the interval of convergence of S.

Remark: In case $A = K$ and $h \in \mathbb{Q}$, the closed disk in $\mathbb{C}_p$ centered at the origin and having radius p^h is a K-affinoid variety, whose K-affinoid algebra is the ring of all power series in $K[[T]]$ satisfying condition (c) above. A Fredholm series satisfies both conditions (b) and (c) if and only if it is a unit in this affinoid algebra.

Remark: In case A is an affinoid algebra, it is easy to see that if Q has leading coefficient a multiplicative unit and slope $\leq h$, then the same is true of any specialization of Q.

Proposition 4.3.3 *Let $h \in \mathbb{R}$ and $F = Q \cdot S$ be a slope $\leq h$ factorization of the power series $F \in A[[T]]$. Then*

$$\mathcal{N}(F) = \mathcal{N}(Q) + \mathcal{N}(S).$$

Moreover, $\mathcal{N}(Q) + \mathcal{N}(S)$ is the slope $\leq h$ decomposition of $\mathcal{N}(F)$.

Proof. We first prove the inclusion $\mathcal{N}(F) \subseteq \mathcal{N}(Q) + \mathcal{N}(S)$. In fact, this inclusion is valid for any pair of power series $Q, S \in A[[T]]$. Indeed, let $Q = \sum_{n\geq 0} q_n T^n$, $S = \sum_{n\geq 0} s_n T^n$ and $F := Q \cdot S = \sum_{n\geq 0} c_n T^n$. Then for any $m \in I_F$, we have $c_m = \sum_{i=0}^{m} q_i s_{m-i}$ and we can therefore choose an i with $0 \leq i \leq m$ such that $v_A(c_m) - v_A(q_i s_{m-i}) \geq 0$. Since $v_A(q_i s_{m-i}) \geq v_A(q_i) + v_A(s_{m-i})$, we have $t := v_A(c_m) - (v_A(q_i) + v_A(s_{m-i})) \geq 0$. Thus

$$(m, v_A(c_m)) = (0, t) + (i, v_A(q_i)) + (m - i, v_A(s_{m-i})) \in \mathcal{N}(Q) + \mathcal{N}(S).$$

This proves $\mathcal{S}(F) \subseteq \mathcal{N}(Q) + \mathcal{N}(S)$ and since $\mathcal{N}(Q) + \mathcal{N}(S)$ is a sup-convex set, it follows that $\mathcal{N}(F) \subseteq \mathcal{N}(Q) + \mathcal{N}(S)$.

For the converse inclusion, we suppose Q and S satisfy the hypotheses of the proposition. Let $d := \deg(Q)$. Let $x_d := (d, v_A(a_d))$ be the last vertex of $\mathcal{N}(Q)$. Our conditions on Q and S are easily seen to imply

$$\mathcal{N}(Q) + \mathcal{N}(S) = \mathcal{N}(Q) \cup \Big(x_d + \mathcal{N}(S) \Big).$$

So we need to show $\mathcal{N}(Q) \subseteq \mathcal{N}(F)$ and also $x_d + \mathcal{N}(S) \subseteq \mathcal{N}(F)$.

Let $P := (n, v_A(q_n))$ be an arbitrary vertex of $\mathcal{N}(Q)$ and suppose i is an integer satisfying $0 < i \leq n$. Then from (a) we have $v_A(q_n) - v_A(q_{n-i}) \leq hi$ and from (b) we have $hi < v_A(s_i)$. Thus $v_A(q_n) < v_A(q_{n-i} s_i)$ for all i with $0 < i \leq n$. From the equality $c_n = q_n + \sum_{i=1}^{n} q_{n-i} s_i$ it then follows that $v_A(c_n) = v_A(q_n)$. Hence

$$P = (n, v_A(c_n)) \in \mathcal{S}(F).$$

It follows that every vertex of $\mathcal{N}(Q)$ is in $\mathcal{N}(F)$ and therefore

$$\mathcal{N}(Q) \subseteq \mathcal{N}(F).$$

Let $I_h = \left\{ n \geq 0 \;\middle|\; v_A(s_m) - mh > v_A(s_n) - nh \,, \forall m > n \right\}$ and consider the set

$$\mathcal{S}_h := \left\{ (n, v_A(s_n)) \;\middle|\; n \in I_h \right\}.$$

From conditions (b) and (c) of definition 4.3.2 one easily checks that

$$\mathcal{N}(S) = \mathcal{H}_+(\mathcal{S}_h).$$

Indeed, the inclusion $\mathcal{H}_+(\mathcal{S}_h) \subseteq \mathcal{N}(S)$ is trivial. For the opposite inclusion, let $P = (n, v_A(s_n)) \in \mathcal{S}(S)$. We will show $P \in \mathcal{H}_+(\mathcal{S}_h)$. By (c), we have $v_A(s_r) - rh \to \infty$ as $r \to \infty$ and there is therefore a largest integer $m \geq n$ for which $v_A(s_m) - mh \leq v_A(s_n) - nh$. The point $P' := (m, v_A(s_m))$ is clearly in $\mathcal{S}_h$. Now consider the triangle $\triangle OPP'$. By definition of P', the line segment $\overline{PP'}$ has slope $\leq h$. But by condition (b) we know that the slope of $\overline{OP'}$ is $> h$. Since $n \leq m$ it follows that P lies over $\overline{OP'}$, hence $P \in \mathcal{H}_+(\overline{OP'}) \subseteq \mathcal{H}_+(\mathcal{S}_h)$.

Now let $P := (n, v_A(s_n))$ be an arbitrary element of $\mathcal{S}_h$ and let $m = d + n$. Then we have

$$c_m = q_d s_n + \sum_{i=1}^{d} q_{d-i} s_{n+i}.$$

From assumption (a) we see that for every i with $0 < i \leq d$ we have $v_A(q_d) - v_A(q_{d-i}) \leq hi$. Since $n \in I_h$ we also have $v_A(s_{n+i}) - (n+i)h > v_A(s_n) - nh$. Thus

$$v_A(q_d) - v_A(q_{d-i}) \leq hi < v_A(s_{n+i}) - v_A(s_n).$$

Hence $v_A(q_d s_n) = v_A(q_d) + v_A(s_n) < v_A(q_{d-i}) + v_A(s_{n+i}) \leq v_A(q_{d-i}s_{n+i})$ for every $i = 1, \ldots, d$ and we conclude $v_A(c_m) = v_A(q_d s_n) = v_A(q_d) + v_A(s_n)$. Thus we have

$$x_d + P = (m, v_A(c_m)) \in \mathcal{N}(F).$$

Since P is an arbitrary element of $\mathcal{S}_h$, it follows that $x_d + \mathcal{S}_h \subseteq \mathcal{N}(F)$ and therefore that also $x_d + \mathcal{H}_+(\mathcal{S}_h) \subseteq \mathcal{N}(F)$. But we have already noted that $\mathcal{H}_+(\mathcal{S}_h) = \mathcal{N}(S)$. Hence we have proved

$$x_d + \mathcal{N}(S) \subseteq \mathcal{N}(F),$$

as desired. This completes the proof of proposition 4.3.3.

4.4 A Weierstrass preparation theorem.

This section is devoted to stating and proving a converse (Theorem 4.4.2 below) to Proposition 4.3.3. First note that if $F \in A[[T]]$, $h \in \mathbb{Q}$ and p^h is in the interval of convergence for F, then $\mathcal{N}(F)$ has a (unique) slope $\leq h$ decomposition.

Definition 4.4.1 *Let $F \in A[[T]]$. A vertex $P = (d, v_A(a_d))$ of $\mathcal{N}(F)$ is said to be a distinguished vertex of F if a_d is a multiplicative unit in A.*

Theorem 4.4.2 *Let $F = \sum_n a_n T^n \in A[[T]]$ and $h \in \mathbb{Q}$ be such that p^h is in the interval of convergence for F. Write the slope $\leq h$ decomposition of $\mathcal{N}(F)$ as*

$$\mathcal{N}(F) = \mathcal{N}_h(F) + \mathcal{N}_h^*(F).$$

Suppose the leading vertex $(d, v_A(a_d))$ of $\mathcal{N}_h(F)$ is a distinguished vertex of F. Then there is a unique slope $\leq h$ factorization

$$F = Q \cdot S$$

in $A[[T]]$. Moreover, the leading coefficient of Q is a multiplicative unit and

$$\mathcal{N}(Q) = \mathcal{N}_h(F), \quad \text{and} \quad \mathcal{N}(S) = \mathcal{N}_h^*(F).$$

The theorem is an easy consequence of the above discussion and the following version of the Weierstrass Preparation Theorem. Recall that if R is a topological ring, then the ring of *restricted* power series over R is the ring $R\langle T\rangle$ consisting of all power series $\sum_{n\geq 0} r_n T^n \in R[[T]]$ for which $r_n \to 0$ as $n \to \infty$.

Lemma 4.4.3 *Let R be a ring and suppose R is separated and complete in the π-adic topology for a fixed element $\pi \in R$. Let $F \in R\langle T\rangle$ and suppose that F mod π is a unitary polynomial (i.e. the leading coefficient is a unit) of degree d in $(R/\pi)[T]$. Then there is a unique factorization*

$$F = Q \cdot S$$

in $R\langle T\rangle$ with the following properties:

(1) $Q \in R[T]$ *is a unitary polynomial of degree d;*

(2) $S \in R\langle T\rangle$ *is a Fredholm series.*

Proof. Since $R\langle T\rangle \cong \varprojlim (R/\pi^n)[T]$, it suffices to prove the lemma under the additional assumption that π is nilpotent. We proceed by induction on the smallest positive integer n for which $\pi^n = 0$. The lemma is trivially true in the special case $\pi = 0$. So by way of induction we suppose $\pi^{n+1} = 0$ in R for some $n \geq 1$, and that the lemma is true over any ring where $\pi^n = 0$.

We first prove the existence assertion. By the induction hypothesis there are polynomials $Q_1, S_1 \in R[T]$ such that $F \equiv Q_1 \cdot S_1$ modulo π^n, $Q_1(T)$ is a unitary polynomial of degree d, and $S_1(T) \in 1 + \pi T R[T]$. Now write $F = Q_1 \cdot S_1 + \pi^n G$ for some $G \in R[T]$. Since Q_1 is unitary of degree d, we can find polynomials $v, r \in R[T]$ such that $S_1^{-1}G = Q_1 v + r$ and $\deg(r) < d$. Hence, letting $u = S_1 v$ we have

$$G = S_1 r + Q_1 u, \quad \text{with} \quad \deg(r) < d.$$

Now set $Q = Q_1 + \pi^n r$ and $S = S_1 + \pi^n u$. We see at once that Q, S satisfy the conclusions of the lemma. This proves the existence assertion.

To prove uniquess, we simply note that if $F = Q \cdot S = Q' \cdot S'$ are two such factorizations, then $Q = Q'u$ where $u = S'S^{-1}$. But since Q and Q' are unitary, it follows that u is a constant, hence that $u \in R^\times$. Hence $S'(T) = u \cdot S(T)$, and since $S(0) = S'(0) = 1$, we have $u = 1$. Thus $S = S'$ and consequently $Q = Q'$. This completes the proof of Lemma 4.4.3.

Proof of Theorem 4.4.2: We first consider the special case where $h = 0$ and $a_d = 1$. In this case, our conditions imply $F \in R[[T]]$ where $R = A^{\mathrm{o}}$, the closed unit ball in A. Let π be a uniformizer in K. Then R is separated and complete in the π-adic topology, and our hypotheses imply F is congruent to a unitary polynomial of degree d modulo π. From Lemma 6.3.3 we then obtain a unique factorization

$$F = Q \cdot S$$

satisfying the conclusions of the theorem. This proves the theorem in the special case $h = 0$ and $a_d = 1$.

In the general case, we choose a finite galois extension L of K and an element $b \in L^\times$ with $|b| = p^{-h}$. Let

$$F^*(T) = b^d a_d^{-1} F(b^{-1}T) = \sum_{n=0}^{\infty} a_n^* T^n \in A_L[[T]].$$

Then $a_d^* = 1$ and $(d, 0)$ is a distinguished vertex of slope ≤ 0 of F^*. It follows from our hypotheses that $1 = p^0$ is in the interval of convergence for F^* and that the slope ≤ 0 decomposition of $\mathcal{N}(F^*)$ has degree d. Hence by the special case considered above we deduce that there is a unique factorization

$$F^* = Q^* \cdot S^*$$

in $A_L^{\mathrm{o}}[[T]]$ such that Q^* is a unitary polynomial of degree d and $S^* \in A_L^{\mathrm{o}}\langle T\rangle$ is a restricted Fredholm series. Thus p^0 is in the interval of convergence of S^*. Moreover we have $\mathcal{N}(Q^*) = \mathcal{N}_0(F^*)$ and $\mathcal{N}(S^*) = \mathcal{N}_0^*(F^*)$. Now put

$$Q(T) := b^{-d} a_d Q^*(bT) \quad \text{and} \quad S(T) := S^*(bT).$$

Then we have $Q(T) \cdot S(T) = F(T)$ in $A_L[[T]]$. Moreover, the slopes of Q are all $\leq h$, the slopes of S are all $> h$, and p^h is in the interval of convergence for S. Thus

$$\mathcal{N}(F) = \mathcal{N}(Q) + \mathcal{N}(S)$$

and this is the slope $\leq h$ decomposition of $\mathcal{N}(F)$. This proves existence of the desired factorization in $A_L[[T]]$. Uniqueness follows from the uniqueness of the slope ≤ 0 factorization of F^*. To see that Q^*, $S^* \in A[[T]]$, we note that for any $\sigma \in Gal(L/K)$ we have the factorization

$$\sigma(Q) \cdot \sigma(S) = F$$

in $A_L[[T]]$ and that this is again a slope $\leq h$ factorization. From the uniqueness, it follows that $\sigma(Q) = Q$ and $\sigma(S) = S$ for every $\sigma \in Gal(L/K)$. Hence Q, $S \in A[[T]]$. This completes the proof of the theorem.

4.5 Slope $\leq h$ factorizations of power series over affinoids.

Let Ω be a K-affinoid variety and $A := A_K(\Omega)$ be the associated K-affinoid algebra, endowed with the spectral norm $|\cdot|_A$. A power series $F \in A[[T]]$ will sometimes be called a *K-power series over Ω*. For any K-power series F over Ω and any K-point $x_0 \in \Omega(K)$, we define the specialization of F at x_0 to be the power series $F_{x_0} \in K[[T]]$ obtained from F by evaluating each coefficient of F at x_0. More generally, if $\Omega_0 \subseteq \Omega$ is a K-subaffinoid then we define the restriction of F to Ω_0 to be the power series $F_{\Omega_0} \in A(\Omega_0)[[T]]$ obtained by restricting the coefficients of F to Ω_0.

Theorem 4.5.1 *Let $A = A_K(\Omega)$ be the affinoid algebra of a K-affinoid variety Ω and let $x_0 \in \Omega(K)$. Let $F \in A\{\{T\}\}$ be an entire power series over Ω and fix $h \in \mathbb{R}$. We suppose also that $F_{x_0} \neq 0$ and let*

$$F_{x_0} = Q_0 \cdot S_0$$

be the slope $\leq h$ factorization of F_{x_0} in $K\{\{T\}\}$. Then there is a K-affinoid subdomain $\Omega_0 \subseteq \Omega$ containing the point x_0 such that

(1) F_{Ω_0} has a slope $\leq h$ factorization $F_{\Omega_0} = Q \cdot S$ with S entire over Ω_0; and

(2) $Q_{x_0} = Q_0$ and $S_{x_0} = S_0$.

Moreover, Q and S are relatively prime over Ω_0. (By this we mean the ideal generated by Q and S in $A_K(\Omega_0)\{\{T\}\}$ is the unit ideal.)

Proof. Let d be the degree of Q_0 and write $F = \sum_{n \geq 0} a_n T^n \in A\{\{T\}\}$. Then $a_d(x_0) \in K^\times$. Choose $\lambda \in \mathbb{Q}$ such that $p^{-\lambda} = |a_d(x_0)|$. Since F is entire, there is an integer $N \geq d$ such that $v_A(a_m) - mh > v_A(a_d) - dh$ for all $m \geq N$. Now fix a positive $\epsilon \in \mathbb{Q}$ and let Ω_0 be the K-affinoid subdomain of Ω defined by the conditions

$$x \in \Omega_0 \Longleftrightarrow \begin{cases} |a_d(x)| = p^{-\lambda}, & \text{and} \\ \\ \dfrac{v_0 - v(a_n(x))}{d-n} \leq h & \text{for } n = 0, 1, \ldots, d-1, \text{ and} \\ \dfrac{v(a_n(x)) - v_0}{n-d} \geq h + \epsilon & \text{for } d < n \leq N. \end{cases}$$

Then a_d is a multiplicative unit over Ω_0 and the Newton polygon of $F_{\Omega_0} \in A(\Omega_0)\{\{T\}\}$ has a slope $\leq h$ decomposition of degree d. By Theorem 4.4.2 it follows that F_{Ω_0} has a slope $\leq h$ factorization $F_{\Omega_0} = Q \cdot S$ over Ω_0. Moreover, since $|a_n(x_0)| \leq |a_n|_{\Omega_0}$ for every n and since also $|a_d(x_0)| = |a_d|_{\Omega_0}$, the specialization to x_0 of this factorization $F_{x_0} = Q_{x_0} \cdot S_{x_0}$ is a slope $\leq h$ factorization in $K\{\{T\}\}$.

By the uniqueness of slope $\leq h$ factorizations, we then conclude that $Q_{x_0} = Q_0$ and $S_{x_0} = S_0$. Thus we have proved (1) and (2).

To complete the proof, we have only to show that Q, S are relatively prime. For this we use Coleman's resolvent $\rho := Res(Q, S) \in A(\Omega_0)$. For any $x \in \Omega_0$ we have $\rho(x) = Res(Q_x, S_x) \in \overline{K}$. But Q_x and S_x clearly have no common zeroes in $\overline{K}$. Hence $\rho(x) \in K^\times$. Thus ρ has no zeroes on Ω_0 and we conclude that $\rho \in A(\Omega_0)^\times$. But ρ is in the ideal generated by Q, S in $A(\Omega_0)\{\{T\}\}$ (see [Co]). This proves Q, S are relatively prime and the theorem is proved.

4.6 Slope $\leq h$ decompositions of cochains and cohomology.

If $Q \in R[T]$ is a polynomial of degree d over a ring R, recall that $Q^* \in R[T]$ is defined by

$$Q^*(T) = T^d \cdot Q(1/T).$$

Note that Q^* is monic if and only if Q is Fredholm.

Definition 4.6.1 Let $(A, |\cdot|_A)$ be a K-Banach algebra and let H be an A-module with an A-linear endomorphism $u : H \longrightarrow H$. We do not assume that H has a topological structure. An element $x \in H$ is said to have slope $\leq h$ with respect to u (for some $h \in \mathbb{Q}$) if there is a polynomial $Q \in A[T]$ with the following properties:

(1) $Q^*(u) \cdot x = 0$;

(2) the leading coefficient of Q is multiplicative with respect to $|\cdot|_A$; and

(3) the slope of Q is $\leq h$.

We let $H^{(h)}$ be the set of all elements of H having slope $\leq h$. A submodule $M \subseteq H$ is said to have slope $\leq h$ if $M \subseteq H^{(h)}$.

Proposition 4.6.2 *$H^{(h)}$ is an A-submodule of H.*

Proof. It suffices to prove $H^{(h)}$ is closed under addition. Let $x_1, x_2 \in H^{(h)}$ and let $x := x_1 + x_2$. Choose polynomials $Q_i \in A[T]$ satisfying (2) and (3) of Definition 4.6.1 such that $Q_i^*(u) \cdot x_i = 0$ for $i = 1, 2$. Let $Q = Q_1 \cdot Q_2$. Clearly, $Q^*(u) \cdot x = 0$ and the leading coefficient of Q is multiplicative. So it suffices to prove that every slope of Q is $\leq h$.

Write $Q_1(T) = \sum_{n=0}^{d_1} a_n T^n$, $Q_2(T) = \sum_{n=0}^{d_2} b_n T^n$, and $Q(T) = \sum_{n=0}^{d} c_n T^n$ where $d = d_1 + d_2$. Choose $r, s \geq 0$ with $r + s = m$ such that $v_A(c_m) \geq v_A(a_r b_s)$. Since $v_A(a_r b_s) \geq v_A(a_r) + v_A(b_s)$ and $v_A(c_d) = v_A(a_{d_1} b_{d_2}) = v_A(a_{d_1}) + v_A(b_{d_2})$, we have

$$\begin{aligned} \tfrac{v_A(c_d) - v_A(c_m)}{d-m} &= \tfrac{(v_A(a_{d_1}) - v_A(a_r)) + (v_A(b_{d_2}) - v_A(b_s))}{d-m} \\ &= \lambda_1 \tfrac{(v_A(a_{d_1}) - v_A(a_r))}{d_1 - r} + \lambda_2 \tfrac{(v_A(b_{d_2}) - v_A(b_s)}{d_2 - s} \end{aligned}$$

where $\lambda_1 = \frac{d_1 - r}{d-m}$ and $\lambda_2 = \frac{d_2 - s}{d-m}$. Since all slopes of Q_1 and Q_2 are $\leq h$, it follows that

$$\frac{v_A(c_d) - v_A(c_m)}{d-m} \leq \lambda_1 \cdot h + \lambda_2 \cdot h = (\lambda_1 + \lambda_2)h = h.$$

This proves every slope of Q is $\leq h$ and the proposition is proved.

Definition 4.6.3 *A slope $\leq h$ decomposition of H is an $A[u]$-module decomposition*

$$H = H_h \oplus H_h^*$$

such that

(1) H_h is a finitely generated A-module with slope $\leq h$; and

(2) for every polynomial $Q \in A[T]$ with leading coefficient a multiplicative unit and of slope $\leq h$, the map $Q^*(u) : H_h^* \longrightarrow H_h^*$ is an isomorphism of A-modules.

Lemma 4.6.4 *Assume that A is commutative and noetherian. Let $\mathcal{R} = A[u]$. Let $\mathcal{S}$ be the multiplicative subset of $\mathcal{R}$ consisting of $Q^*(u)$ where Q runs over all polynomials in $A[T]$ satisfying:*
(a) the leading coefficient of Q is a multiplicative unit, and
(b) Q has slope $\leq h$.
Then a slope $\leq h$ decomposition of H is the same thing as an $\mathcal{S}$-decomposition of H. Therefore H has at most one slope $\leq h$ decomposition. If, moreover,

$$H = H_h \oplus H_h^*$$

is a slope $\leq h$ decomposition, then $H_h = H^{(h)}$.

Proof. It is easy to see that the set of Q defined above is multiplicatively closed. Setting $H_h = H_{\mathcal{S}}$ and $H_h^* = H'$ in Definition 4.1.1 (with $R = A$), we see that a slope $\leq h$ decomposition of H is the same thing as an $\mathcal{S}$-decomposition of H.

Thus by Proposition 4.1.2(a), H has at most one slope $\leq h$ decomposition. To see the last assertion, suppose $H = H_h \oplus H_h^*$ is a slope $\leq h$ decomposition. Since $H_h \subset H^{(h)}$, to prove $H_h = H^{(h)}$ it suffices to prove $H^{(h)} \cap H_h^* = 0$. So let $x \in H^{(h)} \cap H_h^*$. Then there is a polynomial Q with leading coefficient a multiplicative unit and of slope $\leq h$ such that $Q^*(u) \cdot x = 0$. But $Q^*(u) : H_h^* \longrightarrow H_h^*$ is an isomorphism, hence $x = 0$. This proves $H_h = H^{(h)}$.

We now return to the set-up of Chapter 2.

First we work over $A = A(\tilde{\Omega}, K)$ where $\tilde{\Omega}$ is an open K-affinoid in the weight space $\mathcal{X}_T$. So A is a commutative, noetherian K-Banach algebra whose norm group is the same as that of K.

Next choose a coefficient module $\mathbb{D}$ as in section 1.4 (with $R = A$). We assume that $\mathbb{D}$ is an ON-able Banach module over A and that $\Sigma_{\mathbb{A}_f}$ acts completely continuously on it (definition 2.7.3). We fix a $\sigma \in \Sigma_{\mathbb{A}_f}$ and assume that σ acts completely continuously on $\mathbb{D}$ (definition 2.7.1).

We defined in chapter 2.6 the cochain complexes

$$C^*(\mathbb{D}) = \bigoplus_i \mathrm{Hom}_{\Gamma(x_i)}(S_*(\mathbf{H}), \mathbb{D}(x_i))$$

and

$$\widetilde{C}^*(\mathbb{D}) = \bigoplus_i \mathrm{Hom}_{\Gamma(x_i)}(F_*^{[i]}(\mathbf{H}), \mathbb{D}(x_i)).$$

Also, having chosen homotopy equivalences between each $F_*^{[i]}$ and $S_*(\mathbf{H})$, we defined in §2.6(8) a lift of the Hecke operator h_σ on cohomology to the cochain level, called H_σ. It is an A-module endomorphism.

For any K-affinoid subdomain $\Omega \subset \tilde{\Omega}$ we define the $A(\Omega)$-module $\mathbb{D}_\Omega := \mathbb{D}\hat{\otimes}_A A(\Omega)$ where the tensor product is taken with respect to the natural map (restriction of functions) $A \to A(\Omega)$. Then σ also acts completely continuously on $\mathbb{D}_\Omega$.

Now $\mathbb{D} = \mathbb{D}_{\tilde{\Omega}}$ is an ON-able A-algebra and hence $\mathbb{D}_\Omega$ is an ON-able $A(\Omega)$-algebra. So is $\widetilde{C}^*(\mathbb{D}_\Omega)$. By Proposition 2.7.6, H_σ acts completely continuously on $\widetilde{C}^*(\mathbb{D}_\Omega)$. We thus obtain the characteristic power series $P_\Omega(T)$ of H_σ on $\widetilde{C}(\mathbb{D}_\Omega) := \oplus_i \widetilde{C}^i(\mathbb{D}_\Omega)$.

Take u_Ω to be the operator h_σ acting on the cohomology $H(\mathbb{D}_\Omega) := \oplus_i H^i(D_\Omega)$. Thus $H(\mathbb{D}_\Omega)$ is an $A(\Omega)[u_\Omega]$-module and we may therefore apply the concepts of this section to these cohomology groups.

Theorem 4.6.5 *Let $x_0 \in \mathcal{X}_T(K)$ be arbitrary. Then there is an admissible K-affinoid subdomain $\Omega \subseteq \mathcal{X}_T$ containing x_0 such that both the cochains $\widetilde{C}^*(\mathbb{D}_\Omega)$ and the cohomology $H(\mathbb{D}_\Omega)$ admit a slope $\leq h$ decomposition over Ω with respect to u_Ω.*

Proof. In theorem 4.5.1 we proved there is a K-affinoid subdomain $\Omega \subseteq \tilde{\Omega}$ containing x_0 and a slope $\leq h$ factorization

$$P_\Omega = Q \cdot S$$

over Ω with respect to H_σ and Q and S are relatively prime over Ω. Set $U = H_\sigma$ acting on $C := \widetilde{C}(\mathbb{D}_\Omega)$. By construction,

$$P_\Omega(T) := \det(1 - T \cdot U) \in A(\Omega)\{\{T\}\}$$

(where the curly braces denote entire power series).

Since Q and S are relatively prime over Ω, we may apply Theorem 4.2 of [Co] to obtain a unique $A(\Omega)[U]$-module decomposition

$$C = N_\Omega(Q) \oplus F_\Omega(Q) \tag{23}$$

into a direct sum of closed submodules, where $Q^*(U) \cdot N_\Omega(Q) = 0$ and $Q^*(U)$ is invertible on $F_\Omega(Q)$.

We claim that (23) is a slope $\leq h$ decomposition of C.

Indeed, since $Q^*(U)$ annihilates $N_\Omega(Q)$, $N_\Omega(Q) \subset C^{(h)}$. Now let $Z \in A(\Omega)[T]$ be an arbitrary polynomial with leading coefficient a multiplicative unit and of slope $\leq h$. As in the proof of theorem 4.5.1, we conclude that Z is relatively prime to S. By Lemma 4.0 in [Co], it follows that $Z^*(U)$ acts invertibly on $F_\Omega(Q)$. Finally, $N_\Omega(Q)$ is finitely generated as an $A(\Omega)$-module, by Theorem 4.3 in [Co].

Since a slope $\leq h$ decomposition is also an $\mathcal{S}$-decomposition, it follows from Proposition 4.1.2(e) that we obtain a slope $\leq h$ decomposition on the cohomology with respect to u_Ω.

Remark: We don't know that the coboundaries are closed in the cocycles in our complex C. However, a posteriori, since $H(\mathbb{D}_\Omega)_h$ is finitely generated as $A(\Omega)$-module, we know it has a Banach module structure.

Since the characteristic power series of U is obtained via a limiting process from the reduction modulo higher and higher powers of p of $\det(1 - UT)$, the following proposition is clear.

Proposition 4.6.6

(a) Suppose $\Omega' \subseteq \Omega$ is another admissible K-affinoid subdomain neighborhood of x_0 satisfying the conclusion of Theorem 4.6.5. Let $P_\Omega(T)$ and $P_{\Omega'}(T)$ be the characteristic power series constructed in the proof of that Theorem. Then $P_{\Omega'}(T)$ is obtained from $P_\Omega(T)$ by restricting all the coefficients from Ω to Ω'.

(b) Suppose for the $\mathbb{Q}_p$-affinoid Ω we know that the matrix of U with respect to some ON basis has entries in $\mathbb{Z}_p$. Then $P_\Omega(T) \in \mathbb{Z}_p[[T]]$.

4.7 Control series.

We recall some of the notations connected to the big cell. In §3.1(9) we defined the Hecke pairs (I^s, Σ^s), $s \in \Lambda^+$. In the notation of Section 2.1, fix compact open subgroups K_v for $v \neq \ell$, $v < \infty$ and set $K_p = I^s$, so that $K_{\mathbb{A}_f} := \prod K_\ell$ is an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$, depending on s. Let $M_s = \mathbf{M}_K$ as in §2.1(1) for this choice of $K_{\mathbb{A}_f}$.

Similarly, we define the semigroup $\Sigma^s_{\mathbb{A}_f}$ to be a product with factors fixed for all $v \neq p$ and equal to Σ^s at p. Then the local Hecke algebra $\mathcal{H}(I^s, \Sigma^s)$ will act on the cohomology of M_s with coefficients in an appropriate sheaf $\widetilde{\mathbb{D}}$.

Definition 4.7.1 *We denote the cohomology $H^i(M_s, \widetilde{\mathbb{D}})$ together with its structure as $\mathcal{H}(I^s, \Sigma^s)$-module by $H^i(I^s, \mathbb{D})$. Also, let $H(I^s, \mathbb{D}) = \oplus_i H^i(I^s, \mathbb{D})$.*

Note: $H^i(I^s, \mathbb{D})$ is not the group cohomology of I^s, but just a convenient mnenomic notation.

There is a canonical isomorphism $\mathcal{H} := \mathcal{H}(I, \Sigma) \to \mathcal{H}(I^s, \Sigma^s)$ for all $s \in \Lambda^+$ which we use to identify all these Hecke algebras to $\mathcal{H}$. We defined X^s in §3.6(15). Recall that K denotes a fixed finite extension of $\mathbb{Q}_p$. Let $\mathcal{D}^s := \mathcal{D}(X^s, K)$ be the space of all locally analytic K-valued distributions on X^s. If $s = 1$, we omit it from the notation, so that for example $\mathcal{D} = \mathcal{D}(X, K)$.

We note that T acts on the left on $\mathcal{D}$ and therefore that $\mathcal{D}$ inherits a natural structure of Λ_T-module, where Λ_T is the completed group ring $\mathbb{Z}_p[[T]]$.

For each admissible open K-affinoid $\Omega \subset \mathcal{X}_T$ recall that $A(\Omega)$ is its K-affinoid algebra. We then define $\mathcal{D}^s_\Omega := \mathcal{D}^s \hat{\otimes}_{\mathcal{D}(T)} A(\Omega)$, the $A(\Omega)$-module of all locally analytic distributions on X^s over Ω.

Let $\mathcal{X} = \mathcal{X}_T$ denote weight space, and let $A := A(\mathcal{X})$ be the ring of rigid analytic functions on $\mathcal{X}_T$ over $\mathbb{Q}_p$. Then for any Ω as in the preceding paragraph, we have the restriction map $A \otimes K \to A(\Omega)$. If we are given $\{f_\Omega \in A(\Omega)\}$ with $\{\Omega\}$ an admissible open cover of $\mathcal{X}_T$ by $\mathbb{Q}_p$-affinoids, and if $f_{\Omega_1}|\Omega_2 = f_{\Omega_2}$ whenever $\Omega_1 \subset \Omega_2$, then the f_Ω glue together to an $f \in A$. If moreoever the coefficients of each power series f_Ω lie in $\mathbb{Q}_p$ and if each f_Ω has norm ≤ 1 in $A(\Omega)$, then $f \in \Lambda_T$.

Let $\mathcal{O}_\mathcal{X}$ denote the structure sheaf on $\mathcal{X}$ so that $\mathcal{O}_\mathcal{X}(\mathcal{X}) = A$.

Definition 4.7.2 *Let H be a presheaf of $\mathcal{O}_\mathcal{X}$-modules with an $\mathcal{O}_\mathcal{X}$-endomorphism u. A Fredholm series $P \in \Lambda_T[[T]]$ will called a controlling Fredholm series of u acting on H if the following properties are satisfied:*

(a) P is entire, i.e. the restriction of P to any admissible open K-affinoid Ω is in $A(\Omega)\{\{T\}\}$.

(b) For every $k \in \mathcal{X}_T(K)$ and $h \in \mathbb{Q}^{\geq 0}$ there exists an admissible open K-affinoid $\Omega \subseteq \mathcal{X}$ and a slope $\leq h$ factorization

$$P_\Omega = Q_\Omega \cdot R_\Omega$$

over Ω.

(c) $Q^*(u)$ annihilates $H(\Omega)_h$.

Remark: Clearly, a controlling Fredholm series is not unique. It may be hoped that in favorable situations, the ideal of all controlling Fredholm series for a given u acting on a given H may be used to cut out a “spectral variety” for u. However, we do not pursue this in this paper.

Proto-theorem 4.7.3 *Fix $t \in \Lambda^+$ and homotopy equivalences between each $F_*^{[i]}$ and $S_*(\mathbf{H})$. Let U_t denote the corresponding lift of the Hecke operator h_t on cohomology to the cochain level as in §2.6(8).*

Let (H, u) denote either $(\widetilde{C}(\mathcal{D}^t), U_t)$ or $(H(I^t, \mathcal{D}^t), h_t)$. Then there is a controlling Fredholm series for u on H.

We will state this theorem more carefully and prove it in section 6.3. Here we make a few further comments to motivate the contents of chapter 5.

The assertion for the cohomology follows immediately from the assertion for the cochains. In order to prove the latter (say for $t = 1$), we would like to write $\mathcal{D}$ as a projective limit of ON-able Banach spaces, parametrized by $s \in \Lambda^+$ and use the characteristic Fredholm power series of u on the cochains for each Banach space. Unfortunately, we cannot get Banach spaces unless we restrict to open affinoids in weight space, in which case we have the isomorphisms

$$\mathcal{D}_\Omega \xrightarrow{\sim} \varprojlim_{s \geq s(\Omega)} \mathbb{D}_\Omega[s].$$

One problem is that neither Σ nor even Σ^s acts on $\mathbb{D}_\Omega[s]$ in an obvious way. Rather $r \in \Lambda^+$ sends $\mathbb{D}_\Omega[s]$ to $\mathbb{D}_\Omega[sr]$. We will have to compensate for this by using a Hecke operator to average back from $\mathbb{D}_\Omega[sr]$ to $\mathbb{D}_\Omega[s]$, thereby defining something we call the $\star$ action. We carefully track the action of the Hecke operators on everything.

Using the $\star$ action, we then have the characteristic Fredholm power series of u, call it P_Ω^s, on the cochains with values in $\mathbb{D}_\Omega[s]$. However, as Ω varies, we have to be prepared to see $s(\Omega)$ get arbitrarily large. So we will have to prove some kind of independence of P_Ω^s from s. Then by the paragraphs preceding Definition 4.7.2, we will be able to glue these together as Ω varies to obtain the desired controlling power series.

To perform both these tasks we need a rather complicated algebraic study of induced modules and Hecke operators. The key ingredient is Shapiro's lemma. This is the topic of the next chapter.

5 Induced modules and cohomology

Because the framework we develop in this chapter should be useful for a number of different problems, we keep our formulation as general as possible. The reader should keep in mind two cases. In the first case Γ will be an arithmetic group and H^* will be the ordinary group cohomology. In the second case, Γ will be a subgroup of the Iwahori group I and H^* will denote the cohomology of the Shimura manifold as in Definition 4.7.1. In either case, most of the verifications are routine and will be omitted. When we do prove some of the details, we will do it only for the group cohomology case: the Shimura manifold cohomology case is similar. The main point in the latter case is that if one formally writes down Shapiro's lemma for H^*, and interprets the resulting formula in terms of the notation of Definition 4.7.1, one has a true statement.

The goal of this chapter is to understand how the cochains, the cohomology, the Hecke operators and their lifts to cochains all behave as we change the level at p.

5.1 Hecke algebras and cohomology.

Throughout this chapter G is a group and R a commutative ring with identity.

A Hecke pair in G is, by definition, a pair (Γ, S) consisting of a submonoid $S \subseteq G$ and a subgroup $\Gamma \subseteq S$ such that every double Γ-coset $\Gamma\sigma\Gamma$ in S is the union of finitely many right Γ-cosets and also the union of finitely many left Γ-cosets.

By an S-module we mean an R-module M endowed with an action of S *on the left*. We shall, however, also need to consider *right* actions of S. We will use the following conventions. By a *contravariant* S-module, we mean a left S^{-1}-module, where $S^{-1} := \{ s^{-1} \mid s \in S \}$. If M is a contravariant S-module, then for $m \in M$ and $\sigma \in S$ we will use the notational convention $m|\sigma := \sigma^{-1}m$. We let

$$\mathcal{M}od_S, \quad \mathcal{M}od_S^{\mathrm{o}}$$

denote the category of S-modules, respectively the category of contravariant S-modules, with S-morphisms. With our conventions, we have $\mathcal{M}od_S^{\mathrm{o}} \cong \mathcal{M}od_{S^{-1}}$.

The *covariant* and *contravariant* R-Hecke algebras

$$\mathcal{H} := \mathcal{H}_\Gamma(S), \qquad \text{respectively} \qquad \mathcal{H}^* := \mathcal{H}_\Gamma(S^{-1}) \tag{24}$$

are the rings of R-valued bi-Γ-invariant functions on G that are supported on a compact subset of S, respectively S^{-1}, with the usual convolution product defined by

$$(f * g)(y) = \sum_{x \in G/\Gamma} f(x) \cdot g(x^{-1}y)$$

for $f, g \in \mathcal{H}$ (respectively $\mathcal{H}^*$). We note that the map $\mathcal{H} \longrightarrow \mathcal{H}^*$, defined by $f \longmapsto f^*$ with $f^*(\sigma^{-1}) := f(\sigma)$, defines an isomorphism of the opposite algebra of $\mathcal{H}$ with $\mathcal{H}^*$.

Let $M \in \mathcal{M}od_S$ and $\widetilde{M} \in \mathcal{M}od_S^{\mathrm{o}}$. We let $\mathcal{H}$ act on the R-module $H^0(\Gamma, M)$ and contravariantly on $H^0(\Gamma, \widetilde{M})$ by

$$f \cdot x := \sum_{\gamma \in S/\Gamma} f(\gamma) \cdot \gamma x \quad \text{and} \quad x'|f := f^* \cdot x' = \sum_{\gamma \in \Gamma \backslash S} f(\gamma) \cdot x'|\gamma \tag{25}$$

for $x \in H^0(\Gamma, M)$ and $x' \in H^0(\Gamma, \widetilde{M})$. We let $\mathcal{H}$ act on the higher cohomology by devissage as follows. Let R be the trivial S-module and fix a resolution R_*, respectively L_*, of R by free *contravariant* S-modules, respectively *covariant* S-modules. Define complexes $K^*(M)$, $K^*(\widetilde{M})$ by

$$K^*(M) := \mathrm{Hom}_R(R_*, M) \quad \text{and} \quad K^*(\widetilde{M}) := \mathrm{Hom}_R(L_*, \widetilde{M}).$$

For $X = M$ or $X = \widetilde{M}$, $K^*(X)$ is a complex of S-modules, with S-action defined by $(\sigma f)(z) = \sigma \cdot f(z|\sigma)$ in the covariant case (respectively by $(f|\sigma)(z) = f(\sigma z)|\sigma$ in the contravariant case) for $f \in K^*(X)$ and $z \in R_*$ (respectively $z \in L_*$).

Applying the functor $H^0(\Gamma, *)$ to $K^*(X)$ we obtain a complex of $\mathcal{H}$-modules $C^*(\Gamma, X)$. Since R_* is a free S-resolution, it is also a free Γ-resolution. Hence, the homology of $C^*(\Gamma, X)$ is canonically isomorphic to the group cohomology $H^*(\Gamma, X)$. Moreover, the $\mathcal{H}$-structure of $C^*(\Gamma, X)$ induces an $\mathcal{H}$-structure on $H^*(\Gamma, X)$. One easily checks that these $\mathcal{H}$-modules are independent of all choices and are natural as functors of X in both the covariant or contravariant categories. Thus, letting $\mathcal{M}od$ denote the category of R-modules we see that formation of cohomology gives us functors

$$H^*(\Gamma, *) : \mathcal{M}od_S \longrightarrow \mathcal{M}od \quad \text{and} \quad H^*(\Gamma, *)^{\mathrm{o}} : \mathcal{M}od_S^{\mathrm{o}} \longrightarrow \mathcal{M}od$$

together with homomorphisms

$$\mathcal{T} : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(\Gamma, *) \Big) \quad \text{and} \quad \mathcal{T}^* : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(\Gamma, *)^{\mathrm{o}} \Big)^{\mathrm{opp}}, \tag{26}$$

where, for any $\mathcal{M}od$-valued functor F, $\mathrm{End}(F)$ denotes the R-algebra of natural transformations from F to F and $\mathrm{End}(F)^{\mathrm{opp}}$ is the opposite R-algebra.

For $f \in \mathcal{H}$, we let

$$\mathcal{T}_f : H^*(\Gamma, *) \longrightarrow H^*(\Gamma, *) \quad \text{and} \quad \mathcal{T}_f^* : H^*(\Gamma, *)^{\mathrm{o}} \longleftarrow H^*(\Gamma, *)^{\mathrm{o}}$$

be the natural transformations associated to $f \in \mathcal{H}$ and $f^* \in \mathcal{H}^*$ by the above discussion. In particular, for $s \in S$, we define

$$\mathcal{T}_s := \mathcal{T}_{[\Gamma s \Gamma]} \quad \text{and} \quad \mathcal{T}_s^* := \mathcal{T}_{[\Gamma s \Gamma]}^*$$

where $[\Gamma s \Gamma] \in \mathcal{H}$ denotes the characteristic function of $\Gamma s \Gamma$.

5.2 Generalities on Hecke operators and induced modules.

For a Γ-module X we consider the induced S-modules

$$\begin{aligned} \mathrm{Ind}_\Gamma^S(X) \quad &:= \left\{ \phi : S \longrightarrow X \;\middle|\; \phi(\gamma x) = \gamma \phi(x) \text{ for all } \gamma \in \Gamma,\, x \in S \right\}, \\ \text{c-}\mathrm{Ind}_\Gamma^S(X) \quad &:= \left\{ \phi \in \mathrm{Ind}_\Gamma^S(X) \;\middle|\; \phi \text{ has compact support} \right\} \end{aligned}$$

where each is endowed with the usual covariant and contravariant S-actions defined by

$$(\sigma\phi)(x) = \phi(x\sigma) \quad \text{and} \quad (\phi|\sigma)(x) = \phi(x\sigma^{-1}) \tag{27}$$

for $\phi \in \mathrm{Ind}_\Gamma^S(X)$ and $\sigma, x \in S$, where in the latter case we define $\phi(y) := 0$ for any $y \in G$ with $y \notin S$. Thus we obtain functors

$$\mathrm{Ind}_\Gamma^S : \mathcal{M}od_\Gamma \longrightarrow \mathcal{M}od_S \quad \text{and} \quad \mathrm{Ind}_\Gamma^S : \mathcal{M}od_\Gamma^{\mathrm{o}} \longrightarrow \mathcal{M}od_S^{\mathrm{o}}$$

and similarly for $\text{c-}\mathrm{Ind}_\Gamma^S$. Composing with passage to Γ-cohomology we obtain functors

$$\begin{aligned} H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*)) \quad &: \mathcal{M}od_\Gamma \longrightarrow \mathcal{M}od \\ \text{and} \quad H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*))^{\mathrm{o}} \quad &: \mathcal{M}od_\Gamma^{\mathrm{o}} \longrightarrow \mathcal{M}od^{\mathrm{o}} \end{aligned}$$

and pulling back $\mathcal{T}$ and $\mathcal{T}^*$ (see §5.1(26)) by Ind_Γ^S we obtain R-algebra morphisms

$$\tau : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*)) \Big) \quad \text{and} \quad \tau^* : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*))^{\mathrm{o}} \Big)^{\mathrm{opp}}.$$

Definition 5.2.1 For each $s \in S$, we let τ_s and τ_s^* be the Hecke operators defined by $\tau_s := \tau_{[\Gamma s \Gamma]}$ and $\tau_s^* := \tau_{[\Gamma s \Gamma]}^*$. These define natural transformations

$$\begin{aligned} \tau_s : H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) \quad &\longrightarrow H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) \\ \text{and} \quad \tau_s^* : H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) \quad &\longleftarrow H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) \end{aligned}$$

where M and $\widetilde{M}$ run over $\mathcal{M}od_S$ and $\mathcal{M}od_S^{\mathrm{o}}$ respectively.

The definition of τ_s and τ_s^* make use of the actions of S on $\mathrm{Ind}_\Gamma^S(*)$ associated to right translation of functions. But the elements of Ind_Γ^S also enjoy a certain Γ-invariance under *left* translation. This fact allows us to define *another* action of $\mathcal{H}$ on the Γ-cohomology of $\mathrm{Ind}_\Gamma^S(X)$ for an S-module X in either $\mathcal{M}od_S$ or $\mathcal{M}od_S^{\mathrm{o}}$.

More precisely, for $M \in \mathcal{M}od_S$ and $\widetilde{M} \in \mathcal{M}od_S^{\mathrm{o}}$ we have bilinear maps

$$\mathcal{H} \times \mathrm{Ind}_\Gamma^S(M) \longrightarrow \mathrm{Ind}_\Gamma^S(M) \quad \text{and} \quad \mathcal{H}^* \times \mathrm{Ind}_\Gamma^S(\widetilde{M}) \longrightarrow \mathrm{Ind}_\Gamma^S(\widetilde{M})$$

defined as in §5.1(25) by

$$f \cdot \phi := \sum_{x \in S/\Gamma} f(x) \cdot \left[x\phi(x^{-1}y)\right] \quad \text{and} \quad f^* \cdot \widetilde{\phi} := \sum_{x \in \Gamma\backslash S} f(x) \cdot \left[\widetilde{\phi}(xy)|x\right]$$

for $f \in \mathcal{H}$ and $\phi \in \mathrm{Ind}_\Gamma^S(M)$ (respectively $\widetilde{\phi} \in \mathrm{Ind}_\Gamma^S(\widetilde{M})$). This defines an $\mathcal{H}$-structure on $\mathrm{Ind}_\Gamma^S(M)$ and an $\mathcal{H}^*$-structure on $\mathrm{Ind}_\Gamma^S(\widetilde{M})$. These structures are functorial in M and $\widetilde{M}$. In other words, they induce R-algebra morphisms

$$\mathcal{H} \longrightarrow \mathrm{End}\Big(\mathrm{Ind}_\Gamma^S(*)\Big) \quad \text{and} \quad \mathcal{H} \longrightarrow \mathrm{End}\Big(\mathrm{Ind}_\Gamma^S(*)^{\mathrm{o}}\Big)^{\mathrm{opp}}. \tag{28}$$

For any $f \in \mathcal{H}$ we let $h(f)$, $h^*(f)$ denote respectively the images of f under these morphisms. We see at once from the definitions that $h(f)$, $h^*(f)$ commute with the Γ-action on $\mathrm{Ind}_\Gamma^S(*)$ and therefore respect passage to the Γ-cohomology. We therefore also obtain R-algebra morphisms

$$h : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*))\Big) \quad \text{and} \quad h^* : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*))^{\mathrm{o}}\Big)^{\mathrm{opp}}.$$

Definition 5.2.2 *For each $s \in S$, we let h_s and h_s^* be the Hecke operators defined by $h_s := h([\Gamma s\Gamma])$ and $h_s^* := h^*([\Gamma s\Gamma])$. These define natural transformations*

$$\begin{array}{rl} h_s : H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) & \longrightarrow H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) \\ \textit{and} \quad h_s^* : H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) & \longleftarrow H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) \end{array}$$

where M and $\widetilde{M}$ run over $\mathcal{M}od_S$ and $\mathcal{M}od_S^{\mathrm{o}}$ respectively.

To compare the Hecke operators $\mathcal{T}_s$ on $H^*(\Gamma, *)$ with the operators τ_s, h_s acting on $H^*(\Gamma, \mathrm{Ind}_\Gamma^S(*))$ we define Γ-morphisms

$$i : X \hookrightarrow \mathrm{Ind}_\Gamma^S(X) \quad \text{and} \quad i^* : \mathrm{Ind}_\Gamma^S(X) \longrightarrow X$$

for $X \in \mathcal{M}od_S$, respectively $X \in \mathcal{M}od_S^{\mathrm{o}}$, by $i(x)(s) = sx$ (respectively $i(x) = x|s$) if $s \in \Gamma$ and $i(x)(s) = 0$ if $s \notin \Gamma$, and $i^*(\phi) = \phi(1)$. Note that while i and i^* are both Γ-morphisms, neither commutes with the action of S.

Proposition 5.2.3 *For any covariant S-module $M \in \mathcal{M}od_S$ and any $s \in S$, the operator $\mathcal{T}_s$ is given by the commutativity of the diagram*

$$\begin{tikzcd} H^*(\Gamma, M) \arrow[r, "i"] \arrow[dd, "\mathcal{T}_s"'] & H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) \arrow[dr, "h_s"] & \\ & & H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) \arrow[dl, "\tau_s"] \\ H^*(\Gamma, M) & H^*(\Gamma, \mathrm{Ind}_\Gamma^S(M)) \arrow[l, "i^*"'] & \end{tikzcd}$$

Dually, for any contravariant S-module $\widetilde{M} \in \mathcal{M}od_S^{\mathrm{o}}$, the operator $\mathcal{T}_s^*$ is given by the commutativity of the diagram

$$\begin{tikzcd}
H^*(\Gamma, \widetilde{M}) & H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) \arrow[l, "i^*"'] & \\
& & H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) \arrow[ul, "h_s^*"'] \\
H^*(\Gamma, \widetilde{M}) \arrow[uu, "\mathcal{T}_s^*"] \arrow[r, "i"] & H^*(\Gamma, \mathrm{Ind}_\Gamma^S(\widetilde{M})) \arrow[ur, "\tau_s^*"] &
\end{tikzcd}$$

More is true than is expressed in the last proposition. Namely, we can compare the operators h_s, τ_s with the restriction and corestriction that are customarily used to define the Hecke operators. To formulate this precisely, we use the following definition.

Definition 5.2.4 Let X be a Γ-module. For $s \in S$, we define the "induced chunk" $X[s] \subseteq \mathrm{Ind}_\Gamma^S(X)$ by

$$X[s] := \left\{ \phi \in \mathrm{Ind}_\Gamma^S(X) \;\middle|\; \mathrm{supp}\,(\phi) \subseteq \Gamma s \Gamma \right\}.$$

The chunks are Γ-submodules of $\mathrm{Ind}_\Gamma^S(X)$. Indeed, we have a natural Γ-module decomposition

$$\mathrm{Ind}_\Gamma^S(X) = \prod_{s \in \Gamma \backslash S / \Gamma} X[s], \tag{29}$$

which induces a corresponding R-decomposition of the cohomology:

$$H^*(\Gamma, \mathrm{Ind}_\Gamma^S(X)) \quad = \textstyle\prod_{s \in \Gamma \backslash S / \Gamma} H^*(\Gamma, X[s]). \tag{30}$$

In general, the action of S does not respect the decomposition (29), nor do the morphisms $h_s, h_s^*, \tau_s, \tau_s^*$ respect the decomposition (30). However, as the reader can easily check, the operators h_s, τ_s^* do induce operators

$$h_s : H^*(\Gamma, M[1]) \longrightarrow H^*(\Gamma, M[s]) \quad \text{ and } \quad \tau_s^* : H^*(\Gamma, \widetilde{M}[1]) \longrightarrow H^*(\Gamma, \widetilde{M}[s]),$$

a fact we will need for the formulation of proposition 5.2.5 below.

For any $s \in G$, we define the Hecke pair (Γ^s, S^s) by

$$\Gamma^s := \Gamma \cap s^{-1} \Gamma s, \quad \text{respectively} \quad S^s := S \cap s^{-1} S s.$$

For $M \in \mathcal{M}od_S$, respectively $\widetilde{M} \in \mathcal{M}od_S^{\mathrm{o}}$, we let $M^s \in \mathcal{M}od_{S^s}$, respectively $\widetilde{M}^s \in \mathcal{M}od_{S^s}^{\mathrm{o}}$, denote the S^s-module whose underlying R-module is M, respectively $\widetilde{M}$, with S^s-action defined by

$$\alpha \cdot_s m := (s \alpha s^{-1}) \cdot m, \quad \text{respectively} \quad \widetilde{m}|_s \alpha := \widetilde{m}|(s \alpha s^{-1}),$$

for $\alpha \in S^s$ and $m \in M^s$, respectively $\widetilde{m} \in \widetilde{M}$.

If $s \in S$, then s acts on M and on $\widetilde{M}$, and this operator induces an S^s-morphism

$$\psi_s : M \xrightarrow{\ s\ } M^s, \quad \text{respectively} \quad \widetilde{\psi}_s : \widetilde{M}^s \xrightarrow{\ s\ } \widetilde{M}.$$

Corresponding to these, we have restriction and corestriction maps

$$\begin{aligned} res_s &: H^*(\Gamma, M) \longrightarrow H^*(\Gamma^s, M^s) \\ cor_s &: H^*(\Gamma^s, \widetilde{M}^s) \longrightarrow H^*(\Gamma, \widetilde{M}) \end{aligned}$$

defined by the commutativity of the diagrams

$$\begin{tikzcd} H^*(\Gamma^s, M^s) & \\ H^*(\Gamma, M) \arrow[u, "res_s"] \arrow[r, "res"] & H^*(\Gamma^s, M); \arrow[ul, "\psi_s"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & H^*(\Gamma^s, \widetilde{M}^s) \arrow[d, "cor_s"'] \arrow[dl, "\widetilde{\psi}_s"'] \\ H^*(\Gamma^s, \widetilde{M}) \arrow[r, "cor"] & H^*(\Gamma, \widetilde{M}). \end{tikzcd}$$

Proposition 5.2.5 *Let $s \in S$.*

(1) We have canonical Γ-isomorphisms

$$M[s] \cong \mathrm{Ind}_{\Gamma^s}^{\Gamma}(M^s) \quad \text{and} \quad \widetilde{M}[s] \cong \mathrm{Ind}_{\Gamma^s}^{\Gamma}\left(\widetilde{M}^s\right).$$

(2) In the covariant case we have the commutative diagram

$$\begin{tikzcd} H^*(\Gamma, M[s]) \arrow[r, "\sim"] & H^*(\Gamma^s, M^s) \arrow[r, phantom, "="] & H^*(\Gamma^{s^{-1}}, M) \arrow[r, "\sim"] \arrow[d, "cor"'] & H^*(\Gamma, M[s]) \arrow[d, "\tau_s"] \\ H^*(\Gamma, M[1]) \arrow[u, "h_s"] & H^*(\Gamma, M) \arrow[l, "i"'] \arrow[u, "res_s"'] \arrow[r, "\mathcal{T}_s"] & H^*(\Gamma, M) & H^*(\Gamma, \mathrm{Ind}_{\Gamma}^{S}(M)). \arrow[l, "i^*"'] \end{tikzcd}$$

Dually, in the contravariant case we have the commutative diagram

$$\begin{tikzcd} H^*(\Gamma, \widetilde{M}[s]) \arrow[d, "h_s^*"'] & H^*(\Gamma^s, \widetilde{M}^s) \arrow[l, "\sim"'] \arrow[r, phantom, "="] \arrow[d, "cor_s"] & H^*(\Gamma^{s^{-1}}, \widetilde{M}) & H^*(\Gamma, \widetilde{M}[s]) \arrow[l, "\sim"'] \\ H^*(\Gamma, \mathrm{Ind}_{\Gamma}^{S}(\widetilde{M})) \arrow[r, "i^*"] & H^*(\Gamma, \widetilde{M}) & H^*(\Gamma, \widetilde{M}) \arrow[l, "\mathcal{T}_s^*"'] \arrow[u, "res"] \arrow[r, "i"] & H^*(\Gamma, \widetilde{M}[1]). \arrow[u, "\tau_s^*"'] \end{tikzcd}$$

Proof: This is a straightforward verification. We give the details only for the first claim and only in the covariant case. Define $M[s] \longrightarrow \mathrm{Ind}_{\Gamma^s}^{\Gamma}(M^s)$ by $\phi \longmapsto f_\phi$ where $f_\phi(x) = \phi(sx)$ for any $x \in \Gamma$. If $\alpha \in \Gamma^s$ then $f_\phi(\alpha x) = \phi(s\alpha x) = s\alpha s^{-1} \cdot \phi(sx) = \alpha \cdot_s f_\phi(x)$, so $f_\phi \in \mathrm{Ind}_{\Gamma^s}^{\Gamma}$. Moreover, for $\gamma \in \Gamma$ we have $\gamma\phi \longmapsto f_{\gamma\phi}$ where $f_{\gamma\phi}(x) = (\gamma\phi)(sx) = \phi(sx\gamma) = f_\phi(x\gamma) = (\gamma f_\phi)(x)$. Hence the map commutes with the action of Γ. To show our map is a bijection we define a map $\mathrm{Ind}_{\Gamma^s}^{\Gamma}(M^s) \longrightarrow M[s]$ by $f \longmapsto \phi_f$ where ϕ_f is defined on a typical element $\gamma_1 s \gamma_2 \in IsI$ by $\phi_f(\gamma_1 s \gamma_2) = \gamma_1 f(\gamma_2)$. A straightforward verification shows that ϕ_f is well-defined and that the maps $f \longmapsto \phi_f$ and $\phi \longmapsto f_\phi$ are inverses of one another.

Note that the isomorphisms in the top lines of the two diagrams are given by (1) plus Shapiro's lemma.

5.3 Graded Hecke algebras, and the $\star$-action.

In the last section we defined, for $M \in \mathcal{M}od_S$ and $\widetilde{M} \in \mathcal{M}od_S^{\circ}$, the induced modules $\mathrm{Ind}_\Gamma^S(M)$ and $\mathrm{Ind}_\Gamma^S(\widetilde{M})^{\circ}$ as well as their chunk decompoositions

$$\mathrm{Ind}_\Gamma^S(M) = \prod_{s\in\Gamma\backslash S/\Gamma} M[s] \quad \text{and} \quad \mathrm{Ind}_\Gamma^S(\widetilde{M})^{\circ} = \prod_{s\in\Gamma\backslash S/\Gamma} \widetilde{M}[s]. \tag{31}$$

Each of these induced modules is endowed with both an S-structure §5.2(27) and an $\mathcal{H}$-structure §5.2(28). There are three related technical difficulties with our picture.

(i) The action of S does not, in general, respect the decompositions (31), nor do the Hecke operators $h(s), h^*(s)$, $s \in S$, respect these decompositions.

(ii) The S-structure §5.2(27) need not commute with the $\mathcal{H}$-structure §5.2(28).

(iii) The S-structure on the one hand, and the $\mathcal{H}$-structure on the other, give rise to *two* natural actions of $\mathcal{H}$ on the cohomology of the induced modules (31). But in view of (ii), these two actions need not commute with one another. This may happen even in cases where $\mathcal{H}$ is commutative.

To deal with these difficulties, we have introduced the notion of a graded Hecke pair in chapter 2. We will now switch over our notation, replacing Γ with I and S with Σ.

For the rest of this chapter, (I, Σ, Λ) will denote a fixed graded Hecke pair and $\mathcal{H} := \mathcal{H}_I(\Sigma)$, $\mathcal{H}^* := \mathcal{H}_I(\Sigma^{-1})$ will be the associated Hecke algebras as defined in §5.1(24). Let $R[\Lambda^+]$ be the semigroup algebra of Λ^+ and for any subset X of G let $[X]$ be the characteristic function of X. Then for each $\sigma \in \Sigma$ we have $[I\sigma I] \in \mathcal{H}$ and $[I\sigma^{-1}I] \in \mathcal{H}^*$. With these conventions, there is a unique ring homomorphism

$$R[\Lambda^+] \longrightarrow \mathcal{H}$$

sending s to $[IsI]$. One easily checks that this is an isomorphism of rings. In particular, we see that $\mathcal{H}$ is a polynomial ring over R.

Proposition 5.3.1 *Let $r, s, t \in \Lambda^+$ and $\sigma \in \Sigma$ with $s := \delta(\sigma)$. Then for any Σ-module M and contravariant Σ-module $\widetilde{M}$ we have the following commutative diagrams:*

$$\begin{array}{ccc} M[rs] & \xrightarrow{h(t)} & M[rst] \\ \sigma\big\downarrow & & \sigma\big\downarrow \\ M[r] & \xrightarrow{h(t)} & M[rt] \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{M}[rt] & \xrightarrow{\sigma} & \widetilde{M}[rst] \\ h(t)\big\downarrow & & h(t)\big\downarrow \\ \widetilde{M}[r] & \xrightarrow{\sigma} & \widetilde{M}[rs]. \end{array}$$

Proof: The proof is a straightforward application of property §2.5(6). We give the details for the first diagram. Let $\phi \in M[rs]$ and let $y \in \Sigma$. Then

$$\begin{aligned} (h(t) * (\sigma\phi))(y) &= \textstyle\sum_{x\in ItI/I} x \cdot \left[(\sigma\phi)(x^{-1}y)\right] \\ &= \textstyle\sum_{\substack{x\in ItI/I \\ x^{-1}y\in\Sigma}} x \cdot \left[\phi(x^{-1}y\sigma)\right]. \end{aligned}$$

However, since $\phi \in M[rs]$ we see that $\phi(x^{-1}y\sigma) \neq 0 \Longrightarrow x^{-1}y\sigma \in IrsI$ and this implies $\delta(y) = \delta(x)r$ and $x^{-1}y \in \Sigma^{-1}\Sigma \cap \Sigma\Sigma^{-1}$. Thus, by (§2.5(6)), we have $\phi(x^{-1}y\sigma) \neq 0 \Longrightarrow x^{-1}y \in \Sigma$ and it follows that the above sum simplifies to the following:

$$\begin{aligned}(h(t) * (\sigma\phi))(y) &= \sum_{\substack{x \in ItI/I \\ x^{-1}y \in \Sigma}} x \cdot \left[\phi(x^{-1}y\sigma)\right] \\ &= \sum_{x \in ItI/I} x \cdot \left[\phi(x^{-1}y\sigma)\right] \\ &= (\sigma \cdot (h(t) * \phi))(y).\end{aligned}$$

This proves the commutativity of the first diagram. The commutativity of the second diagram is proved similarly.

Using proposition 5.3.1 we can now define a *third* natural action of $\mathcal{H}$ on the cohomology of $\mathrm{Ind}_I^\Sigma(M)$, respectively $\mathrm{Ind}_I^\Sigma(\widetilde{M})$. We do this by first defining a new action of Σ on $\mathrm{Ind}_I^\Sigma(M)$, respectively $\mathrm{Ind}_I^\Sigma(\widetilde{M})$, called the $\star$-action, as follows. For $\sigma \in \Sigma$ with $s := \delta(\sigma) \in \Lambda^+$ and $\phi \in \mathrm{Ind}_I^\Sigma(M)$, respectively $\widetilde{\phi} \in \mathrm{Ind}_I^\Sigma(\widetilde{M})$, we define

$$\sigma \star \phi := \sigma \cdot h(s)\phi, \quad \text{respectively} \quad \widetilde{\phi} \star \sigma := (\widetilde{\phi}|\sigma)|h(s).$$

Proposition 5.3.2 *The pairing $\star$ defines a monoid action of Σ on $\mathrm{Ind}_I^\Sigma(M)$, respectively on $\mathrm{Ind}_I^\Sigma(\widetilde{M})$. Moreover, in both cases, the $\star$-action is a graded action of trivial degree.*

According to (5.3.2), the $\star$ action endows each chunk $M[t]$, respectively $\widetilde{M}[t]$, ($t \in \Lambda^+$) with the structure of a Σ-module, respectively contravariant Σ-module. Thus we may regard these as objects in the corresponding categories:

$$M[t] \in \mathcal{M}od_\Sigma \quad \text{and} \quad \widetilde{M}[t] \in \mathcal{M}od_\Sigma^{\mathrm{o}}.$$

Thus, as before, we obtain R-algebra homomorphisms

$$\mathcal{T} : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(I, M[t])\Big) \quad \text{and} \quad \mathcal{T}^* : \mathcal{H} \longrightarrow \mathrm{End}\Big(H^*(I, \widetilde{M}[t])^{\mathrm{o}}\Big)^{\mathrm{opp}}.$$

Following our previous conventions, we let $\mathcal{T}_f$, $\mathcal{T}_f^*$ be the natural transformations associated to an element $f \in \mathcal{H}$ and make the following definition.

Definition 5.3.3 For each $s \in \Sigma$, we let $\mathcal{T}_s$ and $\mathcal{T}_s^*$ be the Hecke operators defined by $\mathcal{T}_s := \mathcal{T}_{[IsI]}$ and $\mathcal{T}_s^* := \mathcal{T}_{[IsI]}^*$ (using the $\star$ action). These define natural graded transformations of trivial degree

$$\begin{aligned}\mathcal{T}_s : H^*(I, \mathrm{Ind}_I^\Sigma(M)) &\longrightarrow H^*(I, \mathrm{Ind}_I^\Sigma(M)) \\ \text{and} \quad \mathcal{T}_s^* : H^*(I, \mathrm{Ind}_I^\Sigma(\widetilde{M})) &\longleftarrow H^*(I, \mathrm{Ind}_I^\Sigma(\widetilde{M}))\end{aligned}$$

where M and $\widetilde{M}$ run over $\mathcal{M}od_\Sigma$ and $\mathcal{M}od_\Sigma^{\mathrm{o}}$ respectively.

We summarize the main results of this section in the following theorem.

Theorem 5.3.4 *Let (I, Σ, Λ) be a graded Hecke pair and for each $s \in \Lambda^+$ let $\mathcal{T}_s, \tau_s, h_s$ be the covariant Hecke operators on $H^*(I, \mathrm{Ind}_I^\Sigma(M))$, and $\mathcal{T}_s^*, \tau_s^*, h_s^*$ be the contravariant Hecke operators on $H^*(I, \mathrm{Ind}_I^\Sigma(\widetilde{M}))$ defined above. These operators have the following properties:*

(a) *Each of the operators $\mathcal{T}_s, \tau_s, h_s, \mathcal{T}_s^*, \tau_s^*, h_s^*$ is graded. Indeed, we have*

$$\begin{aligned} \deg(\mathcal{T}_s) = \deg(\mathcal{T}_s^*) &= 1 \\ \deg(h_s) = \deg(\tau_s^*) = s \quad \textit{and} \quad \deg(h_s^*) &= \deg(\tau_s) = s^{-1}. \end{aligned}$$

(b) $\mathcal{T}_s = \tau_s \circ h_s$ *and* $\mathcal{T}_s^* = h_s^* \circ \tau_s^*$.

(c) $\mathcal{T}_s$ *commutes with* τ_s *and* h_s, *and dually* $\mathcal{T}^*$ *commutes with* τ_s^* *and* h_s^*.

(d) *For* $t \in \Lambda^+$ *the diagrams*

$$\begin{tikzcd} H^*(I, M[st]) \arrow[r, "\mathcal{T}_s"] & H^*(I, M[st]) \\ H^*(I, M[t]) \arrow[u, "h_s"] \arrow[r, "\mathcal{T}_s"'] \arrow[ur, phantom] & H^*(I, M[t]) \arrow[u, "h_s"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} H^*(I, \widetilde{M}[st]) \arrow[d, "h_s^*"'] & H^*(I, \widetilde{M}[st]) \arrow[l, "\mathcal{T}_s^*"'] \arrow[d, "h_s^*"] \\ H^*(I, \widetilde{M}[t]) & H^*(I, \widetilde{M}[t]) \arrow[l, "\mathcal{T}_s^*"] \end{tikzcd}$$

are commutative.

5.4 Locally constant functions and distributions.

Now we adapt this framework to give a construction of universal distribution modules that makes clear how the Hecke algebra acts on their cohomology. At the same time, we similarly describe modules of locally finite functions.

As in the last section, we let (I, Σ, Λ) be a graded Hecke pair. In §5.2(28) we defined natural actions of h, h^* of $\mathcal{H}$ on the induced modules of Σ-modules and in proposition (5.3.4), we showed that $\mathcal{H}$ commutes with the $\star$-action of Σ on these induced modules. This gives us R-algebra homomorphisms

$$\mathcal{H}[\Sigma] \longrightarrow \mathrm{End}\Big(\mathrm{Ind}_I^{\Sigma}(*)\Big) \quad \text{and} \quad \mathcal{H}[\Sigma] \longrightarrow \mathrm{End}\Big(\mathrm{Ind}_I^{\Sigma}(*)^{\mathrm{o}}\Big)^{\mathrm{opp}}$$

thus endowing the induced modules with natural $\mathcal{H}[\Sigma]$-structures.

Recall that we have a canonical isomorphism $R[\Lambda^+] \cong \mathcal{H}$. Let $\epsilon : \mathcal{H} \longrightarrow R$ be the augmentation morphism associated to the trivial homomorphism $\Lambda^+ \longrightarrow R^\times$ and let $\mathfrak{I} := \ker(\epsilon) \subseteq \mathcal{H}$ be the augmentation ideal.

Definition 5.4.1 *We define two functors*

$$\mathfrak{A} : \mathcal{M}od_\Sigma \longrightarrow \mathcal{M}od_\Sigma \quad \textit{and} \quad \mathfrak{D} : \mathcal{M}od_\Sigma^o \longrightarrow \mathcal{M}od_\Sigma^o$$

as follows:

(a) For $M \in \mathcal{M}od_\Sigma$ *we define*

$$\mathfrak{A}(M) := \text{c-Ind}_I^{\Sigma}(M) \otimes_{\mathcal{H},\epsilon} R$$

and call this the Σ*-module of locally constant* M*-valued functions.*

(b) *For $\widetilde{M} \in \mathcal{M}od^o_\Sigma$ we define*

$$\mathfrak{D}(\widetilde{M}) := \mathrm{Ind}_I^\Sigma(\widetilde{M})^o[\mathfrak{I}] := \left\{ \phi \in \mathrm{Ind}_I^\Sigma(\widetilde{M})^o \;\middle|\; \phi|\alpha = 0 \text{ for all } \alpha \in \mathfrak{I} \right\}.$$

and call this the Σ-module of locally constant $\widetilde{M}$-valued distributions.

Equivalently, we may define these modules as inductive and projective limits respectively. In the covariant case, the maps $h(st^{-1}) : M[t] \longrightarrow M[s]$ for $t \leq s$ give us an inductive system of Σ-modules. Dually, in the contravariant case the maps $h^*(st^{-1}) : \widetilde{M}[s] \longrightarrow \widetilde{M}[t]$ give us a projective system of Σ-morphisms in the contravariant category. We have the following proposition.

Proposition 5.4.2 *We have canonical isomorphisms*

$$\varinjlim_s M[s] \xrightarrow{\sim} \mathfrak{A}(M) \qquad \text{and} \qquad \mathfrak{D}(\widetilde{M}) \xrightarrow{\sim} \varprojlim_s \widetilde{M}[s].$$

Moreover, the functor $\mathfrak{A} : \mathcal{M}od_\Sigma \longrightarrow \mathcal{M}od_\Sigma$ is exact.

Proof: The first two assertions are immediate from the definitions, while the latter follows from the fact that inductive limit functors are exact in any abelian category.

Now let R be a flat $\mathbb{Z}_p$-algebra, which we assume is separated and complete in the p-adic topology. Define the full subcategory of $\mathcal{M}od^{cc}_\Sigma$ of $\mathcal{M}od^o_\Sigma$ to consist of those modules on which every strictly positive element of Λ^+ acts completely continuously (cf. definition 2.7.1). Then we have the following Proposition, which we state without proof, since we will not need it in this paper.

Proposition 5.4.3 *The restriction of $\mathfrak{D}$ to $\mathcal{M}od^{cc}_\Sigma$ is an exact functor.*

Of great importance to us is the following simple proposition.

Proposition 5.4.4 *For any contravariant Σ-module $\widetilde{M}$ we have a canonical morphism*

$$H^*(I, \mathfrak{D}(\widetilde{M})) \to \varprojlim_s H^*(I^s, \widetilde{M}^s)$$

where the transition maps in the projective limit are the appropriate corestriction maps.

The module on the right is sometimes called the "universal norm" module associated to $\widetilde{M}$.

Note: On the finite slope part of the cohomology, this map will be an isomorphism for the simple reason that on that part, the transition maps on the right are all *isomorphisms.* See Theorem 5.5.3 below.

Proof: Since formation of cohomology commutes with the transition maps in the projective limit, we have an morphism

$$H^*(I, \mathfrak{D}(\widetilde{M})) \xrightarrow{\sim} \varprojlim_s H^*(I, \widetilde{M}[s])$$

where the transition maps on the right are the maps

$$h_{st^{-1}} : \widetilde{M}[s] \longrightarrow \widetilde{M}[t], \quad (s \geq t).$$

Apply Proposition 5.2.5 to the I^t-module $\widetilde{M}^t$ and note that the map labeled i^* on the bottom line of that proposition is an isomorphism by Shapiro's lemma. We obtain, for each $s \geq t$, a commutative diagram of $\mathcal{H}$-modules:

$$\begin{array}{ccc} H^*(I, \widetilde{M}[s]) & \xrightarrow{h^*_{st^{-1}}} & H^*(I, \widetilde{M}[t]) \\ \sim\Big\downarrow & & \sim\Big\downarrow \\ H^*(I^s, \widetilde{M}^s) & \xrightarrow{cor_{st^{-1}}} & H^*(I^t, \widetilde{M}^t). \end{array}$$

This completes the proof.

5.5 Finite slope spaces.

As in the last chapter, we let (I, Σ, Λ^+) be a graded Hecke pair. If we are in the covariant case, we impose the additional condition that

$$I \text{ is a topologically finitely generated topological group.} \tag{32}$$

Fix a strictly positive element π of Λ^+. The Hecke operators (from Definition 5.3.3)

$$U := \mathcal{T}_\pi \in \operatorname{End}\Big(H^*(I,*)\Big) \quad \text{and} \quad U^* := \mathcal{T}^*_\pi \in \operatorname{End}\Big(H^*(I,*)^{\mathrm{o}}\Big) \tag{33}$$

and the related operators (from Definition 5.2.1) $u := \tau_\pi$ and $u^* := \tau^*_\pi$,

$$u \in \operatorname{End}\Big(H^*\left(I, \operatorname{Ind}_I^\Sigma(*)\right)\Big) \quad \text{and} \quad u^* \in \operatorname{End}\Big(H^*\left(I, \operatorname{Ind}_I^\Sigma(*)\right)^{\mathrm{o}}\Big), \tag{34}$$

play a special role in our theory. In particular, when $\widetilde{M}$ is a contravariant completely continuous Σ-module, we will use the operator U to cut out certain finitely generated submodules, namely the slope $\leq h$ parts, of its I-cohomology. Note that U acts via the $\star$ action.

The key commutative diagrams (from Theorem 5.3.4(d)) are the following (in the covariant and contravariant cases, respectively).

$$\begin{array}{ccccccc} H^*(I, M[\pi t]) & \xrightarrow{U} & H^*(I, M[\pi t]) & & H^*(I, \widetilde{M}[\pi t]) & \xleftarrow{U^*} & H^*(I, \widetilde{M}[\pi t]) \\ h_\pi \Big\uparrow & \overset{u}{\searrow} & h_\pi \Big\uparrow & & h^*_\pi \Big\downarrow & \overset{u^*}{\nwarrow} & h^*_\pi \Big\downarrow \\ H^*(I, M[t]) & \xrightarrow{U} & H^*(I, M[t]) & & H^*(I, \widetilde{M}[t]) & \xleftarrow{U^*} & H^*(I, \widetilde{M}[t]) \end{array} \tag{35}$$

Let K be a finite extension of $\mathbb{Q}_p$ as usual, and R a K-Banach algebra. Recall that a polynomial $Q \in R[T]$ is said to be Fredholm if $Q(0) = 1$.

Definition 5.5.1 *Let H be an R-module and U be an R-endomorphism of H.*

(a) For an arbitrary $Q \in R[T]$, we define

$$H_Q := \left\{ \xi \in H \;\middle|\; Q(U) \cdot \xi = 0 \right\}.$$

(b) *We say that an element $\xi \in H$ has finite slope (with respect to U) if $\xi \in H_Q$ for some Fredholm polynomial $Q \in R[T]$. We define*

$$H^{\#} := \left\{ \xi \in H \;\middle|\; \xi \text{ has finite slope} \right\}.$$

The main theorem of this section is the following. As always, M is a covariant Σ-module and $\widetilde{M}$ is a contravariant Σ-module.

Theorem 5.5.2 *Let $s, t \in \Lambda^+$. Then h_s induces canonical isomorphisms:*

$$H^0(I, M[t])^{\#} \xrightarrow{\sim} H^0(I, M[st])^{\#} \quad \text{and} \quad H^0(I, \widetilde{M}[st])^{\#} \xrightarrow{\sim} H^0(I, \widetilde{M}[t])^{\#}.$$

Moreover, the canonical maps $M[s] \longrightarrow \mathfrak{A}(M)$, $\mathfrak{D}(\widetilde{M}) \longrightarrow \widetilde{M}[s]$ induce isomorphisms

$$H^0(I, M[s])^{\#} \xrightarrow{\sim} H^0(I, \mathfrak{A}(M))^{\#} \quad \text{and} \quad H^0(I, \mathfrak{D}(\widetilde{M}))^{\#} \xrightarrow{\sim} H^0(I, \widetilde{M}[s])^{\#}.$$

Proof: We first consider the covariant case. We simplify the notation and, for any $r \in \Lambda^+$, let $H[r] := H^0(I, M[r])$. Note that in the group cohomology case, $H[r] = M[r]^I$, the R-submodule of I-invariant elements of $M[r]$. In the adelic cohomology case, in the notation of chapter 2, $H[r] = \oplus_i M[r](x_i)^{\Gamma(x_i)}$.

First, we show that the transition map

$$h_\pi : H[t]^{\#} \longrightarrow H[\pi t]^{\#} \tag{36}$$

is an isomorphism. (Recall that π is a fixed strictly positive element of Λ^+.)

Let $Q \in R[T]$ be any Fredholm polynomial. Then we may write $Q = 1 - T \cdot P(T)$ with $P \in R[T]$. Then U and $P(U)$ are inverses of each other on X_Q for any $R[U]$-module X. It follows at once from this and (35) that the composition $H[t]_Q \overset{h_\pi P(U)}{\longrightarrow} H[\pi t]_Q$ and $H[\pi t]_Q \xrightarrow{u} H[t]_Q$ are inverse morphisms commuting with the action of $\mathcal{H}$. Since $H^{\#}$ is, by definition, the union of the H_Q as Q ranges over all Fredholm polynomials, this proves that (36) is an isomorphism.

Now let $s \in \Lambda^+$ be arbitrary and choose a positive integer n sufficiently large so that $\pi^n \geq s$. Writing $\pi^n = rs$ with $r \in \Lambda^+$ we have, by (36), that the compositions

$$H[t]^{\#} \xrightarrow{h_s} H[st]^{\#} \xrightarrow{h_r} H[\pi^n t]^{\#} \tag{37}$$

is an isomorphism. And also

$$H[st]^{\#} \xrightarrow{h_r} H[\pi^n t]^{\#} \xrightarrow{u^n} H[t]^{\#} \xrightarrow{h_s} H[st]^{\#} \tag{38}$$

is an isomorphism. To see this, use Theorem 5.3.4 and the fact that

$$h_s \tau_{\pi^n} h_r = h_s \tau_s \tau_r h_r = \mathcal{T}_s \mathcal{T}_r = \mathcal{T}_{\pi^n} = U^n,$$

and we saw above that U is an isomorphism on $X^{\#}$ for any $R[U]$-module X.

So h_s on $H[t]^{\#}$ is both injective and surjective and we have proved the first assertion of the theorem in the covariant case. The contravariant case is proved dually.

We now turn to the covariant case of the last assertion of the theorem. Under hypothesis (32) the canonical map

$$\varinjlim_s H[s] \longrightarrow H^0(I, \mathfrak{A}(M))$$

is an isomorphism and therefore the same is true for the finite slope subspaces

$$\varinjlim_s H[s]^{\#} \longrightarrow H^0(I, \mathfrak{A}(M))^{\#}.$$

The second assertion of the theorem is therefore a consequence of the first. The contravariant case is proved similarly, except that we don't need (32). This completes our proof of Theorem 5.5.2.

We will apply the previous theorem in an adelic context where the cochains used to compute the cohomology may be chosen to have slope $\leq h$ decompositions. So we prove the following corollary.

Theorem 5.5.3 *Let π be a fixed strictly positive element of Λ^+. Let $\widetilde{M}$ be a contravariant Σ-module $\widetilde{M}$. Suppose that for each $s \in \Lambda^+$, the cochain complex $\widetilde{C}^*(\widetilde{M}[s])$ from section 2.6 (which computes $H^*(I, \widetilde{M}[s])$ possesses a slope $\leq h$ decomposition with respect to H_π for each degree $*$ and for every $h \in \mathbb{Q}^{\geq 0}$, for some H_π which lifts the Hecke operator T_π on cohomology to the cochain level. Also, assume that H_π satisfies the conclusion of Lemma 5.5.4 below.*

Then the canonical morphism of Proposition 5.4.4 induces an isomorphism of $\mathcal{H}$-modules on the slope $\leq h$ parts:

$$H^*(I, \mathfrak{D}(\widetilde{M}))_h \xrightarrow{\sim} \varprojlim_s H^*(I^s, \widetilde{M}^s)_h. \tag{39}$$

The transition maps on the right hand side are all isomorphisms of $\mathcal{H}$-modules and for each s, the projection

$$H^*(I, \mathfrak{D}(\widetilde{M}))_h \xrightarrow{\sim} H^*(I^s, \widetilde{M}^s)_h. \tag{40}$$

is an isomorphism.

Moreover, these statements are also true on the cochain level where "$\mathcal{H}$-modules" is replaced by "$R[H_\pi]$-modules".

Proof: First note that by Shapiro's lemma, $H^*(I^s, \widetilde{M}^s) = H^*(I, \widetilde{M}[s])$ and this is even true on the cochain level (cf. Proposition 5.2.5): $\widetilde{C}^*(I^s, \widetilde{M}^s) = \widetilde{C}^*(I, \widetilde{M}[s])$.

Since Hom commutes with projective limits, and since $\widetilde{M} = \lim \widetilde{M}[s]$ we have that $\widetilde{C}^*(I, \widetilde{M}) = \lim \widetilde{C}^*(I, \widetilde{M}[s])$. By hypothesis we have slope $\leq h$ decompositions $\widetilde{C}^*(I, \widetilde{M}[s]) = \widetilde{C}^*[s]_h \oplus \widetilde{C}^*[s]'$. From Proposition 4.1.2(a) on $\mathcal{S}$-decompositions, we know that the transition maps send $\widetilde{C}^*[s]_h \to \widetilde{C}^*[st]_h$ and $\widetilde{C}^*[s]' \to \widetilde{C}^*[st]'$. Also, by Proposition 4.1.2(e), we have for each s the corresponding slope $\leq h$ decomposition on cohomology: $H^*(I, \widetilde{M}[s]) = H^*(I, \widetilde{M}[s])_h \oplus H^*(\widetilde{C}^*[s]')$.

We now use the following lemma, to be proved later.

Lemma 5.5.4 *There exists a lift H_π such that on the finite slope parts with respect to H_π, the transition maps $\widetilde{C}^*(I, \widetilde{M}[s])^{\#} \to \widetilde{C}^*(I, \widetilde{M}[st])^{\#}$ are isomorphisms.*

Since the slope $\leq h$ part is contained in the finite slope part, we get that $\widetilde{C}^*[s]_h \to \widetilde{C}^*[st]_h$ are isomorphisms.

Now $\widetilde{C}^*(I, \widetilde{M}) = \lim \widetilde{C}^*(I, \widetilde{M}[s]) = \lim \widetilde{C}^*[s]_h \oplus \lim \widetilde{C}^*[s]'$. We claim this is a slope $\leq h$ decomposition. In definition 4.6.3, (1) is clear since $\widetilde{C}^*(I, \widetilde{M}[s])$ is a finitely generated R-module for any s and they are all isomorphic, so $\lim \widetilde{C}^*(I, \widetilde{M}[s])$ is a finitely generated R-module. As for (2), let $Q(T)$ be a polynomial as in definition 4.6.3. We must show that $Q^*(H_\pi)$ induces an isomorphism $\lim \widetilde{C}^*[s]' \to \lim \widetilde{C}^*[s]'$. This is clear because $Q^*(H_\pi)$ induces an isomorphism on each term of the limit.

From Proposition 4.1.2(e) we obtain the corresponding slope $\leq h$ decomposition of the cohomology: $H^*(I,\widetilde{M}) = \lim H^*(I,\widetilde{M}[s]) = \lim H^*(I,\widetilde{M}[s])_h \oplus H^*(\lim \widetilde{C}^*[s]')$. Note that in the slope $\leq h$ part of the right hand side, taking cohomology commuted with projective limits because all the transition maps were isomorphisms.

It is now clear that (39) and (40) hold for cohomology and that the analogous statements hold on the level of cochains.

Remark: We do not assert any isomorphism for the transition maps on the non-finite slope part of the cochains or cohomology, nor is one likely to hold.

Proof of Lemma 5.5.4: First we have to prove a sublemma.

Sublemma 5.5.5 *Let R be a Banach algebra over a finite extension K of $\mathbb{Q}_p$ and let A and B be $R[U]$-modules. Let $A^{\#}$ and $B^{\#}$ be the finite slope subspaces (with respect to U) of A and B respectively.*

Suppose $h : A \to B$ is an $R[U]$-morphism and that $u : B \longrightarrow A$ is an R-morphism such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{U} & A \\ h\downarrow & \overset{u}{\nearrow} & h\downarrow \\ B & \xrightarrow{U} & B \end{array}$$

is commutative. Then the map $h : A^{\#} \to B^{\#}$ is an isomorphism.

Proof: It is enough to show that for any Fredholm polynomial $Q \in R[T]$ that $h : A_Q \to B_Q$ is an isomorphism. Write $Q(T) = 1 - TP(T)$ so that $UP(U) = 1$ on C_Q for any $R[U]$-module C.

To see that h is injective, let $a \in A_Q$ such that $ha = 0$. Then $a = UP(U)a = uhP(U)a = uP(U)ha = 0$.

To see that h is surjective, let $b \in B_Q$. Then $b = UP(U)b = huP(U)b$ so all we need to show is that $\alpha := uP(U)b$ is in A_Q. But $hQ(U)\alpha = Q(U)h\alpha = Q(U)b = 0$ and since h is injective, we get that $Q(U)\alpha = 0$. This completes the proof of the sublemma.

Now recall from section 2.6 the cochain maps

$$f : C^*(\widetilde{M}) \longrightarrow \widetilde{C}^*(\widetilde{M}) \qquad \text{and} \qquad g : \widetilde{C}^*(\widetilde{M}) \longrightarrow C^*(\widetilde{M}).$$

For any $s \in \Lambda^+$, let $\widetilde{M}(s) = \bigoplus_{t \geq s} \widetilde{M}[t]$. Then we have from (35) applied to H^0 the commutative diagram

$$\begin{array}{ccc} C^*(\widetilde{M}(s\pi)) & \xrightarrow{U} & C^*(\widetilde{M}(s\pi)) \\ h\downarrow & \overset{u}{\nearrow} & \downarrow h \\ C^*(\widetilde{M}(s)) & \xrightarrow{U} & C^*(\widetilde{M}(s)) \end{array} \tag{41}$$

where U and u are given at (33) and (34) (applied to the cochains which can be interpreted as an H^0: $C^*(\widetilde{M}) = H^0(I, \mathrm{Hom}_{\mathbb{Z}}(S_*, \widetilde{M}))$ – see Section 2.3) and the vertical maps h are induced by the maps h_π on the coefficient modules (see Definition 5.2.2 and Proposition 5.3.1). In particular, U is a lift of the Hecke operator T_π to the cochain level as given for example by (§2.4(5)).

Now consider the diagram

$$\begin{tikzcd} \widetilde{C}^*(\widetilde{M}(s\pi)) \arrow[r, "\widetilde{U}"] \arrow[d, "h"'] & \widetilde{C}^*(\widetilde{M}(s\pi)) \arrow[d, "h"] \\ \widetilde{C}^*(\widetilde{M}(s)) \arrow[r, "\widetilde{U}"] \arrow[ur, "\widetilde{u}"] & \widetilde{C}^*(\widetilde{M}(s)) \end{tikzcd}$$

where the vertical maps h are induced by h_π on the coefficient modules and

$$\begin{aligned} \widetilde{u} &= f \circ u \circ g \\ \widetilde{U} &= h \circ \widetilde{u}. \end{aligned}$$

Note that $\widetilde{U}$ also is a lift of the Hecke operator T_π to the cochain level because h commutes with f and $hu = U$: thus $\widetilde{U} = hfug = fhug = fUg$.

So to complete the proof of the lemma, we need to check that the above diagram is commutative, i.e. that

$$\widetilde{u} \circ h = \widetilde{U} \quad \text{on } \widetilde{C}^*(\widetilde{M}(s\pi)).$$

This is achieved by a straightforward computation. Explicitly, let $\varphi \in \widetilde{C}^*(\widetilde{M}(s\pi))$ be arbitrary and set $\xi := g(\varphi) \in C^*(\widetilde{M}(s\pi))$. We have

$$\begin{aligned} (\widetilde{u} \circ h)(\varphi) &= (h \cdot \varphi)|\widetilde{u} \\ &= f\big(g(h \cdot \varphi)|u\big) \\ &= f\left((h \cdot g(\varphi)) \bigm| u\right) \\ &= f\big((h \cdot \xi) \bigm| u\big) \\ &= f\big(h \cdot (\xi \bigm| u)\big) \quad \text{(by the commutativity of (41))} \\ &= h \cdot f\big(\xi \bigm| u\big) \\ &= h \cdot f\big(g(\varphi) \bigm| u\big) \\ &= h \cdot \big(f \circ u \circ g(\varphi)\big) \\ &= h \cdot \big(\varphi|\widetilde{u}\big) \\ &= \varphi|\widetilde{U}. \end{aligned}$$

Thus we can take $H_\pi = \widetilde{U}$. This completes the proof.

5.6 Seeding the machine.

To apply the ideas in this section to derive our main theorems, we need the following proposition.

Proposition 5.6.1 *Let $t, s \in \Lambda^+$ and Ω be an affinoid open subset of $\mathcal{X}_T$ such that $s \geq s(\Omega)$ (Definition 3.6.5). If we set $\widetilde{M} = \mathbb{D}_\Omega[s]$ then $\widetilde{M}^t \simeq \mathbb{D}_\Omega(X^t[st])$, $\widetilde{M}[t] \simeq \mathbb{D}_\Omega[st]$ and $\mathfrak{D}(\widetilde{M}) \simeq \mathcal{D}_\Omega$.*

Proof: Recall that by definition, $\mathbb{D}_\Omega[s] = \mathbb{D}_\Omega(X[s]) = \mathbb{D}_\Omega(X[s,s])$. By Proposition 3.6.2, right translation by t induces an isomorphism of strict p-adic manifolds $X[s] \to X^t[s, st]$. If f is a strict analytic function on $X^t[s, st]$, then $t \cdot f$ is a strict analytic function on $X[s]$, where $(t \cdot f)(x) := f(xt)$. We get an isomorphism $\widetilde{M}^t \to \mathbb{D}_\Omega(X^t[s, st])$ by sending $m \mapsto \mu$ according to the formula $\mu(f) = m(t \cdot f)$. It is easy to see this is a map of Σ^t-modules. Then use Lemma 3.6.6 to see that $\mathbb{D}_\Omega(X^t[s, st]) = \mathbb{D}_\Omega(X^t[st])$.

If we induce both sides of the last equality from I^s to I we get that $\widetilde{M}[t] = \mathbb{D}_\Omega[st]$. If we now take the projective limit in t of both sides, we get $\mathfrak{D}(\widetilde{M}) = \mathcal{D}_\Omega$.

6 Theorems of control and comparison

In this chapter, we state and prove our three main theorems: the existence of a controlling characteristic power series for the U-operator; the control theorem which aligns the slope $\leq h$ part of the cohomology of $\mathcal{D}$ with the same part for $\mathcal{D}_k$ for each $k \in \mathcal{X}_T$; and the comparison theorem which for arithmetic k compares the slope $\leq h$ part of the cohomology of $\mathcal{D}_k$ with that of the same part for the finite dimensional V_k.

6.1 A Ring theoretic lemma.

In this section we prove a purely algebraic lemma that enables us to compare systems of Hecke eigenvalues on various cohomology groups in an efficient way. It and its proof are very similar to Theorem 5.1 and its proof in [APS].

For any ring A, define A_{red} to be the reduction of A, i.e. A modulo its nilradical. The following theorem holds true whether we interpret H^* as group cohomology or cohomology of the appropriate Shimura manifold.

Theorem 6.1.1 *Let R be a noetherian ring, (I, Σ) a Hecke pair, and denote the Hecke algebra over R as $\mathcal{H}_R := \mathcal{H}(I, \Sigma) \otimes R$. We assume $\mathcal{H}_R$ is commutative. Let $\mathcal{S}$ be a multiplicative subset of $\mathcal{H}_R$. Let M be an $R[\Sigma]$-module, so that the cohomology $H(M) := H(I, M) := \bigoplus H^*(I, M)$ is an $\mathcal{H}_R$-module.*

Let $\mathcal{I}$ be an ideal in R that is generated by a finite M-regular sequence.

(i) If $H(M)$ has an $\mathcal{S}$-decomposition, then so has $H(M/\mathcal{I}M)$.

(ii) Let $\widetilde{\mathcal{R}}(M) = Im\ (\mathcal{H}_R \to End_R(H(M)_{\mathcal{S}})$ and $\mathcal{R}(M) = \widetilde{\mathcal{R}}(M)_{red}$. Then there is a natural isomorphism

$$(\mathcal{R}(M)/\mathcal{I}\mathcal{R}(M))_{red} \cong \mathcal{R}(M/\mathcal{I}).$$

Proof: Our proof is by induction on the length of an M-regular sequence of generators for $\mathcal{I}$. We begin with the case where $\mathcal{I}$ is principal, generated by an M-regular element $\alpha \in R$, i.e. an element α such that M has no α-torsion. In this case, we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} M \longrightarrow M/\alpha M \longrightarrow 0.$$

(i) Pass to the long exact cohomology sequence and use Proposition (4.1.2)(c) to see that $H(M/\alpha M)$ has an $\mathcal{S}$-decomposition.
(ii) Looking again at the long exact cohomology sequence and using again Proposition (4.1.2)(c), we obtain an exact sequence

$$0 \longrightarrow H/\alpha H \longrightarrow H(M/\alpha M)_{\mathcal{S}} \longrightarrow H[\alpha] \longrightarrow 0$$

where $H := H(M)_{\mathcal{S}}$ and $H[\alpha]$ is the α-torsion in H.

We first construct a homomorphism $\widetilde{\mathcal{R}}(M) \longrightarrow \mathcal{R}(M/\alpha M)$. For this, we let $x \in \mathcal{H}_R$ be such that x annihilates H. Then the last exact sequence implies x^2 annihilates $H(M/\alpha M)_{\mathcal{S}}$. Hence x maps to an element of the nilradical of $\widetilde{\mathcal{R}}(M/\alpha M)$ and it follows that x maps to 0 in $\mathcal{R}(M/\alpha M)$. Thus the canonical map $\mathcal{H}_R \longrightarrow \mathcal{R}(M/\alpha M)$ factors through the canonical map $\mathcal{H} \longrightarrow \widetilde{\mathcal{R}}(M)$. Hence we have a canonical surjective map

$$\tilde{\varphi} : \widetilde{\mathcal{R}}(M) \longrightarrow \mathcal{R}(M/\alpha M).$$

Since
$$(\mathcal{R}(M)/\mathcal{I}\mathcal{R}(M))_{\mathrm{red}} \cong (\widetilde{\mathcal{R}}(M)/\mathcal{I}\widetilde{\mathcal{R}}(M))_{\mathrm{red}},$$
it suffices to show that $\ker(\tilde{\varphi})$ = the radical of $\mathcal{I}\widetilde{\mathcal{R}}(M)$ in $\widetilde{\mathcal{R}}(M)$. The inclusion $\supseteq$ is obvious.

So let $x \in \ker(\tilde{\varphi})$. From the above exact sequence, we conclude that some power of x annihilates $H/\alpha H$. Since we are only trying to prove that some power of x is in $\mathcal{I}\widetilde{\mathcal{R}}(M)$, we may assume that x annihilates $H/\alpha H$. Thus, $xH \subseteq \alpha H$.

Since, by definition of an $\mathcal{S}$-decomposition, H is finitely generated over R, there is a positive integer m such that $H[\alpha^{m+1}] = H[\alpha^m]$. In particular, we see that $\alpha^m H$ has no α-torsion and from this it follows that $\alpha : \alpha^m H \longrightarrow \alpha^{m+1} H$ is an isomorphism of R-modules. Let $\beta : \alpha^{m+1} H \longrightarrow \alpha^m H$ be the inverse map. Let $y \in \mathrm{End}_R(H)$ be the composition
$$y : H \xrightarrow{x^{m+1}} \alpha^{m+1} H \xrightarrow{\beta} \alpha^m H \subseteq H.$$
Then $\alpha y = x^{m+1}$. Note that $\alpha^{m+1} y = \alpha^m x^{m+1}$.

Now define
$$B := \{\, w \in \mathrm{End}_R(H) \mid \exists\; k \geq 0, z \in \widetilde{\mathcal{R}}(M) \text{ such that } \alpha^k w = \alpha^m z \,\}.$$
The endomorphism y constructed in the last paragraph is an element of B. Clearly, B is a finite R-algebra containing $\widetilde{\mathcal{R}}(M)$. Moreover, for every element of $\beta \in B$ we have $\alpha^k \beta \in \widetilde{\mathcal{R}}(M)$ for some $k \geq 0$. Since B is finitely generated as a module over R, there is an exponent N such that $\alpha^N B \subseteq \widetilde{\mathcal{R}}(M)$.

Now consider $x^{(m+1)(N+1)} = (\alpha y)^{N+1} = \alpha(\alpha^N y^{N+1})$. Since $y^{N+1} \in B$ we have $\alpha^N y^{N+1} \in \widetilde{\mathcal{R}}(M)$. Hence $x^{(m+1)(N+1)} \in \alpha\widetilde{\mathcal{R}}(M)$. This proves $x \in Rad_{\widetilde{\mathcal{R}}(M)}(\alpha)$. Hence $\ker(\tilde{\varphi}) \subseteq$ the radical of $\mathcal{I}\widetilde{\mathcal{R}}(M)$ in $\widetilde{\mathcal{R}}(M)$. This completes the proof in the special case where $\mathcal{I}$ is generated by a single M-regular element of R.

Now we suppose $r \geq 1$ and that the theorem is true whenever $\mathcal{I}$ is generated by an M-regular sequence of length r. Let $\alpha_1, \ldots, \alpha_r, \alpha$ be an M-regular sequence of length $r+1$. Let $\mathcal{I}$ be the ideal generated by $\alpha_1, \ldots, \alpha_r$ in Λ and let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ and α.

(i) We have an exact sequence
$$0 \longrightarrow M/\mathcal{I}M \xrightarrow{\alpha} M/\mathcal{I}M \longrightarrow M/\mathcal{J}M \longrightarrow 0.$$
Pass to the long exact cohomology sequence and use Proposition (4.1.2)(c) and the inductive hypothesis to see that $H(M/\mathcal{J}M)$ has an $\mathcal{S}$-decomposition.

(ii) We define ψ to be the composition of the natural surjective homomorphisms
$$\psi : \widetilde{\mathcal{R}}(M) \xrightarrow{\varphi} \widetilde{\mathcal{R}}(M/\mathcal{I}M) \longrightarrow \mathcal{R}(M/\mathcal{J}M).$$
(They are surjective because everything in an $\widetilde{\mathcal{R}}$-ring is induced by a Hecke operator in $\mathcal{H}_R$.)

Let $x \in \ker(\psi)$. Then by the principal case proved in the last paragraph, we have $y := \varphi(x) \in Rad_{\widetilde{\mathcal{R}}(M/\mathcal{I}M)}(\alpha)$. Thus there is an $m \geq 0$ such that $y^m \in \alpha\widetilde{\mathcal{R}}(M/\mathcal{I}M)$. This means $\varphi(x^m) \in \alpha\widetilde{\mathcal{R}}(M/\mathcal{I}M)$. So there is an element $z \in \widetilde{\mathcal{R}}(M)$ such that $\varphi(x^m) = \alpha\varphi(z)$. It then follows from the induction hypothesis that $x^m - \alpha z \in Rad_{\widetilde{\mathcal{R}}(M)}(\mathcal{I})$. There is therefore a positive integer N such that
$$(x^m - \alpha z)^N \in \mathcal{I}\widetilde{\mathcal{R}}(M).$$
From this we see at once that $x^{mN} \in \mathcal{J}\widetilde{\mathcal{R}}(M)$. Hence $x \in Rad_{\widetilde{\mathcal{R}}(M)}(\mathcal{J})$. This proves $\ker(\psi) \subseteq Rad_{\widetilde{\mathcal{R}}(M)}(\mathcal{J})$. But the opposite inclusion is immediate. Hence $\ker(\psi) = Rad_{\widetilde{\mathcal{R}}(M)}(\mathcal{J})$. This completes the proof of Theorem 6.1.1.

6.2 The control theorem.

In this section we prove a result comparing the adelic cohomology with coefficients in the distributions $\mathcal{D}_\Omega$ over an admissible K-affinoid open $\Omega \subset \mathcal{X}_T$ of weight space with the cohomology with coefficients in the specialization $\mathcal{D}_k$ at any point $k \in \Omega(K)$.

Fix a strictly positive elements $\pi \in \Lambda^+$. We get a theorem only on the finite slope part (with respect to the Hecke operator T_π) of the cohomology. More accurately, we don't compare the cohomologies but rather the image of the Hecke algebra $\mathcal{H}$ in the ring of $A(\Omega)$-linear endomorphisms of the slope $\leq h$ part of the cohomology.

Theorem 6.2.1 *Let K be a finite extension of $\mathbb{Q}_p$ and let $k \in \mathcal{X}_T(K)$.*

For any admissible K-affinoid open $\tilde{\Omega} \subset \mathcal{X}_T$ let $R = A(\tilde{\Omega})$ and J_k be the ideal in R consisting of all functions in R that vanish at k. Let (I, Σ, Λ) be the graded Hecke pair from Theorem 2.5.3. Let $\mathcal{H}_R$ be the Hecke algebra $\mathcal{H}(I, \Sigma) \otimes R$.

Let $\mathcal{D}_{\tilde{\Omega}}$ be the $R[\Sigma]$-module defined in Definition 3.6.3, so that the cohomology $H(I, \mathcal{D}_{\tilde{\Omega}}) := \bigoplus H^*(I, \mathcal{D}_{\tilde{\Omega}})$ is an $\mathcal{H}_R$-module.

Fix $h \in \mathbb{Q}^{\geq 0}$ and a strictly positive $\pi \in \Lambda^+$. Let "h-decomposition" mean with respect to the Hecke operator T_π.

For any $R[\Sigma]$-module M such that the cohomology $H(I, M)$ has an h-decomposition, let $\widetilde{\mathcal{R}}(M, h) = Im\ (\mathcal{H}_R \to End_R(H(I, M)_h)$ and $\mathcal{R}(M, h) = \widetilde{\mathcal{R}}(M, h)_{red}$.

Then there exists an admissible K-affinoid open $\Omega \subset \mathcal{X}_T$ containing k such that:

(i) $H(\mathcal{D}_\Omega)$ has an h-decomposition, and so has $H(\mathcal{D}_k)$.

(ii) There is a natural isomorphism

$$(\mathcal{R}(\mathcal{D}_\Omega, h)/J_k\mathcal{R}(\mathcal{D}_\Omega, h))_{red} \cong \mathcal{R}(\mathcal{D}_k, h).$$

Proof: (i) First choose an $\tilde{\Omega}$ and an $s \geq s(\tilde{\Omega})$. We will apply the machinery in section 5.5 to the case where $\widetilde{M} = \mathbb{D}_{\tilde{\Omega}}[s]$ – see Proposition 5.6.1.

Let $\widetilde{U}$ be the lift of the Hecke operator T_π to the cochains $\widetilde{C}^*(\mathbb{D}_{\tilde{\Omega}}[s])$ as defined in the proof of Lemma 5.5.4. Referring to the notation in that proof, we have that $\widetilde{U} = h \circ \widetilde{u}$, where $\widetilde{u} = f \circ u \circ g$ is the Hecke operator τ_π^* induced on the cochains by $\pi : \mathbb{D}_{\tilde{\Omega}}[s] \to \mathbb{D}_{\tilde{\Omega}}[s\pi]$ and formula §5.1(25).

By Proposition 3.6.7, $\mathbb{D}_{\tilde{\Omega}}[s\pi]$ and $\mathbb{D}_{\tilde{\Omega}}[s]$ are ON-able $A(\tilde{\Omega})$-modules, $\pi : \mathbb{D}_{\tilde{\Omega}}[s] \to \mathbb{D}_{\tilde{\Omega}}(X^\pi[s\pi])$ is completely continuous and each element of I induces a map of norm ≤ 1: $\mathbb{D}_{\tilde{\Omega}}[s\pi] \to \mathbb{D}_{\tilde{\Omega}}[s\pi]$.

By the dicussion in the paragraph preceding Proposition 2.7.6, $\widetilde{C}^*(\mathbb{D}_{\tilde{\Omega}}[s])$ is an ON-able $A(\tilde{\Omega})$-module. From Proposition 2.7.6 itself it follows that $\widetilde{u}$ is a completely continuous map. Since $\widetilde{U} = h \circ \widetilde{u}$, $\widetilde{U}$ is a completely continuous endomorpism of $\widetilde{C}^*(\mathbb{D}_{\tilde{\Omega}}[s])$. We can now apply Theorem 4.6.5 to deduce the existence of Ω as in the statement of our theorem such that the cochains $\widetilde{C}^*(\mathbb{D}_\Omega[s])$ have an h-decomposition.

Then combining Proposition 5.6.1 and Theorem 5.5.3 we obtain a natural isomorphism on the slope $\leq h$ parts (with respect to $\widetilde{U}$) of the cochains:

$$\widetilde{C}^*(I, \mathcal{D}_{\tilde{\Omega}})_h \simeq \widetilde{C}^*(I, \mathbb{D}_{\tilde{\Omega}}[s])_h.$$

Note that $R = A(\Omega)$ is a noetherian ring. Recall that an h-decomposition is a type of $\mathcal{S}$-decomposition where $\mathcal{S}$ is the multiplicative subset of $\mathcal{H}_R$ defined in Lemma 4.6.4. So by Proposition 4.1.2(e) we can pass to cohomology and obtain isomorphisms:

$$H(I, \mathcal{D}_{\tilde{\Omega}})_h \simeq H(I, \mathbb{D}_{\tilde{\Omega}}[s])_h.$$

Since $\mathcal{X}_T$ is a disjoint union of open balls, it is easy to that J_k is generated by a finite regular sequence in $A(\Omega)$. Since $\mathcal{D}_\Omega \approx \mathcal{D}(N)\hat{\otimes}_K A(\Omega)$ (see the proof of Lemma 3.6.6) it follows that this is also a $\mathcal{D}_\Omega$-regular sequence.

From the exact sequence of Theorem 3.7.4, we deduce that $\mathcal{D}_k \simeq \mathcal{D}_\Omega/J_k\mathcal{D}_\Omega$. It then follow from (i) of Theorem 6.1.1 that $H(\mathcal{D}_k)$ also has an h-decomposition.

(ii) This follows immediately from (ii) of Theorem 6.1.1.

6.3 The existence of a controlling Fredholm power series.

In this section we prove a global result. This is the result promised as Proto-theorem 4.7.3. We state and prove the result only for $t = 1$, leaving the generalization to the reader.

Theorem 6.3.1 *Fix a strictly positive element $\pi \in \Lambda^+$. Then there exists a Fredholm power series entire on weight space $\mathcal{X}_T$ and with coefficients in the Iwasawa algebra Λ_T which controls (Definition 4.7.2) the finite slope part (with respect to the Hecke operator T_π) of the cohomology of the Shimura manifold $\mathbf{M}_K$ (§2.1(2)) with coefficients in the sheaf of distributions $\mathcal{D}$.*

Proof: Let $\widetilde{U}$ be the lift of T_π to the cochain level given in Lemma 5.5.4. For each Ω as in the conclusion of Theorem 6.2.1, and for any $s \geq s(\Omega)$, let $P_\Omega^{[s]}(T)$ be the characteristic Fredholm series of $\widetilde{U}$ acting on the cochains $\widetilde{C}^*(I, \mathbb{D}_\Omega[s])$. From the construction of $P_\Omega^{[s]}(T)$ in terms of the determinants of matrices with respect to an ON basis coming from monomials in the coordinate functions on N^s, it is clear that each coefficient of $P_\Omega^{[s]}(T)$ is itself a power series over Ω with coefficients in $\mathbb{Q}_p$, and that it has norm ≤ 1 in $A(\Omega)$ and it is entire. By Theorem 5.5.3, we see that this power series in independent of s, so we call it simply $P_\Omega(T)$.

If $\Omega_2 \subset \Omega_1$, by construction it is obvious that $P_{\Omega_1}|\Omega_2 = P_{\Omega_2}$. By the discussion in the paragraph preceding Definition 4.7.2, it follows that the $P_\Omega(T)$ glue together into an entire power series $P(T) \in \Lambda_T$. We claim this is a controlling Fredholm series for $\widetilde{U}$ acting on the cochains: $\widetilde{C}^*(I, \mathcal{D})$.

We have already checked (a) of Definition 4.7.2. From the proof of Theorem 4.6.5, we see that (b) and (c) of that definition also hold on the cochain level, given the fact that $\widetilde{C}^*(I, \mathcal{D}_\Omega)_h \to \widetilde{C}^*(I, \mathbb{D}_\Omega[s])_h$ is an isomorphism for each h, Ω and $s \geq s(\Omega)$ (by Shapiro's lemma for cochains, Theorem 5.5.3 and Proposition 5.6.1). Then (c) on the cohomology level follow immediately.

6.4 The comparison theorem.

In this section we connect our main theorems with automorphic cohomology, that is, the cohomology of the Shimura manifold with coefficients in a finite dimensional module.

Theorem 6.4.1 *Fix a strictly positive elements $\pi \in \Lambda^+$. Fix an arithmetic weight $k \in \mathcal{X}_T(K)$ locally algebraic of level s and with algebraic character ψ (definition 3.5.5). Let $h < m_\psi(\pi)$ §3.11(21). Then the map $\mathcal{D}_k^s \to V_k^s$ in §3.11(20) induces an isomorphism on the slope $\leq h$ part (with respect to T_π) of the adelic cohomology:*

$$H^*(I^s, \mathcal{D}_k^s)_h \xrightarrow{\sim} H^*(I^s, V_k^s)_h.$$

Under these conditions, $\mathcal{R}(\mathcal{D}_k^s, h) = \mathcal{R}(V_k^s, h)$ in the notation of Theorem 6.2.1.

Proof: We have the exact sequence of §3.11(20):

$$0 \longrightarrow \mathbb{K}_k^s \longrightarrow \mathbb{D}_k^s[s] \longrightarrow V_k^s \longrightarrow 0$$

of K-Banach modules of level s. From Theorem 3.11.1, we have that the norm of π on $\mathbb{K}_k^s$ is $\leq p^{-m_\psi(\pi)}$. We pass to the long exact sequence of cohomology:

$$\cdots \to H^j(I^s, \mathbb{K}_k^s) \longrightarrow H^j(I^s, \mathbb{D}_k^s[s]) \longrightarrow H^j(I^s, V_k^s) \longrightarrow H^{j+1}(I^s, \mathbb{K}_k^s) \to \cdots$$

Just as in the proof of Theorem 6.2.1, we see that $H^*(I^s, \mathcal{D}_k^s) \simeq H^*(I^s, \mathbb{D}_k^s[s])$ has a slope $\leq h$ decomposition (with respect to T_π) and by elementary linear algebra, so does $H^*(I^s, V_k^s)$, which is a finite dimensional vector space over K.

It follows from Proposition 4.1.2(c) that $H^j(I^s, \mathbb{K}_k^s)$ also has a slope $\leq h$ decomposition and we have the exact sequence

$$\cdots \to H^j(I^s, \mathbb{K}_k^s)_h \longrightarrow H^j(I^s, \mathbb{D}_k^s[s])_h \longrightarrow H^j(I^s, V_k^s)_h \longrightarrow H^{j+1}(I^s, \mathbb{K}_k^s)_h \to \cdots$$

Let $P_k(T)$ be the specialization at k of the controlling Fredholm power series from Theorem 6.3.1. Then it has a slope $\leq h$ factorization $P_k = QR$ and Q is a polynomial of slope $\leq h$ such that $Q^*(T_\pi)$ annihilates $H^*(I^s, \mathbb{D}_k^s[s])_h$. There is another polynomial Q_1 of slope $\leq h$ such that $Q_1^*(T_\pi)$ annihilates $H^*(I^s, V_k^s)_h$.

By Proposition 2.7.5 both $Q^*(T_\pi)$ and $Q_1^*(T_\pi)$ act invertibly on $H^*(I^s, \mathbb{K}_k^s)$ and hence on $H^*(I^s, \mathbb{K}_k^s)_h$. Now let $x \in H^j(I^s, \mathbb{K}_k^s)_h$. Then $x|Q^*(T_\pi)$ goes to 0 in $H^j(I^s, \mathbb{D}_k^s[s])_h$. Therefore it is the image of some $y \in H^{j-1}(I^s, V_k^s)_h$. It follows that $x|Q^*(T_\pi)Q_1^*(T_\pi)$ is the image of $y|Q_1^*(T_\pi) = 0$. Hence $x|Q^*(T_\pi)Q_1^*(T_\pi) = 0$ and so $x = 0$.

We have shown that $H^*(I^s, \mathbb{K}_k^s)_h = 0$ and the theorem follows.

Let K and k be as in Theorem 6.4.1 and Ω, h as in Theorem 6.2.1. If we have a K-point of the reduced Hecke algebra $\xi : \mathcal{R}(\mathcal{D}_\Omega, h) \to K$, we say that ξ has weight k if ξ factors through $\mathcal{R}(\mathcal{D}_\Omega, h)/J_k\mathcal{R}(\mathcal{D}_\Omega, h)$.

We say that ξ is arithmetic of weight k if the system of Hecke eigenvalues $\{\xi(T) \mid T \in \mathcal{H}\}$ occurs in the finite-dimensional arithmetic cohomology $H^*(I^s, V_k^s)$. That is, there exists an eigenclass $\phi \in H^*(I^s, V_k^s)$ such that for any $T \in \mathcal{H}$, $\phi|T = \xi(T)\phi$.

Putting together Theorem 6.2.1 and Theorem 6.4.1 we obtain:

Corollary 6.4.2 *Let $\xi : \mathcal{R}(\mathcal{D}_\Omega, h) \to K$ be a ring homomorphism of weight k, and suppose $h < m_\psi(\pi)$. Then ξ is arithmetic of weight k.*

Proof: Note that by Proposition 3.6.2, $\mathcal{D}_\Omega \simeq \mathcal{D}_\Omega^s$ and $\mathcal{D}_k \simeq \mathcal{D}_k^s$. From the definition we have $\xi : \mathcal{R}(\mathcal{D}_\Omega, h)/J_k\mathcal{R}(\mathcal{D}_\Omega, h) \to K$. By Theorem 6.2.1, we have $\mathcal{R}(\mathcal{D}_\Omega, h)/J_k\mathcal{R}(\mathcal{D}_\Omega, h) \simeq \mathcal{R}(\mathcal{D}_k^s, h)$. By Theorem 6.4.1, $\mathcal{R}(\mathcal{D}_k^s, h) \simeq \mathcal{R}(V_k^s, h)$.

Therefore, ξ factors through $\mathcal{R}(V_k^s, h) \to K$, which by linear algebra gives the corollary.

As we let k vary so that the highest weight ψ gets large in an archimedean sense, a fixed h becomes less then $m_\psi(\pi)$. Specialization at such k gives us a map $\mathcal{R}(\mathcal{D}_\Omega, h) \to \mathcal{R}(V_k^s, h)$. If $\Xi : \mathcal{R}(\mathcal{D}_\Omega, h) \to A(\Omega)$ is any family of Hecke generalized eigenvalues, its specialization at k will either be the 0 map, or it will yield a ring homomorphism $\xi : \mathcal{R}(\mathcal{D}_k^s, h) \to K$ of weight k. (A ring homomorphism has to take 1 to 1.) The corollary then implies that if specialized to a sufficiently large k (in the archimedean sense), Ξ will become either 0 or arithmetic.

Conversely, any system of Hecke eigenvalues occurring in $\mathcal{R}(V_k^s, h)$ for k sufficiently large will lift to a ring homomorphism $\xi : \mathcal{R}(\mathcal{D}_\Omega, h) \to K$. Therefore, the spectrum of $\mathcal{R}(\mathcal{D}_\Omega, h)$ will be our candidate for an eigenvariety of slope $\leq h$ in the neighborhood of an arithemtic k, as long as h is sufficiently large.

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