K-Theory and Eisenstein series

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1 Background on Distributions

1.1 Basic Definitions

Throughout these notes, V will be a finite dimensional $\mathbb{Q}$-vector space. By a lattice in V we mean a finitely generated sybmodule that generates V as a $\mathbb{Q}$-vector space. A subset $X \subseteq V$ is said to be *bounded* if X is contained in a lattice. For a given function f on V and a vector $\ell \in V$ we say ℓ is a period of f if $f(x+\ell) = f(x)$ for all $x \in V$. The set of all periods of f will be denoted L_f. If L_f is a lattice, we say f is uniformly locally constant.

We define the group of *test functions* on V to be the group

$$\mathcal{S}(V) := \{\, f : V \longrightarrow \mathbb{Z} \,|\, f \text{ is uniformly locally constant and of bounded support} \,\}$$

For A an arbitrary abelian group, we define the group of A-valued distributions on V to be the group

$$\mathrm{Dist}(V, A) := \mathrm{Hom}(\mathcal{S}(V), A).$$

To keep track of support and period lattices we introduce the partially ordered set

$$\mathcal{L}_V := \left\{ (M, L) \;\middle|\; L \subseteq M \text{ is an inclusion of lattices in } V \right\},$$

where for two elements $\Lambda_i = (M_i, L_i) \in \mathcal{L}_V$ $(i = 1, 2)$, we say

$$\Lambda_1 \leq \Lambda_2 \Longleftrightarrow L_2 \subseteq L_1 \subseteq M_1 \subseteq M_2.$$

We also define

$$\deg(\Lambda) := [M : L]$$

which is, of course, a positive integer. We sometimes speak of $\mathcal{L}_V$ as the category whose objects are the elements of $\mathcal{L}_V$ and whose morphisms $\Lambda_1 \longrightarrow \Lambda_2$ are inequalities $\Lambda_1 \leq \Lambda_2$.

For $f \in \mathcal{S}(V)$ we let M_f be the lattice generated by the support of f and set

$$\Lambda_f := (M_f, L_f) \in \mathcal{L}_V.$$

For each Λ in $\mathcal{L}_V$ we then define

$$\mathcal{S}_V(\Lambda) = \left\{ f \in \mathcal{S}(V) \;\middle|\; \Lambda \geq \Lambda_f \right\},$$

which is a free finitely generated abelian group of rank $\deg(\Lambda)$. Indeed, for $\Lambda = (M, L)$ we have a natural identification

$$\mathcal{S}(\Lambda) \cong \{\, f : M/L \longrightarrow \mathbb{Z} \,\}.$$

If $\Lambda_i = (M_i, L_i)$ $(i = 1, 2)$ and $\Lambda_2 \geq \Lambda_1$ then we have an inclusion

$$\mathcal{S}_V(\Lambda_1 \to \Lambda_2) : \mathcal{S}_V(\Lambda_1) \hookrightarrow \mathcal{S}_V(\Lambda_2).$$

Thus

$$\mathcal{S}_V : \mathcal{L}_V \longrightarrow \mathrm{Ab}$$

defines a covariant functor. Moreover, we have a canonical isomorphism

$$\mathcal{S}(V) \cong \varinjlim_{\mathcal{L}_V} \mathcal{S}_V.$$

Now let $\mathcal{A}_V : \mathcal{L}_V \longrightarrow \mathrm{Ab}$ be an arbitrary covariant functor from $\mathcal{L}_V$ to Ab. Then we define

$$\mathrm{Dist}(V, \mathcal{A}_V) := \mathrm{Hom}(\mathcal{S}_V, \mathcal{A}_V)$$

to be the additive group of natural transformations $\mu : \mathcal{S}_V \longrightarrow \mathcal{A}_V$. More precisely, for each $\Lambda \in \mathcal{L}_V$,

$$\mu_\Lambda : \mathcal{S}_V(\Lambda) \longrightarrow \mathcal{A}_V(\Lambda)$$

is an additive homomorphism and for any morphism $\Lambda_1 \longrightarrow \Lambda_2$ the diagram

$$\begin{array}{ccc}
\mathcal{S}_V(\Lambda_2) & \stackrel{\mu_{\Lambda_2}}{\longrightarrow} & \mathcal{A}_V(\Lambda_2) \\
\uparrow & & \uparrow \\
\mathcal{S}_V(\Lambda_1) & \stackrel{\mu_{\Lambda_1}}{\longrightarrow} & \mathcal{A}_V(\Lambda_1).
\end{array}$$

Finally, let Vect_0 be the category whose objects are finite dimensional vector spaces and whose morphisms from $W \longrightarrow V$ are pairs (ι^*, ι) of linear maps $\iota^* : W^* \longrightarrow V^*$ and $\iota : W \longrightarrow V$ such that

$$\Big\langle w^*,\, w \Big\rangle_W = \Big\langle \iota^*(w),\, \iota(w^*) \Big\rangle_V$$

for all $(w^*, w) \in W^* \times W$.

If $M \subseteq V$ is a full lattice in V. We define

$$\mathcal{S}(M) := \left\{\, f \in \mathcal{S}(V) \;\middle|\; \mathrm{supp}\,(f) \subseteq M \,\right\}$$

and let $\chi_M \in \mathcal{S}(M)$ be the characteristic function of M. For each $f \in \mathcal{S}(V)$ we let $f_M := f \cdot \chi_M \in \mathcal{S}(M)$. Clearly, the map $f \mapsto f_M$ is an idempotent on $\mathcal{S}(V)$ with image $\mathcal{S}(M)$. Dually, we also have an idempotent $\mu \mapsto \mu_M$ on $\mathrm{Dist}(V, A)$ given by

$$\mu_M(f) := \mu(f_M)$$

for $f \in \mathcal{S}(V)$. We define

$$\operatorname{Dist}(M, A) := \left\{ \mu \in \operatorname{Dist}(V, A) \;\middle|\; \mu_M = \mu \right\}$$

and note that we have a canonical isomorphism

$$\operatorname{Dist}(M, A) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathcal{S}(M), A).$$

The following simple proposition will be useful in the next section.

Proposition. *The canonical map*

$$\begin{array}{rcl} \mathit{Dist}(V, A) & \longrightarrow & \varprojlim_{M} \mathit{Dist}(M, A) \\ \mu & \longmapsto & \{\mu_M\}_M \end{array}$$

is an isomorphism.

It will sometimes be handy to think functorially. For this it is convenient to introduce the category Vect_0 of finite dimensional vector spaces V over $\mathbb{Q}$ together with *injective* linear maps as morphisms. Then the test functor $\mathcal{S} : V \longmapsto \mathcal{S}(V)$ is a contravariant functor from Vect_0 to the category Ab of abelian groups. Namely, if $W \xrightarrow{\iota} V$ is a morphism in Vect_0 then pullback

$$\mathcal{S}(V) \xrightarrow{\iota^*} \mathcal{S}(W)$$

is a morphism of abelian groups. If moreover, $\mathcal{A} : \mathrm{Vect}_0 \longrightarrow \mathrm{Ab}$ is a covariant functor, then by

$$(V, V') \longmapsto \operatorname{Dist}(V, \mathcal{A}(V'))$$

is a covariant functor $\mathrm{Vect}_0 \times \mathrm{Vect}_0 \longrightarrow \mathrm{Ab}$.

1.2 Fourier Transforms

a natural surjective homomorphism

$$\mathcal{S}_V(\Lambda_1/\Lambda_2) : \mathcal{S}_V(\Lambda_1) \longrightarrow \mathcal{S}_V(\Lambda_2)$$

given by

$$\mathcal{S}_V(\Lambda_1/\Lambda_2)(f)(x + L_2) := \sum_{\substack{y \in M_1/L_1 \\ y + L_1 \subseteq x + L_2}} f(y + L_1)$$

for every $f \in \mathcal{S}_V(\Lambda_1)$ and $x \in M_2$. Then

$$\mathcal{S}_V : \mathcal{L}_V \longrightarrow \mathrm{Ab}$$

is a covariant functor. Let A be an additive group and $\mu \in \operatorname{Dist}(V^*, A)$ be a disrtibution on the dual space V^* of V. Let $L \subseteq V$ be a lattice and $L^* \subseteq V^*$ the dual lattice. We fix an additive character $\psi : \mathbb{Q} \longrightarrow \mu_\infty$ for which $\ker(\psi) = \mathbb{Z}$ and identify $V^*/L^* = L^\vee$ via ψ. The fourier transform of μ on L is the function

$$\widehat{\mu}_L : L^\vee \longrightarrow A$$

defined by

$$\widehat{\mu}_L(\psi_{x^*}) = \mu(x^* + L^*).$$

Clearly, a distribution μ is completely determined by the collection of all its Fourier transforms on all lattices L.

For any inclusion $L \subseteq M$ of lattices in V, we define the norm $N_{M/L} : \mathcal{F}(M^\vee, A) \longrightarrow \mathcal{F}(L^\vee, A)$ by

$$Tr_{M/L}(F)(\chi) = \sum_{\substack{\widetilde{\chi} \in M^\vee \\ \chi'|_L = \chi}} F(\chi').$$

for any $F \in \mathcal{F}(M^\vee, A)$

Proposition. *For any $\mu \in Dist(V^*, A)$ and any inclusion $L \subseteq M$ of lattices in V we have $Tr_{M/L}(\widehat{\mu}_M) = \widehat{\mu}_L$. Moreover, the map*

$$\begin{array}{ccc} Dist(V^*, A) & \longrightarrow & \varprojlim_L \mathcal{F}(L^\vee, A) \\ \mu & \longmapsto & \{\widehat{\mu}_L\}_L \end{array}$$

is an isomorphism.

1.3 Convolution of distributions.

Now suppose A is a ring (not necessarily commutative). For any full lattice $M \subseteq V$ we define the convolution product

$$\begin{array}{rcl} \mathrm{Dist}(M, A) \times \mathrm{Dist}(M, A) & \longrightarrow & \mathrm{Dist}(M, A) \\ \mu, \nu & \longmapsto & \mu * \nu \end{array}$$

as follows. Fix $\mu, \nu \in \mathrm{Dist}(M, A)$ and let $f \in \mathcal{S}(M)$. Fix a period lattice L of f and note that $L \subseteq M$. We may therefore define

$$(\mu * \nu)(f) := \sum_{x,y \in M/L} f(x+y) \cdot \big(\mu(x+L) \cdot \nu(y+L)\big),$$

which one easily checks easily is independent of the choice of period lattice L. Rewriting the sum as a double sum we obtain

$$(\mu * \nu)(f) := \sum_{y \in M/L} \left(\sum_{x \in M/L} f(x+y) \cdot \mu(x+L) \right) \cdot \nu(y+L),$$

which "explains" the usual "double integral notation" for the convolution product

$$\int_M f d(\mu * \nu) := \int_M \left(\int_M f(x+y) d\mu(x) \right) d\nu(y).$$

Convolution on $\mathrm{Dist}(M, A)$ is an associative and distributive operation on $\mathrm{Dist}(M, A)$.

In the above definition of convolution on $\mathrm{Dist}(M, A)$, all the sums are finite. If we try to replace M with the full vector space V, then in general these sums are no longer finite and the above definition of convolution is meaningless.

Let $\mu, \nu \in \mathrm{Dist}(V, A)$. Then for each lattice M we have $\mu_M, \nu_M \in \mathrm{Dist}(M, A)$ and we may convolve these distributions to obtain a distribution $\mu_M * \nu_M \in \mathrm{Dist}(M, A)$.

Definition. We say $\mu, \nu \in \mathrm{Dist}(V, A)$ are *convolvable* if, for every $f \in \mathcal{S}(V)$, the net $\left\{(\mu_M * \nu_M)(f)\right\}_M$ is eventually constant, i.e. if there is a lattice M for which

$$(\mu_{M'} * \nu_{M'})\,(f) = (\mu_M * \nu_M)\,(f)$$

for every lattice M' containing M. If μ, ν are convolvable, then we define $\mu * \nu \in \mathrm{Dist}(V, A)$ by

$$(\mu * \nu)\,(f) := \lim_M\,(\mu_M * \nu_M)\,(f).$$

for $f \in \mathcal{S}(V)$.

1.4 Convolvability of distributions with linearly disjoint supports

Let $X \subseteq V$ be a subspace and $\mu \in \mathrm{Dist}(V, A)$. We define

$$\mathrm{Dist}(X, A) := \left\{\, \mu \in \mathrm{Dist}(V, A) \;\middle|\; \mu(f) = 0 \text{ whenever } f \text{ is supported outside } X \,\right\}.$$

Theorem. *Let $X, Y \subseteq V$ be two subspaces and let $\mu \in$ Dist(X, A) and $\nu \in$ Dist(Y, A). If $X \cap Y = \{\, 0 \,\}$ then μ and ν are convolvable.*

Proof: Indeed, if $f \in \mathcal{S}(V)$ has period lattice L, then we define

$$(\mu * \nu)(f) := \sum_{x,y \in V/L} f(x + y)\mu(x + L) \cdot \nu(y + L).$$

This is a finite sum, which we see as follows. Let M_f be a support lattice for f and let

$$M := (X + M_f) \cap (Y + M_f).$$

One easily checks that M is a lattice containing M_f. Moreover, for all $x \in X$, $y \in Y$ if $x + y \in M_f$ then $x, y \in M$. Thus for every lattice M' containing M we have

$$\begin{aligned}
(\mu_{M'} * \nu_{M'})(f) &= \sum_{x,y \in M'/L} f(x + y)\mu(x + L) \cdot \nu(y + L) \\
&= \sum_{x,y \in M/L} f(x + y)\mu(x + L) \cdot \nu(y + L) \\
&= (\mu_M * \nu_M)(f).
\end{aligned}$$

This proves μ and ν are convolvable, as claimed.

2 Trigonometric Functions

2.1 Circular algebras.

For each $t \in \mathbb{Q}$, we let $q^t : \mathbb{C} \longrightarrow \mathbb{C}^\times$ be defined by $q^t(z) := e^{2\pi i t z}$ for $z \in \mathbb{C}$ and let $\mathfrak{F}$ be the field of meropmorphic functions generated over $\mathbb{Q}$ by the group μ_∞ of roots of unity in $\mathbb{C}$ and by the functions q^t, $t \in \mathbb{Q}$:

$$\mathfrak{F} := \mathbb{Q}\left(\, \mu_\infty,\ q^t \;\middle|\; t \in \mathbb{Q}\,\right).$$

We call $\mathfrak{F}$ the field of trigonometric functions on $\mathbb{C}$. We let $\mathcal{E}, \mathcal{C} \subseteq \mathfrak{F}^\times$ be the subgroups defined by

$$\mathcal{E} := \left\{\, \zeta q^t \;\middle|\; \zeta \in \mu_\infty, t \in \mathbb{Q} \,\right\}, \quad \mathcal{C}_+ := \left\langle 1 - \zeta q^t \;\middle|\; \zeta \in \mu_\infty,\ t \in \mathbb{Q}^+ \,\right\rangle \quad \text{and} \quad \mathcal{C} := \mathcal{E}\mathcal{C}_+$$

and define the ring of trigonometric functions on $\mathbb{Q}$ to be the subring $\mathfrak{R}$ of $\mathfrak{F}$ generated by $\mathcal{C}$:

$$\mathfrak{R} := \mathbb{Z}[u | u \in \mathcal{C}].$$

Note that every $F \in \mathfrak{R}$ defines a holomorphic function on $\mathbb{C} \setminus \mathbb{Q}$.

More generally, let V be a finite dimensional $\mathbb{Q}$-vector space and V^* be the space of $\mathbb{Q}$-linear functionals on V. Let $\mathcal{H}(V)$ be the complement of the union of all rational affine hyperplanes in $V_\mathbb{C}$:

$$\mathcal{H}(V) := \left\{ z \in V_\mathbb{C} \;\middle|\; \forall x^* \in V^*, x^* \neq 0 \Longrightarrow \langle z, x^* \rangle \notin \mathbb{Q} \right\}.$$

For any $0 \neq x^* \in V$ and $\theta \in \mathfrak{R}$ we define $\theta_{x^*} : \mathcal{H}(V) \longrightarrow \mathbb{C}$ by $\theta_{x^*}(z) := \theta\big(\langle z, x^* \rangle\big)$ and define

$$\mathfrak{R}_V := \mathbb{Z}\left[\theta_{x^*} \;\middle|\; \theta \in \mathfrak{R},\ 0 \neq x^* \in V^* \right].$$

We will call $\mathfrak{R}_V$ the ring of trigonometric functions on $\mathcal{H}(V)$. We also define

$$\mathcal{E}_V := \left\langle \epsilon_{x^*} \;\middle|\; \epsilon \in \mathcal{E},\ x^* \in V^* \right\rangle \quad \text{and} \quad \mathcal{C}_V := \left\langle u,\ 1 - u \;\middle|\; u \in \mathcal{E}_V \setminus \mu_\infty \,\right\rangle$$

and we call these the groups of divisible and circular units on $\mathcal{H}(V)$ respectively. The terminology "divisible" is justified by the following proposition.

Proposition. *$\mathcal{E}_V$ is the maximal divisible subgroup of $\mathfrak{R}_V^\times$.*

For $\Lambda \in \mathcal{L}_V$ and $x \in M_\Lambda$ we define $\chi_x : M_{\Lambda^*} \longrightarrow \mathcal{E}_V$ by

$$\chi_x(x^*)(z) := \exp(\langle z - x, x^* \rangle)$$

for $x^* \in M_{\Lambda^*}$ and $z \in \mathcal{H}(V)$. We define

$$\mathcal{E}_V(\Lambda) := \left\langle \chi_x(x^*) \;\middle|\; (x, x^*) \in M_\Lambda \times M_{\Lambda^*} \right\rangle \quad \text{and} \quad \mathcal{C}_V(\Lambda) := \left\langle u, 1 - u \;\middle|\; u \in \mathcal{E}_V(\Lambda) \setminus \mu_\infty \right\rangle.$$

We then define the subring $\mathfrak{R}_V(\Lambda) \subseteq \mathfrak{R}_V$ by

$$\mathfrak{R}_V(\Lambda) := \mathbb{Z}\left[u \;\middle|\; u \in \mathcal{C}_V(\Lambda) \right].$$

Obviously, whenever $\Lambda_1 \leq \Lambda_2$ we have inclusions $\mathcal{E}_V(\Lambda_1) \subseteq \mathcal{E}_V(\Lambda_2)$, $\mathcal{C}_V(\Lambda_1) \subseteq \mathcal{C}_V(\Lambda_2)$ and $\mathfrak{R}_V(\Lambda_1) \subseteq \mathfrak{R}_V(\Lambda_2)$. So we may regard $\mathfrak{R}_V$ as a covariant functor from $\mathcal{L}_V$ to the category of rings:

$$\mathfrak{R}_V : \mathcal{L}_V \longrightarrow \text{Rings}$$

and $\mathcal{E}_V$ and $\mathcal{C}_V$ as covariant functors to Ab.

Now let $K_*(\mathfrak{R}_V(\Lambda))$ be the Milnor ring of $\mathfrak{R}_V(\Lambda)$ and let $\mathfrak{J}_V(\Lambda) \subseteq K_*(\mathfrak{R}_V(\Lambda))$ be the (homogeneous) ideal generated by the image of $\mathcal{E}_V(\Lambda)$ in $K_1(\mathfrak{R}_V(\Lambda))$. Finally, we let

$$\mathcal{K}_V(\Lambda) = \bigoplus_n \mathcal{K}_V(\Lambda)_n$$

be the graded subring of $K_*(\mathfrak{R}_V(\Lambda))/\mathfrak{J}_V(\Lambda)$ generated by the image of $\mathcal{C}_V(\Lambda)$. It follows at once from the above remarks that $\mathcal{K}_V$ is a covariant functor from $\mathcal{L}_V$ to the category of graded rings. We define

$$\mathcal{K}_*(V) := \varinjlim_{\mathcal{L}_V} \mathcal{K}_V$$

and call this the circular algebra of V.

2.2 The Canonical Circular Distribution

When $V = \mathbb{Q}$ we have $\mathcal{H}(V) = \mathbb{C} \setminus \mathbb{Q}$, $\mathfrak{R}_V = \mathfrak{R}$, $\mathcal{E} = \mathcal{E}_V$, and $\mathcal{C}_V = \mathcal{C}$. The multiplication pairing on $\mathbb{Q}$ identifies V^* with $\mathbb{Q}$. For any $u \in \mathcal{C}$ we define $\delta_u : \mathbb{Q} \longrightarrow \mathbb{Z}$ by $\delta_u : x \longmapsto \operatorname{ord}_{z=x} u(z)$. One sees at once that $\delta_u \in \mathcal{S}(\mathbb{Q})$ and that the map

$$\begin{array}{rcl} \delta : \mathcal{C}(\mathbb{Q}) & \longrightarrow & \mathcal{S}(\mathbb{Q}) \\ u & \longmapsto & \delta_u \end{array}$$

is a surjective homomorphism containing $\mathcal{E}(\mathbb{Q})$ in its kernel. Indeed, we have

Proposition. *The kernel of δ is precisely $\mathcal{E}(\mathbb{Q})$, i.e. we have a canonical isomorphism*

$$\delta : \mathcal{C}(\mathbb{Q})/\mathcal{E}(\mathbb{Q}) \xrightarrow{\sim} \mathcal{S}(\mathbb{Q}).$$

We compose the inverse of δ with the isomorphism $\mathcal{C}(\mathbb{Q})/\mathcal{E}(\mathbb{Q}) \cong \mathcal{K}_1(\mathbb{Q})$ to obtain a canonical isomorphism $\xi : \mathcal{S}(\mathbb{Q}) \longrightarrow \mathcal{K}_1(\mathbb{Q})$ which we regard as a $\mathcal{K}_1(\mathbb{Q})$-valued distribution

$$\xi \in \operatorname{Dist}(\mathbb{Q}, \mathcal{K}_1(\mathbb{Q})).$$

We call ξ the canonical circular distribution.

2.3 Circular Distributions on Vector Spaces.

Let V be a finite dimensional vector space. For any ordered basis $\mathbf{v} = (v_1, v_2, \dots, v_n)$ of V and any character $\chi : L_{\mathbf{v}} \longrightarrow R^\times$ define

$$\xi_{\mathbf{v}}(\chi) := \{1 - \chi(v_1), \dots, 1 - \chi(v_n)\}.$$

be an ordered basis for V and let $\mathbf{v}^* = (v_1^*, v_2^*, \dots, v_n^*)$ be the dual basis of V^*. For $N \in \mathbb{N}$ we let we let $\Lambda_{N,\mathbf{v}} := (\frac{1}{N} L_{\mathbf{v}}, L_{\mathbf{v}})$.

Theorem. *There is a unique distribution $\xi_{\mathbf{v}} \in Dist(V, K_{n,V})$ such that for all $N \in \mathbb{N}$, $t = (t_1, \dots, t_n) \in (\mathbb{Q}^+)^n$, and $x \in \frac{1}{N} L_{t\mathbf{v}}$ we have*

$$\xi_{\mathbf{v}}(\Lambda_{N,t\mathbf{v}})(x + L_{t\mathbf{v}}) = \{1 - \chi_x(v_1^*), \dots, 1 - \chi_x(v_n^*)\}$$

We note that the family $t\Lambda_N$ contains a final subset of $\mathcal{L}_V$.

Let $\pi : V \longrightarrow W$ be a surjective $\mathbb{Q}$-linear map. Then π induces a holomorphic surjection

$$\mathcal{H}(\pi) : \mathcal{H}(V) \longrightarrow \mathcal{H}(W).$$

Moreover, π induces a morphism $\mathcal{L}_V \longrightarrow \mathcal{L}_W$ by $\pi(\Lambda) := \big(\pi(M_\Lambda), \pi(L_\Lambda)\big)$ and pull-back by $\mathcal{H}(\pi)$ induces a ring homomorphism

$$\pi^* : \mathfrak{R}_W(\pi(\Lambda)) \longrightarrow \mathfrak{R}_V(\Lambda).$$

Furthermore, we have π^* sends $\mathcal{E}_W(\pi(\Lambda))$ to $\mathcal{E}_V(\Lambda)$ and $\mathcal{C}_W(\pi(\Lambda))$ to $\mathcal{C}_V(\Lambda)$.

To any $a \in \mathbb{Q}^\times$ we associate the morphism $(a, a^{-1}) : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q} \times \mathbb{Q}$ in $\mathrm{Vect}_0 \times \mathrm{Vect}_0$, which by functoriality induces a homomorphism $[a] : \mathrm{Dist}(\mathbb{Q}, \mathcal{K}_1(\mathbb{Q})) \longrightarrow \mathrm{Dist}(\mathbb{Q}, \mathcal{K}_1(\mathbb{Q}))$.

Proposition. *For all $a \in \mathbb{Q}^\times$ we have $[a] \cdot \xi = \xi$.*

Proof: Let $f \in \mathcal{S}(\mathbb{Q})$. Then $\xi(f) = \{u\} \in \mathcal{K}_1(\mathbb{Q})$ for some $u \in \mathcal{C}(\mathbb{Q})$. Thus $\delta_u = f$. Then $([a] \cdot \xi)(f) = a^{-1} \cdot (\xi(f|a))$ $([a] \cdot u)(z) = u([a]^T z) = u(az)$. So $\mathrm{ord}_{\,z=x}([a] \cdot u) = \mathrm{ord}_{\,z=x} u(az) = \mathrm{ord}_{\,z=ax} u(z) = f(ax) = (\delta_u|a)(x) = (f|a)(z)$. Thus $\xi([a] \cdot f) = [a] \cdot u$ Then for all $z \in \mathbb{C}$ we have $([a] \cdot \xi)(f)(z) = \xi(f \circ [a]))([a]^T(z)) = \xi(f \circ a)(az)$

It follows immediately from the definitions that $\xi|a = \xi$ for any $a \in \mathbb{Q}^\times$.

For arbitrary V, and any non-zero element $v \in V$ we let $i_v : \mathbb{Q} \longrightarrow V$ be the unique linear map sending 1 to v. Orthogonal projection from V to the image of i_v then induces a map $pr_v : V \longrightarrow \mathbb{Q}$ with the property $pr_v \circ i_v = 1$. We define $\Phi(v) \in \mathrm{Dist}(V, \mathcal{K}_1(V))$ to be the composition

$$\Phi(v) : \mathcal{S}(V) \xrightarrow{i_v^*} \mathcal{S}(\mathbb{Q}) \xrightarrow{\xi} \mathcal{K}_1(\mathbb{Q}) \xrightarrow{pr_v^*} \mathcal{K}_1(V).$$

A simple induction shows that if $v_1, \dots, v_m$ are linearly independent vectors in V, then the distributions $\Phi(v_i)$ are convolvable and $\Phi(v_1) * \dots * \Phi(v_m)$ is supported on the space spanned by $v_1, v_2, \dots, v_m$. For each m, we let $\mathcal{D}_m(V) \subseteq \mathrm{Dist}(V, \mathcal{K}_m(V))$ be the additive group generated by distributions of the form $\Phi(v_1) * \dots * \Phi(v_m)$ with $v_1, \dots, v_m$ linearly independent and let

$$\mathcal{D}_*(V) := \bigoplus_m \mathcal{D}_m(V) \subseteq \bigoplus_m \mathrm{Dist}(V, \mathcal{K}_m(V)) = \mathrm{Dist}(V, \mathcal{K}_*(V)).$$

For convenience of calculation we record the following proposition.

Proposition. *Let $e_1, \dots, e_n$ be a basis of V and let $L := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ be the lattice spanned by this basis. Let $x = x_1e_1 + \cdots x_ne_n$ be an arbitrary element of V and let f_x be the characteristic function of $x + L$. Then for any $m \le n$ we have*

$$[\Phi(e_1) * \cdots * \Phi(e_m)]\,(f_x) \sim \Big\{1 - \epsilon(-x_1)q_{e_1}, \dots, 1 - \epsilon(-x_m)q_{e_m}\Big\}.$$

where $\sim$ denotes congruence modulo $\mathfrak{J}_V$ in $K_*(\mathfrak{R}_V)$.

Proof. We simply compute. We have

$$\begin{aligned}
&[\Phi(e_1) * \cdots * \Phi(e_m)]\,(f_x)\\
&\quad\sim \sum_{v_1,\dots,v_m \in V/L} f_x(v_1 + \cdots + v_m)\{\Phi(e_1)(v_1 + L),\ \Phi(e_2)(v_2 + L),\ \dots,\ \Phi(e_m)(v_m + L)\}\\
&\quad= \sum_{\substack{v_1,\dots,v_m \in V/L\\ v_1 + \cdots v_m + L = x + L}} \{\Phi(e_1)(v_1 + L),\ \Phi(e_2)(v_2 + L),\ \dots,\ \Phi(e_m)(v_m + L)\}.
\end{aligned}$$

On the other hand, we have for each i,

$$\Phi(e_i)(v_i + L)(z) \;=\; \xi(f_{v_i+L,e_i})(\langle z, e_i\rangle)$$

where, for $v = \sum a_ie_i$, we have $f_{v+L,e_i} \in \mathcal{S}(\mathbb{Q})$ is given by

$$f_{v+L,e_i}(t) = f_{v+L}(te_i) = \begin{cases} 1 & \text{if } t \in a_i + \mathbb{Z} \text{ and } a_j \in \mathbb{Z} \text{ for all } j \neq i;\\ \\ 0 & \text{otherwise.}\end{cases}$$

In particular, we have

$$f_{v+L,e} = \begin{cases} f_{a+\mathbb{Z}} & \text{if } v + L = ae + L \text{ for some } a \in \mathbb{Q};\\ \\ 0 & \text{if } v + L \not\subseteq \mathbb{Q}e + L.\end{cases}$$

Thus, we have

$$\Phi(e_i)(v_i + L)(z) \;=\; \begin{cases} \xi(f_{a+\mathbb{Z}})(\langle z, e_i\rangle) & \text{if } v_i + L = ae_i + L \text{ for some } a \in \mathbb{Q};\\ \\ 0 & \text{if } v_i + L \not\subseteq \mathbb{Q}e_i + L.\end{cases}$$

Moreover $\xi(f_{a+\mathbb{Z}})(\langle z, e_i\rangle) = 1 - \epsilon(-a)q_{e_i}(z)$. So we conclude:

$$
\begin{aligned}
&[\Phi(e_1) * \cdots * \Phi(e_m)]\,(f_x) \\
\sim\quad & \sum_{\substack{v_1,\dots,v_m \in V/L \\ v_1+\cdots v_m+L=x+L}} \{\Phi(e_1)(v_1+L),\, \Phi(e_2)(v_2+L),\, \dots,\, \Phi(e_m)(v_m+L)\}. \\
=\quad & \sum_{\substack{v_i \in (\mathbb{Q}e_i+L)/L \\ v_1+\cdots v_m+L=x+L}} \{\Phi(e_1)(v_1+L),\, \Phi(e_2)(v_2+L),\, \dots,\, \Phi(e_m)(v_m+L)\}. \\
=\quad & \{\Phi(e_1)(x_1e_1+L),\, \Phi(e_2)(x_2e_2+L),\, \dots,\, \Phi(e_m)(x_me_m+L)\}. \\
=\quad & \{1-\epsilon(-x_1)q_{e_1}, 1-\epsilon(-x_2)q_{e_2},\, \dots, 1-\epsilon(-x_m)q_{e_m}\}.
\end{aligned}
$$

This proves the proposition.

Proposition. *For all non-zero $v \in V$ and $a \in \mathbb{Q}^\times$ we have*

$$\Phi(av) = \Phi(v).$$

Proof. Let $f \in \mathcal{S}(V)$ and $z \in V_{\mathbb{C}}$. Then $a := aI \in G$ and by the last proposition $\Phi|a = \Phi$. Hence $\Phi(av)|a = \Phi(v)$.

$$
\begin{aligned}
\Phi(av)(f)(z) &= (\Phi(v)|a^{-1})(f)(z) \\
&= (\Phi(v)(a^{-1}f)(za) \\
&= \xi(\iota_v(a^{-1}f))(za) \\
&= \xi(a^{-1}\iota_v(f))(a\langle z, v\rangle) \\
&= (\xi|a^{-1})(\iota_v(f))(\langle z, v\rangle) \\
&= \xi(\iota_v(f))(\langle z, v\rangle) \\
&= \Phi(v)(f)(z).
\end{aligned}
$$

This proves $\Phi(av) = \Phi(v)$ as claimed.

2.4 Dedekind Reciprocty

For any basis $v_1, v_2, \dots, v_n$ of V, we define

$$\Psi(v_1, \dots, v_n) := \Phi(v_1^*) * \cdots * \Phi(v_n^*).$$

where $v_1^*, \dots, v_n^*$ is the dual basis of V.

Theorem (Dedekind Reciprocity) *Any two elements of $\mathcal{D}_*(V)$ are convolvable, hence $\mathcal{D}_*(V)$ is an algebra under convolution, generated by the elements $\Phi(v) \in \mathcal{D}_1(V)$ where v runs over non-zero elements of V. Moreover, if $v_0, v_1, \dots, v_m \in V$ are non-zero and linearly dependent, we have*

$$\sum_{i=0}^{m} (-1)^i \Psi(v_0, \dots, \widehat{v}_i, \dots, v_m) = 0.$$

The proof is based on the following proposition.

Proposition. *Let $v_1, v_2, \ldots, v_n \in V$ be a basis for V and let $v_0 = v_1 + v_2 + \cdots + v_n$. Then*

$$\sum_{i=0}^{m} (-1)^i \Psi(v_0, \ldots, \widehat{v}_i, \ldots, v_m) = 0.$$

Proof. Let $L = \mathbb{Z}v_1^* + \cdots \mathbb{Z}v_n^* \subseteq V$ be the lattice spanned by $v_1^*, \ldots, v_n^*$. Let $x \in V$ be given by $x = x_1 v_1^* + \cdots x_n v_n^*$. Let $\gamma \in Aut(V)$ be given by

$$\begin{pmatrix} \gamma v_1 \\ \gamma v_2 \\ \cdots \\ \gamma v_m \end{pmatrix} = \begin{pmatrix} v_0 \\ -v_1 \\ \cdots \\ -v_{m-1} \end{pmatrix}.$$

Note that then $\gamma v_0 = v_m$. Then we have

$$\begin{array}{rcl} \Phi(v_0, v_1, \ldots, v_{m-1})(x+L)(z) & = & (\Phi(v_1, \ldots, v_m)|\gamma^{-1})(x+L)(z) \\ & = & (\Phi(v_1, \ldots, v_m)(\gamma x + L)(z\gamma) \end{array}$$

and more generally

$$\begin{array}{rcl} \Phi(v_0, v_1, \ldots, v_{m-1})(x+L)(z) & = & (\Phi(v_1, \ldots, v_m)|\gamma^{-1})(x+L)(z) \\ & = & (\Phi(v_1, \ldots, v_m)(\gamma x + L)(z\gamma) \end{array}$$

3 A Computation in K-theory

Theorem. *Let R be an arbitrary commutative ring. Let $n \geq 1$ and $u_1, \dots, u_n \in R^\times$. Suppose for all $k = 1, \dots, n$ that $u_1 + \cdots u_k \in R^\times$. Then we have the identity*

$$\sum_{i=0}^{n} (-1)^i \{u_0, \dots, \widehat{u_i}, \dots, u_n\}$$

lies in the ideal J generated by $\{-1\}$.

Proof: We prove this by induction on n. The assertion is obvious if $n = 1$. When $n = 2$ we have

$$\frac{u_1}{u_0} + \frac{u_2}{u_0} = 1.$$

Hence

$$\begin{aligned} 0 &= \left\{ \frac{u_1}{u_0}, \frac{u_2}{u_0} \right\} \\ &= \{u_1, u_2\} - \{u_1, u_0\} - \{u_0, u_2\} + \{u_0, u_0\} \\ &= \{u_0, u_1\} + \{u_1, u_2\} + \{u_0, u_2\} + \{u_0, u_0\} \end{aligned}$$

Now add $\{u_0, -1\}$ to both sides of this equation and use the fact that $\{u_0, -u_0\} = 0$ to conclude that the assertion is true for $n = 2$.

Now suppose $m \geq 2$ and the result is true for $n = m$. We will prove the result for $n = m + 1$. Let $v_1, v_2, \dots, v_{m+1} \in R^\times$ and set

$$v_0 = v_1 + \cdots + v_{m+1}.$$

Also, let

$$u_0 = u_1 + \cdots + u_m \quad \text{with } u_1 = v_1, \dots, u_m = v_m.$$

Then we have

$$(*) \qquad \sum_{i=0}^{m} (-1)^i \{u_0, \dots, \widehat{u_i}, \dots, u_m\} \in J$$

Multiplying on the left by $\{v_{m+1}\}$ we get

$$\sum_{i=0}^{m} (-1)^i \{v_{m+1}, u_0, \dots, \widehat{u_i}, \dots, u_m\} \in J$$

and multiplying (*) on the left by $\{v_0\}$ we get

$$\sum_{i=0}^{m} (-1)^i \{v_0, u_0, \dots, \widehat{u_i}, \dots, u_m\} \in J$$

Now use the fact that $v_0 = u_0 + v_{m+1}$ to deduce that

$$\{v_{m+1}, u_0\} - \{v_0, u_0\} \equiv \{v_{m+1}, v_0\} \pmod{J}.$$

Thus subtracting the above two elements of J tells us that

$$\{v_{m+1}, u_1, \ldots, u_m\} + \sum_{i=1}^{m} (-1)^i \{v_{m+1}, v_0, \ldots, \widehat{u_i}, \ldots, u_m\} - \{v_0, u_1, \ldots, u_m\}$$

is in J, or equivalently, after multiplying both sides by $(-1)^m$ and recalling that $v_i = u_i$ for $i = 1, \ldots, m$, we have

$$\{v_1, \ldots, v_m, v_{m+1}\} + \sum_{i=1}^{m} (-1)^i \{v_0, \ldots, \widehat{v_i}, \ldots, v_m, v_{m+1}\} + (-1)^{m+1}\{v_0, v_1, \ldots, v_m\}$$

is in J. This just says

$$\sum_{i=0}^{m+1} (-1)^i \{v_0, \ldots, \widehat{v_i}, \ldots, v_{m+1}\} \in J.$$

Which is the desired assertion for $n = m + 1$. The theorem follows by induction.

3.1 Explicit Formulas

Let $\beta = (v_1, \ldots, v_n) \in \mathcal{B}_V$ (a row of vectors), let $L_\beta \subseteq V$ be the $\mathbb{Z}$-module spanned by β in V and let $L_\beta^* \subseteq V^*$ be the dual lattice of L_β. For each $x^* \in V^*$ let

$$\epsilon_{x^*} : V \longrightarrow \mu_\infty$$

be the character defined by

$$\epsilon_{x^*}(v) = \epsilon(\langle x^*, v\rangle).$$

We also define $\chi_{x^*} : V \longrightarrow \mathcal{E}_V \subseteq \mathfrak{R}_{V^*}^\times$ by

$$\chi_{x^*}(v) = \bar{\epsilon}_{x^*}(v) q_v.$$

Proposition. *Let $\beta = (v_1, \ldots, v_n)$ be a basis of V. Then for every $x^* \in V^*$ we have*

$$\Psi(\beta)(x^* + L_\beta^*) = \Big\{ 1 - \chi_{x^*}(v_1), 1 - \chi_{x^*}(v_2), \ldots, 1 - \chi_{x^*}(v_n) \Big\} \in \mathcal{K}_n(V).$$

The proof of this is a straightforward computation.

Theorem. *Let $\beta = (v_1, v_2, \ldots, v_n)$ be a basis of V, let $1 \leq m \leq n$, and set $v_0 = v_1 + \cdots + v_m$. Then*

$$\sum_{i=0}^{m} (-1)^i \Psi(v_0, \ldots, \widehat{v_i}, \ldots, v_n) = 0.$$

Proof. For each $i = 0, \ldots, m$ let $\beta_i := (v_0, \ldots, \widehat{v_i}, \ldots, v_n)$. Note that $\beta_0 = \beta$ and also that $L_{\beta_i} = L_\beta$ for all $i = 0, \ldots, m$. It suffices to show

$$\sum_{i=0}^{m} (-1)^i \Psi(\beta_i)(x^* + NL_\beta^*) = 0$$

for every $\beta \in \mathcal{B}_V$, $x \in V^*$ and $N \in \mathbb{N}$. But $L^*_{\frac{1}{N}\beta} = NL^*_\beta$, so it suffices to prove the above identity when $N = 1$. We have

$$\begin{aligned}
\sum_{i=0}^{m}(-1)^i \Psi(\beta_i)(x^* + L^*_\beta) &= \sum_{i=0}^{m}(-1)^i \Psi(v_0, \ldots, \widehat{v_i}, \ldots, v_n)(x^* + L^*_\beta) \\
&\equiv \sum_{i=0}^{m}(-1)^i \Big\{1 - \chi_{x^*}(v_0), \ldots, \widehat{1 - \chi_{x^*}(v_i)}, \ldots, 1 - \chi_{x^*}(v_n)\Big\} \\
&\equiv \sum_{i=0}^{m}(-1)^i \big\{u_0, \ldots, \widehat{u_i}, \ldots, u_n\big\} \pmod{\mathfrak{J}_V}
\end{aligned}$$

where $u_0 = 1 - \chi_{x^*}(v_0)$ and $u_i = \chi_{x^*}(w_{i-1}) - \chi_{x^*}(w_i)$ for $i = 1, \ldots, n$, where $w_j = \sum_{0<i\leq j} v_i$ for $j = 0, \ldots, n$. By our computation in K-theory, we see that the last expression vanishes. This completes the proof.

3.2 The Action of $GL(V)$.

Let $G = GL(V)$ act on the left on $\mathfrak{R}_V$ by

$$(\gamma F)(z) = F(z\gamma)$$

This action induces a left action on $\mathcal{K}_*(V)$. We let G act on the left on $\mathcal{S}(V^*)$ by

$$(\gamma f)(z) = f(z\gamma).$$

We then let G act on the right on $\mathrm{Dist}(V^*, \mathcal{K}_*(V))$ by

$$(\mu|\gamma)(f) = \gamma^{-1} \cdot \mu(\gamma f).$$

We let G act on the left on $W = V^* \times V$ by $\gamma(v^*, v) = (v^*\gamma^{-1}, \gamma v)$. We let G act on the right on

$$\mathcal{F}\Big(V^* \times V, \mathrm{Dist}\big(V^*, \mathcal{K}_*(V)\big)\Big) := \Big\{\text{functions } \Phi : V^* \times V \longrightarrow \mathrm{Dist}(V^*, \mathcal{K}_*(V))\Big\}$$

by

$$(\Phi|\gamma)(w) = \Phi(\gamma w)|\gamma.$$

for $w \in V^* \times V$.

Finally, we let $\mathcal{B}_V$ denote the set of ordered bases of V and let G act on the right on

$$\mathcal{F}\Big(\mathcal{B}_V, \mathrm{Dist}\big(V^*, \mathcal{K}_*(V)\big)\Big) := \Big\{\text{functions } \Psi : \mathcal{B}_V \longrightarrow \mathrm{Dist}(V^*, \mathcal{K}(V))\Big\}$$

by

$$(\Psi|\gamma)(\beta) = \Psi(\gamma\beta)|\gamma.$$

Proposition. *Define $\Phi : V^* \times V \longrightarrow Dist(V^*, \mathcal{K}_1(V))$ by*

$$\Phi(v^*, v)(f) \sim \xi(f_{v^*})_v$$

for $f \in \mathcal{S}(V^*)$. Here $f_{v^*} \in \mathcal{S}(\mathbb{Q})$ is given by $f_{v^*}(t) = f(tv^*)$. Then the following are true.

(1) For all $\gamma \in G(\mathbb{Q})$, $\Phi|\gamma = \Phi$.

(2) For all $a \in \mathbb{Q}^\times$, $\Phi(a^{-1}v^*, av) = \Phi(v^*, v)$.

Proof. This is just a computation. Let $\gamma \in G$ and $f \in \mathcal{S}(V^*)$. Then for any $(v^*, v) \in V^* \times V$, we have

$$\begin{array}{rcl} (\Phi|\gamma)(v^*, v)(f) & = & \Big(\Phi(v^*\gamma^{-1}, \gamma v)|\gamma\Big)(f) \\ & = & \gamma^{-1} \cdot \Big[\Big(\Phi(v^*\gamma^{-1}, \gamma v)\Big)(\gamma f)\Big] \\ & = & \gamma^{-1} \cdot \xi\big((\gamma f)_{v^*\gamma^{-1}}\big)_{\gamma v} \\ & = & \xi\big(f_{v^*}\big)_v \\ & = & \Phi(v^*, v)(f). \end{array}$$

So $\Phi|\gamma = \Phi$ and (1) is proved.

We also have

$$\begin{array}{rcl} \Phi(a^{-1}v^*, av)(f) & = & \xi\big(f_{a^{-1}v^*}\big)_{av} \\ & = & \Big((\xi|a)\big(f_{v^*}\big)\Big)_v \\ & = & \xi\big(f_{v^*}\big)_v \\ & = & \Phi(v^*, v)(f) \end{array}$$

proving (2).

Proposition. *Define $\Psi : \mathcal{B}_V \longrightarrow Dist(V^*, \mathcal{K}_n(V))$ by*

$$\Psi(\beta) = \Phi(v_1^*, v_1) * \Phi(v_2^*, v_2) * \cdots * \Phi(v_n^*, v_n)$$

where $\beta = (v_1, \ldots, v_n)$ and $\beta^* = (v_1^*, \ldots, v_n^*)$ is the dual basis. Then we have the following.

(1) For all $\gamma \in G(\mathbb{Q})$ we have $\Psi|\gamma = \Psi$.

(2) For all $a_1, \ldots, a_n \in \mathbb{Q}^\times$ we have $\Psi(a_1v_1, a_2v_2, \ldots, a_nv_n) = \Psi(v_1, \ldots, v_n)$.

(3) For all $\sigma \in S_n$ we have $\Psi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}) = sgn(\sigma) \cdot \Psi(v_1, \ldots, v_n)$

Proof. This is just a computation. Let $\gamma \in G$. For any $\beta = (v_1, \ldots, v_n) \in \mathcal{B}_V$ let $\beta^* = (v_1^*, \ldots, v_n^*) \in \mathcal{B}_{V^*}$ be the dual basis of V^*. Then we have $(\gamma\beta)^* = \beta^*\gamma^{-1}$ and therefore

$$\begin{array}{rcl} (\Psi|\gamma)(\beta) & = & \Psi(\gamma\beta)|\gamma \\ & = & \Big(\Phi(v_1^*\gamma^{-1}, \gamma v_1) * \Phi(v_2^*\gamma^{-1}, \gamma v_2) * \cdots * \Phi(v_n^*\gamma^{-1}, \gamma v_n)\Big) \Big| \gamma \\ & = & \Big(\Phi(v_1^*\gamma^{-1}, \gamma v_1)|\gamma\Big) * \Big(\Phi(v_2^*\gamma^{-1}, \gamma v_2)|\gamma\Big) * \cdots * \Big(\Phi(v_n^*\gamma^{-1}, \gamma v_n)|\gamma\Big) \\ & = & \Big((\Phi|\gamma)(v_1^*, v_1)\Big) * \Big((\Phi|\gamma)(v_2^*, v_2)\Big) * \cdots * \Big((\Phi|\gamma)(v_n^*, v_n)\Big) \\ & = & \Phi(v_1^*, v_1) * \Phi(v_2^*, v_2) * \cdots * \Phi(v_n^*, v_n) \\ & = & \Psi(\beta). \end{array}$$

So $\Psi|\gamma = \Psi$, proving (1). Assertion (2) follows from (2) of the previous proposition and (3) follows from the skew symmetry of multiplication in $\mathcal{K}_*(V)$.

3.3 The group of $\mathfrak{F}^\times$-valued characters on $\mathbb{Q}$

We consider the group

$$\Xi := \mathrm{Hom}(\mathbb{Q}, \mathfrak{F}^\times)$$

of all characters $\psi : \mathbb{Q} \longrightarrow \mathfrak{F}^\times$. Thus each $\psi \in \Xi$ is a function $\psi : \mathbb{Q} \longrightarrow \mathfrak{F}^\times$ satisfying

$$\psi(r+s) = \psi(r) \cdot \psi(s)$$

for all $r, s \in \mathbb{Q}$. Note that Ξ is a group under pointwise multiplication and that the map

$$\Xi \longrightarrow \mathcal{E}$$

defined by $\psi \longmapsto \psi(1)$ is a surjective homomorphism. As examples, we let $Q, \epsilon \in \Xi$ be the characters defined by

$$Q(t) = q^t \quad \text{and} \quad \epsilon(t) := \exp(2\pi i t) \in \mu_\infty.$$

The kernel of the homomorphism

$$\begin{array}{rcl} \mathrm{ord}_\infty : \Xi & \longrightarrow & \mathbb{Q} \\ \psi & \longmapsto & \mathrm{ord}_\infty(\psi(1)) \end{array}$$

is the group

$$\Xi_0 := \left\{ \psi \in \Xi \;\middle|\; \psi(1) \in \mu_\infty \right\}.$$

From the natural identifications

$$\Xi_0 = \mathrm{Hom}(\mathbb{Q}, \mu_\infty) = \mathrm{Hom}_{cont}(\mathbb{A}_f, \mu_\infty)$$

we see that Ξ_0 has a natural structure as $\mathbb{A}_f$-module. Namely, for $\alpha \in \mathbb{A}_f$ and $\psi \in \Xi_0$ we let $\psi^\alpha \in \Xi_0$ be defined by

$$\psi^\alpha(t) = \psi(\alpha t).$$

We also let

$$\Xi_1 := \left\{ \psi \in \Xi \;\middle|\; \psi(1) = 1 \right\}.$$

Then Ξ_1 is a $\widehat{\mathbb{Z}}$-submodule of Ξ_0.

Proposition.

(1) Ξ_0 *is a free rank one* $\mathbb{A}_f$*-module generated by* ϵ.

(2) Ξ_1 *is a free rank one* $\widehat{\mathbb{Z}}$*-module generated by* ϵ.

Moreover, the map

$$\begin{array}{rcl} \mathbb{A}_f \times \mathbb{Q} & \longrightarrow & \Xi \\ (\beta, t) & \longmapsto & \epsilon^\beta \cdot Q^t \end{array}$$

is an isomorphism.

3.4 The group $Aut(\mathfrak{F})$

The group $G := Aut(\mathfrak{F})$ acts naturally on $\Xi := \mathrm{Hom}(\mathbb{Q}, \mathfrak{F}^\times)$ through its action on $\mathfrak{F}^\times$. For each $\sigma \in G$ we define $t \in \mathbb{Q}^\times$, $\beta \in \mathbb{A}_f$, and $u \in \widehat{\mathbb{Z}}^\times$, by

$$\begin{array}{rcl} \sigma(\epsilon) & = & \epsilon^u. \\ \sigma(Q) & = & \epsilon^\beta \cdot Q^t \end{array}$$

We then define

$$\rho(\sigma) := \begin{pmatrix} u & \beta \\ 0 & t \end{pmatrix} \in B(\mathbb{A}) \subseteq PGL_2(\mathbb{A})$$

where B is the group of upper triangular matrices in PGL_2. Note that $B(\mathbb{A})$ is given as the restricted product:

$$B(\mathbb{A}) = B(\mathbb{R}) \times {\prod_p}' B(\mathbb{Q}_p)$$

and that the groups $B(\mathbb{Q}_p)$ are totally disconnected. Thus the connected component $B(\mathbb{A})^0$ of $B(\mathbb{A})$ is given by

$$B(\mathbb{A})^0 = B(\mathbb{R})^0 \times \{1\}$$

where $B(\mathbb{R})^0$ the connected component of $B(\mathbb{R})$, which is the subgroup consisting of matrices with positive determinant.

Theorem. *The map ρ induces an isomorphism*

$$G \cong B(\mathbb{A})/B(\mathbb{A})^0.$$

3.5 Remarks on Fourier Expansions

Every $f \in \mathfrak{F}$ is periodic mod m for some $m \in \mathbb{N}$ and has two "q-expansions"

$$\widetilde{f}_+ := \sum_{\substack{n \in \mathbb{Z} \\ n >> -\infty}} a_n q^{n/m} \in \mathbb{Q}_{ab}((q^{1/m})) \quad \text{and} \quad \widetilde{f}_- := \sum_{\substack{n \in \mathbb{Z} \\ n << \infty}} b_n q^{n/m} \in \mathbb{Q}_{ab}((q^{-1/m}))$$

with coefficients in $\mathbb{Q}_{ab}$, where the first of these expansions converges to f on some upper half-plane and the second converges to f on some lower half-plane. It follows from this that $\mathbb{Q}_{ab}$ is the algebraic closure of $\mathbb{Q}$ in $\mathfrak{F}$.

We write

$$\mathbb{Q}_{ab}((q^t))_+ := \varinjlim_m \mathbb{Q}_{ab}((q^{1/m})) \quad \text{and} \quad \mathbb{Q}_{ab}((q^t))_- := \varinjlim_m \mathbb{Q}_{ab}((q^{-1/m}))$$

and let $G_{ab} := Gal(\mathbb{Q}_{ab}/\mathbb{Q})$ act on both through the action on the coefficients.

The maps

$$\begin{array}{rcl} \mathfrak{F} & \longrightarrow & \mathbb{Q}_{ab}((q^t))_\pm \\ f & \longmapsto & \widetilde{f}_\pm \end{array}$$

are in injective homomorphisms. Moreover the image of each of these inclusions is easily seen to be invariant under the action of G_{ab}. We therefore obtain a natural inclusion

$$G_{ab} \hookrightarrow G = Aut(\mathfrak{F}).$$

4 The Algebra of Simplicial Cones

Let V be a finite dimensional vector space over $\mathbb{Q}$ and let $V^* = \mathrm{Hom}(V, \mathbb{Q})$. For each $0 \neq \lambda \in V^*$ we let $[\lambda]$ denote the positive ray spanned by λ in V^*. We define $C_1(V)$ to be the free $\mathbb{Z}$-module generated formally by the set of all such rays $[\lambda]$. Thus, a typical element of $C_1(V)$ is a finite formal sum $\sum n_{[\lambda]} \cdot [\lambda]$. We define the graded algebra $C_*(V)$ of simplicial cones on V to be the Grassman algebra on $C_1(V)$. More precisely, for each $m \geq 0$ we let

$$C_m(V) := \bigwedge^m \Big(C_1(V) \Big)$$

and let

$$C_*(V) = \bigoplus_{m \geq 0} C_m(V)$$

with the usual $\wedge$-product as multiplication.

We let the group $GL(V)$ act on V on the right, so that $GL(V)$ acts naturally on $C_*(V)$ on the left.

We also have a natural differential $\partial : C_*(V) \longrightarrow C_*(V)$ of degree -1 defined in the usual way by

$$\partial \Big([\lambda_0] \wedge \cdots \wedge [\lambda_m] \Big) = \sum_{i=0}^{m} (-1)^i \Big([\lambda_0] \wedge \cdots \wedge \widehat{[\lambda_i]} \wedge \cdots \wedge [\lambda_m] \Big).$$

5 The Circular Distribution

5.1 The group of circular units

Let V be a finite dimensional vector space over $\mathbb{Q}$ and let $V^* = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. We let $GL(V)$ be the group automorphisms of V and view the elements of $GL(V)$ as acting on the *right*. Let $\mathcal{A}(V)$ denote the ring of all holomorphic functions $\xi : V_{\mathbb{C}} \longrightarrow \mathbb{C}$ that are translation invariant under some lattice $L \subseteq V$. Let $\mathcal{M}(V)$ be the fraction field of $\mathcal{A}(V)$.

We let $GL(V)$ act on the right on $\mathcal{A}(V)$ (and thus also on $\mathcal{M}(V)$) by

$$(F|\sigma)(z) = F(z\sigma^{-1})$$

for $F \in \mathcal{A}(V)$, $\sigma \in GL(V)$, and $z \in V_{\mathbb{C}}$.

Let W denote the set of all pairs $w = (r, \lambda) \in \mathbb{Q} \times V^*$ with $\lambda \neq 0$. For each $w \in W$, define $\epsilon_w \in \mathcal{A}(V)^\times$ by $\epsilon_w(z) = \exp(2\pi i(\lambda z - r))$ and let

$$\begin{aligned} R(V) &:= \mathbb{Z}\left[\epsilon_w \;\middle|\; w \in W\right]; \\ \mathcal{K}(V) &:= \text{the fraction field of } R(V). \end{aligned}$$

We define the group of circular units on V to be the subgroup $\mathcal{C}(V)$ of $\mathcal{K}(V)^\times$ generated by the elements ϵ_w and $1 - \epsilon_w$ as w runs over W:

$$\mathcal{C}(V) := \langle \epsilon_w,\, 1 - \epsilon_w \,|\, w \in W \rangle.$$

Finally, we define the ring

$$\mathcal{R}(V) := R(V)[\mathcal{C}(V)].$$

5.2 Test functions

We define

$$\mathcal{S}(V) := \{\, f : V \longrightarrow \mathbb{Z} \,|\, f \text{ is uniformly locally constant and of bounded support} \,\}$$

These are what we call the test functions on V. We let the group $GL(V)$ act naturally on $\mathcal{S}(V)$ on the left by

$$(\sigma f)(v) = f(v\sigma).$$

5.3 The circular distribution on $\mathbb{Q}$

In this section we take $V = \mathbb{Q}$. Thus $W = \mathbb{Q} \times \mathbb{Q}^\times$. The divisor of a circular unit $u \in \mathcal{C}(\mathbb{Q})$ is defined to be the function $\delta_u : \mathbb{Q} \longrightarrow \mathbb{Z}$ given by

$$\delta_u(v) := \mathrm{ord}_{\,v}(u)$$

where $\mathrm{ord}_{\,v}(u)$ is the order of vanishing of u at v. We note that $\delta_u \in \mathcal{S}(\mathbb{Q})$ for every $u \in \mathcal{C}(\mathbb{Q})$. Moreover, the function $u \longmapsto \delta_u$ defines a homomorphism

$$\delta : \mathcal{C}(\mathbb{Q}) \longrightarrow \mathcal{S}(\mathbb{Q}).$$

The kernel of δ is the group $\mathcal{E}(\mathbb{Q})$ generated by the elements ϵ_w with $w \in W$.

Complex conjugation induces an involution ι on $R(\mathbb{Q})$ (and on $\mathcal{K}(\mathbb{Q})$) by the formula

$$\xi^\iota(z) = \overline{\xi(\overline{z})}.$$

We say that an element of $\xi \in \mathcal{K}(\mathbb{Q})^\times$ is *positive* if (1) $\xi^\iota = \xi$, and (2) the leading coefficient in the Taylor expansion of ξ at the origin is positive. Note that the set of positive elements is a subgroup of $\mathcal{K}(V)^\times$. We let $\mathcal{C}^+ := \mathcal{C}^+(\mathbb{Q})$ denote the group of positive circular units on $\mathbb{Q}$:

$$\mathcal{C}^+ := \left\{ u \in \mathcal{C}(\mathbb{Q}) \;\middle|\; u \text{ is positive} \right\}.$$

One easily verifies that for every $u \in \mathcal{C}(\mathbb{Q})$ there exists one and only one $\epsilon \in \mathcal{E}(\mathbb{Q})$ such that $\epsilon \cdot u$ is positive. The following theorem is an immediate consequence.

Theorem: *The map $\delta : \mathcal{C}^+(\mathbb{Q}) \longrightarrow \mathcal{S}(\mathbb{Q})$ is an isomorphism of abelian groups.*

Proof: A straightforward calculation.

The composition

$$\eta : \mathcal{S}(Q) \xrightarrow{\delta^{-1}} \mathcal{C}^+(\mathbb{Q}) \hookrightarrow \mathcal{C}(\mathbb{Q})$$

defines a $\mathbb{Q}^+$-invariant distribution $\eta \in \mathrm{Dist}(\mathbb{Q}, \mathcal{C}(\mathbb{Q}))$ such that for all $w = (r, \lambda) \in \mathbb{Q} \times \mathbb{Q}^\times$

$$\eta(r + \lambda^{-1}\mathbb{Z})(z) = 2 \sin\left(\frac{|\lambda| z}{2}\right) = -i \cdot \mathrm{sgn}(\lambda) \cdot \epsilon_{-w/2} \cdot (1 - \epsilon_w).$$

We call η the *circular distribution* on $\mathbb{Q}$.

6 The Manin Relations over $GL_2^+(\mathbb{Q})$

Let $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ with the standard fractional linear action of $GL_2(\mathbb{Q})$ (on the left). Let $D_\infty = \{0\} - \{i\infty\} \in \Delta_0$. Then Δ_0 is generated by D_∞ as a $\mathbb{Z}[GL_2^+(\mathbb{Q})]$-module. Moreover, the annihilator of D_∞ in $\mathbb{Z}[GL_2^+(\mathbb{Q})]$ is the left ideal generated by the set

$$\left\{ \gamma - 1 \;\middle|\; \gamma \text{ a diagonal matrix} \right\} \cup \left\{ 1 + \sigma,\ 1 + \rho + \rho^2 \right\}$$

where

$$\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

¿From this we deduce the following proposition.

Proposition: *Let M be a right $GL_2^+(\mathbb{Q})$-module and let $\mu \in M$ be an element for which the following properties hold.*

1. *$\mu|\gamma = \mu$ for every diagonal matrix $\gamma \in GL_2^+(\mathbb{Q})$;*
2. *$\mu|(1 + \sigma) = 0$;*
3. *$\mu|(1 + \rho + \rho^2) = 0$.*

Then there is a unique $GL_2^+(\mathbb{Q})$-invariant homomorphism

$$\Phi : \Delta_0 \longrightarrow M$$

for which $\Phi(D_\infty) = \mu$.

7 Compactly Supported Cohomology.

Let V be a finite dimensional vector space over $\mathbb{Q}$. For $k \geq 0$, we let $(V^*)^{k+1}$ be the $(k+1)$-fold product of V^* and let $(\mathbb{Q}^+)^{k+1}$ act componentwise on $(V^*)^{k+1}$. The orbits of this action will be called *k-simplices on V*. For $\lambda \in (V^*)^{k+1}$ we let $[\lambda]$ denote the k-simplex represented by λ. A simplex $[\lambda_0, \dots, \lambda_k]$ will be called *non-degenerate* if every m-element subset of $\{\lambda_0, \dots, \lambda_k\}$ with $m \leq n$ is linearly independent.

For each $k \geq 0$, we let $C_k(V)$ denote the free abelian group on the set of non-degenerate k-simplices on V. We let

$$C_*(V) := \bigoplus_{k \geq 0} C_k(V)$$

and define a boundary map ∂ on $C_*(V)$ by

$$\partial\big([\lambda_0, \lambda_1, \dots, \lambda_k]\big) := \begin{cases} \displaystyle\sum_{i=0}^{k} (-1)^i [\lambda_0, \dots, \widehat{\lambda}_i, \dots, \lambda_k] & \text{if } k > 0; \\ 0 & \text{if } k = 0. \end{cases}$$

A standard computation shows that the sequence

$$\cdots \xrightarrow{\partial} C_k(V) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(V) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

is exact.

The group $GL(V)$ acts naturally (on the left) on the complex $(C_*(V), \partial)$. For an arbitrary arithmetic subgroup Γ of $GL(V)$ and an arbitrary Γ-module M, we define the complex

$$C_c^*(\Gamma, M) := \mathrm{Hom}_\Gamma(C_*, M).$$

We have the following theorem (of Borel-Serre (??)).

Theorem *Suppose the order of every torsion element of Γ acts invertibly on M. Then there is a canonical isomorphism*

$$H_c^*(\Gamma, M) \cong H(C_c^*(\Gamma, M)).$$

8 K-groups and higher dimensional circular distributions

Let $V := \mathbb{Q}^n$. For each $i = 1, 2, \ldots, n$ let $e_i : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ be projection to the ith coordinate. Define $\eta_i : \mathcal{S}(\mathbb{Q}) \longrightarrow \mathcal{R}_n^\times$ to be the composition of

$$\eta_i : \mathcal{S}(\mathbb{Q}) \xrightarrow{\eta} \mathcal{R}(\mathbb{Q})^\times \xrightarrow{e_i^*} \mathcal{R}_n^\times.$$

Now take the tensor product of the first k of these to get

$$\eta := \eta_1 \otimes \cdots \otimes \eta_k : \mathcal{S}(\mathbb{Q}^k) \longrightarrow \mathcal{R}_n^\times \otimes \cdots \otimes \mathcal{R}_n^\times$$

Now for any sequence $\lambda_1, \ldots, \lambda_k$ of linearly independent linear functionals on $\mathbb{Q}^n$, we may choose $\sigma \in GL_n(\mathbb{Q})$ such that $\lambda_i = \sigma e_i$ and define

$$\widetilde{\Phi}(\lambda_1, \ldots, \lambda_k) := \eta|\sigma^{-1} : \mathcal{S}(\mathbb{Q}^k) \longrightarrow \mathcal{R}_n^{\otimes k}$$

Note: This definition doesn't make sense since σ doesn't act on $\mathcal{S}(\mathbb{Q}^k)$.
Claim: This is independent of the choice of σ.

To see that this is true, we will check that if $\sigma e_i = t_i e_i$ for all i where $t_i \in \mathbb{Q}^\times$, then $\eta|\sigma^{-1} = \eta$. (Note: it appears that we actually need to know $t_i \in \mathbb{Q}^+$.) It suffices to check that if $f_1 \otimes \cdots \otimes f_k \in \mathcal{S}(\mathbb{Q}^k)$ then $(\eta|\sigma^{-1})(f) = \eta(f)$. So we compute:

$$(\eta|\sigma^{-1})(f) = \eta(\sigma^{-1} f)|\sigma^{-1} \in (\mathcal{R}_n^\times)^{\otimes k}$$

So we first compute $\sigma^{-1} f$. For any $x \in \mathbb{Q}^k$ we have

$$\begin{aligned}(\sigma^{-1} f)(x) = f(x\sigma^{-1}) &= \textstyle\prod_{i=1}^k f_i(x\sigma^{-1} \cdot e_i) \\ &= \textstyle\prod_{i=1}^k f_i(t_i^{-1} x_i) \\ &= \textstyle\prod_{i=1}^k (t_i^{-1} f_i)(x_i) \\ &= \left(\bigotimes_{i=1}^k (t_i^{-1} f_i) \right)(x).\end{aligned}$$

Hence

$$\begin{aligned}\eta(\sigma^{-1} f) &= \bigotimes_{i=1}^k \eta_i(t_i^{-1} f_i) \\ &= \bigotimes_{i=1}^k \left(\eta(t_i^{-1} f_i) \circ e_i\right) \\ &= \bigotimes_{i=1}^k \left(\left(\eta|t_i^{-1}\right)(f_i) \right) \Big| \, t_i \circ e_i \\ &= \bigotimes_{i=1}^k \eta(f_i) \,\big|\, t_i \circ e_i\end{aligned}$$

Thus,

$$(\eta(\sigma^{-1}f)|\sigma^{-1} = \left(\bigotimes_{i=1}^{k}\eta(f_i) \mid t_i \circ e_i\right) \Big| \sigma^{-1} = \left(\bigotimes_{i=1}^{k}\eta(f_i) \mid t_i \circ \sigma e_i\right).$$

Finally, we compute each factor at a point $z \in \mathbb{C}^n$:

$$\left(\eta(f_i) \mid t_i \circ \sigma e_i\right)(z) = (\eta(f_i)|t_i)(z\sigma e_i) = (\eta(f_i)|t_i)(t_i z_i) = \eta(f_i)(z_i) = (\eta_i(f_i))(z).$$

Hence

$$\left(\eta(f_i) \mid t_i \circ \sigma e_i\right) = \eta_i(f_i)$$

and therefore

$$(\eta|\sigma^{-1})(f) = \bigotimes_{i=1}^{k}\eta_i(f_i) = \eta(f)$$

and we have proved $\eta|\sigma^{-1} = \eta$, as claimed.

Note that we have actually proved more. Namely, we have proved: For all $t_1, \dots, t_k \in \mathbb{Q}^\times$ (I think $\mathbb{Q}^+$) we have

$$\widetilde{\Phi}(t_1\lambda_1, \dots, t_k\lambda_k) = \widetilde{\Phi}(\lambda_1, \dots, \lambda_k).$$

Now compose $\widetilde{\Phi}$ with projection to $\mathcal{K}_k(\mathcal{R}_n)$ and define

$$\Phi(\lambda_1, \cdots, \lambda_k) : \mathcal{S}(\mathbb{Q}^k) \longrightarrow \mathcal{K}_k(\mathcal{R}_n)$$

as above. If $\lambda_1, \dots, \lambda_k$ are linearly dependent, we define

$$\Phi(\lambda_1, \cdots, \lambda_k) = 0.$$

Theorem. *Φ has good properties.*

For each $\lambda \in V^*$ we view $\lambda : V \longrightarrow \mathbb{Q}$ and use λ to pull-back functions on $\mathbb{C}$ to functions on $V_{\mathbb{C}}$. In particular λ induces a function

$$\mathcal{R}(\mathbb{Q}) \xrightarrow{\lambda^*} \mathcal{R}(V).$$

In particular, λ induces a homomorphism

$$\lambda^* : \mathrm{Dist}(\mathbb{Q}, \mathcal{R}(\mathbb{Q})^\times) \longrightarrow \mathrm{Dist}(\mathbb{Q}, \mathcal{R}(V)^\times).$$

We will make use of the distributions

$$\lambda^*(\eta) \in \mathrm{Dist}(\mathbb{Q}, \mathcal{R}(V)^\times)$$

where $\eta \in \mathrm{Dist}(\mathbb{Q}, \mathcal{R}(\mathbb{Q})^\times)$ is the circular distribution on $\mathbb{Q}$ defined in a previous section. It follows from the $\mathbb{Q}^+$-invariance of η that $\lambda^*(\eta)$ depends only on λ up to multiplication by a positive rational number. Letting $[\lambda] \in C_0(V)$ be the 0-simplex associated to λ we may therefore define

$$\xi([\lambda]) := \lambda^*(\eta).$$

More generally, if $m \geq 0$ and $[\lambda] = [\lambda_0, \ldots, \lambda_m]$ is an m-simplex on V, we define

$$\xi([\lambda]) := \xi([\lambda_0]) \cup \xi([\lambda_1]) \cup \cdots \cup \xi([\lambda_m]) \in \mathrm{Dist}(\mathbb{Q}^{m+1}, K_{m+1}(\mathcal{R}(V))),$$

where the cup product is defined as the composition

$$\mathcal{S}(\mathbb{Q}^{m+1}) \xrightarrow{\sim} \mathcal{S}(\mathbb{Q}) \otimes \cdots \otimes \mathcal{S}(\mathbb{Q}) \xrightarrow{\bigotimes_i \xi([\lambda_i])} \mathcal{R}(V)^\times \otimes \cdots \otimes \mathcal{R}(V)^\times \xrightarrow{\{\ \}_{m+1}} K_{m+1}(\mathcal{R}(V))$$

where K_n is Milnor's nth K-group and $\{\ \}_n$ is Milnor's n-fold symbol.

Now consider the subgroup $\mathfrak{I}_n$ of $K_n(\mathcal{R}(V))$ generated by elements of the form $\{\eta_1, \ldots, \eta_n\}$ with $\eta_i \in \mathcal{C}(V)$ and at least one $\eta_i \in \mathcal{E}(V)$. We define

$$\widetilde{\mathcal{K}}_n := K_n(\mathcal{R}(V))/\mathcal{E}_n(V).$$

Note that $\mathfrak{I}_n$ is the nth graded part of the ideal of the K-ring generated by the elements $\epsilon_w \in K_1(\mathcal{R}(V))$.

Finally, we define the graded ring $\mathcal{D}(V)$ by

$$\mathcal{D}(V) := \bigoplus_{m \geq 0} \mathrm{Dist}(\mathbb{Q}^{m+1}, \widetilde{\mathcal{K}}_{m+1}(V))$$

with multiplication given by cup product. Then ξ may be viewed as a homomorphism

$$\xi : C_*(V) \longrightarrow \mathcal{D}(V).$$

The group $GL(V)$ acts on both $C_*(V)$ (on the left) and on $\mathcal{D}(V)$ (on the right) and we have the following theorem.

"Theorem": *The map ξ is a $GL(V)$-invariant cocycle. More precisely, for any $\sigma \in C_*(V)$ we have*

$$\xi(\partial\sigma) = 0.$$

and for any $\gamma \in GL(V)$ we have

$$\xi(\gamma\sigma)|\gamma = \xi(\sigma).$$

Remarks: In fact, I have not yet written down a proof of this "theorem". What I *have* done is examine the special case $n = m = 2$ in some detail. I am reasonably certain that those computations will generalize as suggested by the theorem. However, there are a few provisos. Possibly, the correct statement for $GL(V)$-invariance is that each element of $GL(V)$ acts by multiplication by the sign of the determinant. It also seems to me that the statement has not yet been properly formulated. What we would really like to do is define a $\mathbb{Z}$-algebra structure on $C_*(V)$ (or something closely related). Then ∂ would be a derivation on this algebra (over $\mathbb{Z}$). We still need to define precisely the algebra structure of $\mathcal{D}(V)$. Then $\xi : C_*(V) \longrightarrow \mathcal{D}(V)$ will be a $GL(V)$-invariant algebra homomorphism whose "derivative" is zero (whatever that means).

The significance of the above theorem lies in the fact that $\mathcal{D}(V)$ is a very rich algebra, admitting lots of interesting homomorphisms to interesting arithmetically defined modules.

For example, if $V = \mathbb{Q}^n$ and we fix a uniform bounded subset $U \subseteq V$ and let Γ be the arithmetic group stabilizing U, then restriction of distributions to U induces a Γ-invariant map

$$\mathcal{D}(V) \xrightarrow{\rho_U} \mathrm{Dist}(U, \widetilde{\mathcal{K}}_n(\mathbb{Q}^n)).$$

Then any Γ-equivariant homomorphism

$$\varphi : \widetilde{\mathcal{K}}_n(\mathbb{Q}^n) \longrightarrow M$$

to a Γ-module M will induce a Γ-invariant map

$$\xi_\varphi : C_*(V) \longrightarrow M$$

and that, in turn may be viewed as an element of $H^*_c(\Gamma, M)$. Moreover, the $GL(V)$-invariance of ξ implies ξ_φ is a Hecke eigenclass and can even be used to compute the eigenvalues. These eigenvalues will be of Eisenstein type. Celia's calculation will arise as a special case of this general principle (Take $n = 2$, $U = (\mathbb{Z}^2)' \setminus (p\mathbb{Z} \times \mathbb{Z})'$, and $\Gamma = \Gamma_0(p)$; then let $\varphi : \widetilde{\mathcal{K}}_2(\mathbb{Q}^2) \longrightarrow \mathbb{F}_p(\omega^r)$ be defined as in Celia's paper.)

Fantasies: The moral of the story is that ξ parametrizes cohomology classes

$$\xi_\varphi \in H^*(\Gamma, M).$$

I expect (fantasize?) that these families are significant. When φ varies over an analytic family, the family ξ_φ should be analytic as well. Since our construction is global, we expect to see Euler systems as special cases. Finally, this picture feels quite general. Similar constructions should work for modular units (in place of circular units). I believe these constructions deserve to respect representation theoretic constructions, hence one might even dream that they point towards a form of Langlands functoriality.