The Jacquet-Langlands correspondence via ℓ-adic uniformization

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1 Introduction

We fix, once and for all, an integer $M > 3$ and two distinct primes p, ℓ with $(p\ell, M) = 1$. We let B denote the quaternion algebra over $\mathbb{Q}$ ramified exactly at $\ell\infty$. We let $\mathcal{R}$ denote a fixed maximal order in B. For each prime q we let $B_q := B \otimes \mathbb{Q}_q$, and $\mathcal{R}_q := \mathcal{R} \otimes \mathbb{Z}_q$. For each $q \neq \ell$ we fix, once and for all, an isomorphism $\iota_q : B \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for which $\iota_q(\mathcal{R} \otimes \mathbb{Z}_q) = M_2(\mathbb{Z}_q)$. Henceforth we will use these isomorphisms to make the identifications

$$B_q = \mathrm{M}_2(\mathbb{Q}_q) \quad \text{and} \quad \mathcal{R}_q = \mathrm{M}_2(\mathbb{Z}_q) \tag{1.1}$$

whenever $q \neq \ell$.

Recall that a Hecke pair in a group G is, by definition, a pair $(\Sigma, \mathcal{K})$ with $\mathcal{K} \subseteq \Sigma \subseteq G$ where $\mathcal{K}$ is a subgroup of G and Σ is a subsemigroup of G that commensurates $\mathcal{K}$. For any rational prime q we define the Hecke pair $(\Sigma_q, \mathcal{K}_q)$ in $\mathbb{GL}_2(\mathbb{Q}_q)$ by

$$\begin{aligned} \Sigma_q \quad &:= \quad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_q) \;\middle|\; \begin{array}{l} ad - bc \neq 0, \\ a \equiv 1, \ b \equiv 0 \pmod{M\mathbb{Z}_q} \end{array} \right\} \\ \mathcal{K}_q \quad &:= \quad \Sigma_q \cap \mathbb{GL}_2(\mathbb{Z}_q). \end{aligned} \tag{1.2}$$

This Hecke pair will play the role of auxiliary tame level structure.

For any positive integer Q, we let $\beta_Q := \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ and define

$$\Sigma_q(Q) := \Sigma_q \cap \beta_Q \Sigma_q \beta_Q^{-1}, \qquad \text{and} \qquad \mathcal{K}_q(Q) := \mathcal{K}_q \cap \beta_Q \mathcal{K}_q \beta_Q^{-1}. \tag{1.3}$$

We then let

$$\mathcal{H}_q(Q) := \mathcal{H}(\Sigma_q(Q), \mathcal{K}_q(Q)) \tag{1.4}$$

be the convolution algebra generated by double cosets on $\Sigma_q(Q)$ with respect to $\mathcal{K}_q(Q)$.

Let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$ and $\mathbb{A}_f := \mathbb{Q} \otimes \widehat{\mathbb{Z}}$ denote the finite adeles of $\mathbb{Q}$. For $Q \in \mathbb{Z}^+$ we define the Hecke pair $(\Sigma(Q), \mathcal{K}(Q))$ by

$$\Sigma(Q) := \prod_{q \neq \ell}{}' \, \Sigma_q(Q), \qquad \text{and} \qquad \mathcal{K}(Q) := \prod_{q \neq \ell} \mathcal{K}_q(Q). \tag{1.5}$$

We may then define Hecke pairs $(\Sigma_{\mathbb{GL}_2}(Q), \mathcal{K}_{\mathbb{GL}_2}(Q))$ and $(\Sigma_{\mathbb{B}^\times}(Q), \mathcal{K}_{\mathbb{B}^\times}(Q))$ in $\mathbb{GL}_2(\mathbb{A}_f)$ and $\mathbb{B}^\times(\mathbb{A}_f)$, respectively, by

$$\Sigma_{\mathbb{GL}_2}(Q) := \Sigma_\ell(Q) \times \Sigma(Q), \qquad \mathcal{K}_{\mathbb{GL}_2}(Q) := \mathcal{K}_\ell(Q) \times \mathcal{K}(Q) \tag{1.6}$$

and

$$\Sigma_{\mathbb{B}^\times}(Q) := \mathcal{R}'_\ell \times \Sigma(Q), \qquad \mathcal{K}_{\mathbb{B}^\times}(Q) := \mathcal{R}_\ell^\times \times \mathcal{K}(Q). \tag{1.7}$$

where $\mathcal{R}'_\ell$ is the subsemigroup of non-zero elements of $\mathcal{R}_\ell$.

Finally, we define the Hecke algebra

$$\mathcal{H}(Q) := \mathcal{H}(\Sigma(Q), \mathcal{K}(Q)) \cong \bigotimes_{q \neq \ell}{}' \, \mathcal{H}_q(Q), \tag{1.8}$$

where the tensor product is the *restricted* tensor product with respect to the choice of the multiplicative identity $1 \in \mathcal{H}_q(Q)$ for each q.

If we wish to emphasize the tame level M, we will write $\mathcal{H}(M, Q)$ instead of simply $\mathcal{H}(Q)$. We note that in more classical language, we have

$$\mathcal{H}(M, Q) = \mathbb{Z}\left[T_q,\ T_{q,q},\ U_r \;\middle|\; q, r \text{ are primes} \neq \ell, \text{ with } q \nmid MQ \text{ and } r | MQ\right].$$

Note that we have omitted Hecke operators at the prime ℓ. In particular, this algebra acts on the cohomology of *both* $\mathcal{K}(Q)$ and $\mathcal{K}_{\mathbb{B}^\times}(Q)$. The Jacquet-Langlands correspondence, a version of which is described in the next section, is a correspondence that identifies systems of eigenvalues of $\mathcal{H}(Q)$ occurring on the quaternionic side and with *certain* systems of Hecke eigenvalues (those that are new at ℓ) occurring on the $\mathbb{GL}_2$ side.

2 Geometric Jacquet-Langlands

2.1 Cohomology of Modular Curves

We fix, once and for all, embeddings: $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell \subseteq \mathbb{C}$. Let $\overline{\mathbb{Z}}_\ell$ be the ring of integers in $\bar{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$ be the residue field. We have the diagram

$$s := \operatorname{Spec}(\overline{\mathbb{F}}_\ell) \longrightarrow S := \operatorname{Spec}(\overline{\mathbb{Z}_\ell}) \longleftarrow \eta := \operatorname{Spec}(\bar{\mathbb{Q}}_\ell). \tag{2.1}$$

For Z any scheme over $\operatorname{Spec}(\mathbb{Z}_\ell)$ we denote by Z_s, Z_S, Z_η, and $Z_\mathbb{C}$ be the base-changes of Z via the morphisms induced by the ring homomorphisms from $\mathbb{Z}_\ell$ to $\overline{\mathbb{F}}_\ell$, $\overline{\mathbb{Z}}_\ell$, $\bar{\mathbb{Q}}_\ell$, and $\mathbb{C}$ respectively.

Now fix a positive integer Q and let $Y := Y_Q$ be the affine modular curve over $\mathbb{Z}_\ell$ with $\mathcal{K}_{\mathbb{GL}_2}(Q)$-level structure. Let

$$\Gamma := \Gamma_Q := \mathbb{GL}_2{}^+(\mathbb{Q}) \cap \mathcal{K}_{\mathbb{GL}_2}(Q) \tag{2.2}$$

We also choose a section y of $Y/\mathrm{Spec}(\mathbb{Z}_\ell)$. Tensoring with s, S and η over $\mathrm{Spec}(\mathbb{Z}_\ell)$ we obtain sections y_s, y_S and y_η of $Y_s/s, Y_S/S$ and Y_η/η, respectively, and we have the following diagram with cartesian squares:

$$\begin{array}{rcrcr} s & \longrightarrow & S & \longleftarrow & \eta \\ y_s \downarrow & & y_S \downarrow & & y_\eta \downarrow \\ Y_s & \longrightarrow & Y_S & \longleftarrow & Y_\eta. \end{array}$$

For $* = s, S$, or η we then let

$$\Delta_* := \pi_1^{et}(Y_*, y_*)$$

and note that we have canonical group homomorphisms

$$\Delta_s \longrightarrow \Delta_S \longleftarrow \Delta_\eta \hookleftarrow \Gamma$$

Proposition 2.1. *Let p be a prime different from ℓ and let F be a finite abelian p-group endowed with a continuous right action of $\mathcal{K}_p(Q)$ (see (1.3)). Let $\widetilde{F}$ be the local coefficient system on $Y_\mathbb{C}$ associated to F. Then there is a unique finite locally constant sheaf $\mathcal{F}_S$ on the etale site Y_S^{et} whose base-change to $\mathbb{C}$ is $\widetilde{F}$.*

Proof. Consider the p-adic Tate module on Y_S

$$Ta_p(E_S/Y_S) = \varprojlim_n E_S[p^n].$$

This is a projective limit of finite etale group schemes $E_S[p^n]$ over Y_S and therefore determines a representation T of Δ_S. Since the action of Δ_S must respect the level structure on E/Y we have a commutative diagram of group homomorphisms

$$\begin{array}{ccc} & & \Gamma \\ & \swarrow & \downarrow \\ \Delta_S & \longrightarrow & \mathcal{K}_p(Q). \end{array} \tag{2.3}$$

in which the horizontal arrow is continuous and the vertical arrow is the natural inclusion. The action of $\mathcal{K}_p(Q)$ on F therefore induces a continuous action of Δ_S on F which classifies a finite locally constant sheaf $\mathcal{F}_S$ on Y_S. The commutativity of the above diagram implies that the base change of $\mathcal{F}_S$ to $\mathbb{C}$ is $\widetilde{F}$. This proves the existence of the desired sheaf $\mathcal{F}_S$.

Since Δ_S acts trivially on $\overline{\mathbb{Z}}_\ell$, it acts trivially on p-power roots of unity and therefore properties of the Weil pairing imply that the image of Δ_S in $\mathcal{K}_p(Q)$ is contained in the subgroup of elements of determinant 1. But the image of Γ in $\mathcal{K}_p(Q)$ is dense. Thus the action of Γ on F determines the action of Δ_S and uniqueness of $\mathcal{F}_S$ follows. This completes the proof. $\square$

Now assume F is a finite abelian p-group endowed with a continuous (right) action of the semigroup $\Sigma_p(Q)$ and let $\Sigma_{\mathbb{GL}_2}$ act on F via projection to $\Sigma_p(Q)$. Under this assumption we may define Hecke operators on the cohomology of the sheaves $\mathcal{F}_*$ $(* = s, S, \eta)$ as follows.

Let q be an arbitrary prime different from ℓ. Then, over S, we have two finite etale morphisms

$$\pi_1, \pi_2 : Y_{Qq} \longrightarrow Y_Q.$$

The first of these is the morphism associated to forgetting the level q-structure, while the second is associated to dividing by the level q-structure. Now let $\beta_q := \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \in \Sigma_{\mathbb{GL}_2}(Qq)$ and define the group homomorphisms

$$i_1, i_2 : \mathcal{K}_{\mathbb{GL}_2}(Qq) \longrightarrow \mathcal{K}_{\mathbb{GL}_2}(Q)$$

by letting i_1 be the natural inclusion and i_2 be conjugation by β_q: $x \longmapsto \beta_q^{-1} x \beta_q$. Moreover, the map $F \xrightarrow{\beta_q} F$ intertwines the i_1-action with the i_2-action of $\mathcal{K}_{\mathbb{GL}_2}(Qq)$, hence induces a morphism $\rho_q : i_1^* F \longrightarrow i_2^* F$.

It follows immediately from the definitions that the finite locally constant sheaves on $(Y_{Qq})_S$ associated to $i_1^*(F)$, $i_2^*(F)$ by Proposition 2.1 are respectively $\pi_1^*(\mathcal{F}_S)$ and $\pi_2^*(\mathcal{F}_S)$. Moreover, ρ_q induces a morphism of sheaves

$$\rho_q : \pi_1^*(\mathcal{F}_S) \longrightarrow \pi_2^*(\mathcal{F}_S).$$

Definition 2.2. Let $Y = Y_Q$ and F be as above. Let $\mathcal{F}_S$ be the finite locally constant sheaf on Y_S associated to F by Proposition 2.1. Then for q any prime different from ℓ and for $* = s, S, \eta$ we define

$$T_q : H_c^*(Y_*, \mathcal{F}_*) \longrightarrow H_c^*(Y_*, \mathcal{F}_*)$$

to be the composition

$$H_c^*(Y_*, \mathcal{F}_*) \longrightarrow H_c^*((Y_0)_*, \pi_1^*\mathcal{F}_*) \xrightarrow{\rho_q} H_c^*((Y_0)_*, \pi_2^*\mathcal{F}_*) \longrightarrow H_c^*(Y_*, \mathcal{F}_*).$$

Let $X := X_Q$ be the complete modular curve of level $\mathcal{K}_{\mathbb{GL}_2}(Q)$ and let $\alpha : Y \hookrightarrow X$ be the associated affine modular curve, everything over $\mathrm{Spec}(\mathbb{Z}_\ell)$. Consider the fundamental diagram

$$\begin{array}{ccccc} Y_s & \xrightarrow{i_Y} & Y_S & \xleftarrow{j_Y} & Y_\eta \\ \downarrow \alpha_s & & \downarrow \alpha_S & & \downarrow \alpha_\eta \\ X_s & \xrightarrow{i_X} & X_S & \xleftarrow{j_X} & X_\eta. \end{array} \tag{2.4}$$

For $* = s, S$, or η and a finite locally constant sheaf $\mathcal{F}_*$ on Y_*^{et} define

$$H_c^n(Y_*, \mathcal{F}_*) := H^n(X_*, \alpha_{*!}(\mathcal{F}_*)). \tag{2.5}$$

2.2 Vanishing Cycles for Modular Curves

In this section we fix a positive integer Q that is not divisible by ℓ^2 and consider the modular curve $X := X_Q$ as well as the affine modular curve $Y := Y_Q$, as defined in the last section. We note that is X is flat and proper over $\mathrm{Spec}(\mathbb{Z}_\ell)$ with connected geometric fibers, smooth generic fiber, and semi-stable reduction, and that we have an open immersion

$$\alpha : Y_S \hookrightarrow X_S$$

whose image is obtained from X_S by removing the finite set C_S of smooth sections of X_S/S associated to the cusps of X_S.

Now fix a locally constant sheaf $\mathcal{F}_S$ of finite abelian p-groups on Y_S and let

$$\mathcal{F}_s := i_Y^*\mathcal{F}_S \quad \text{and} \quad \mathcal{F}_\eta := j_Y^*\mathcal{F}_S.$$

Then $\mathcal{F}_s$, $\mathcal{F}_S$, and $\mathcal{F}_\eta$ are locally constant sheaves on the etale sites Y_s^{et}, Y_S^{et}, and Y_η^{et}, respectively.

Finally, let Σ denote the finite set of singular points in Y_s and let $(\mathcal{F}_s)_\Sigma$ denote the skyscraper sheaf whose stalk at an arbitrary point y of Y_s is given by

$$((\mathcal{F}_s)_\Sigma)_{(y)} = \begin{cases} (\mathcal{F}_s)_{(y)} & \text{if } y \in \Sigma, \\ \\ 0 & \text{if } y \notin \Sigma. \end{cases}$$

Theorem 2.3. *We have a canonical exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_c(Y_s,\mathcal{F}_s) & \longrightarrow & H^1_c(Y_\eta,\mathcal{F}_\eta) & \longrightarrow & H^0(Y_s,(\mathcal{F}_s)_\Sigma) & \longrightarrow \\ & \longrightarrow & H^2_c(Y_s,\mathcal{F}_s) & \longrightarrow & H^2_c(Y_\eta,\mathcal{F}_\eta) & \longrightarrow & 0. & \end{array}$$

Proof. The proof is an immediate consequence of the following two lemmas from the theory of vanishing cycles on proper curves over $\mathbb{Z}_\ell$ with semistable reduction, together with the fact that $H^1(Y_s,(\mathcal{F}_s)_\Sigma)=0$. $\square$

Lemma 2.4. *Let $X_s \overset{i_X}{\hookrightarrow} X_S \overset{\jmath_X}{\hookleftarrow} X_\eta$ be as above and let $\mathcal{L}$ be a constructible sheaf of finite $\mathbb{Z}/p^m\mathbb{Z}$-modules on X_η^{et}. Let*

$$R^n\Psi(\mathcal{L}) := \iota_X^*\big(R^n\jmath_{X,*}(\mathcal{L})\big),\ \ (n\geq 0)$$

be the complex of vanishing cycles sheaves on X_s (see [SGA7]). Then we have a canonical long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(X_s,R^0\Psi(\mathcal{L})) & \longrightarrow & H^1(X_\eta,\mathcal{L}) & \longrightarrow & H^0(X_s,R^1\Psi(\mathcal{L})) \\ & \longrightarrow & H^2(X_s,R^0\Psi(\mathcal{L})) & \longrightarrow & H^2(X_\eta,\mathcal{L}) & \longrightarrow & H^1(X_s,R^1\Psi(\mathcal{L})). \end{array}$$

Proof. Since the morphism $X_S \longrightarrow S$ is proper of relative dimension one, the above exact sequence is induced by the vanishing cycles spectral sequence (see [SGA7] for details). $\square$

Now let $\mathcal{L}_\eta$ and $\mathcal{L}_s$ be the constructible sheaves on X_η^{et} and X_s^{et}, respectively, given by

$$\mathcal{L}_\eta := (\alpha_\eta)_!\mathcal{F}_\eta \quad\text{and}\quad \mathcal{L}_s := (\alpha_s)_!\mathcal{F}_s.$$

Lemma 2.5. *The sheaves $R^n\Psi(\mathcal{L}_\eta)$ $(n\geq 0)$ on X_s are given by*

(a) For each $n\geq 0$ we have

$$R^n\Psi_X((\alpha_\eta)_!\mathcal{F}_\eta)\cong(\alpha_s)_!R^n\Psi_Y(\mathcal{F}_\eta).$$

(b) $R^0\Psi(\mathcal{L}_\eta)=\mathcal{L}_s$;

(c) $R^1\Psi(\mathcal{L}_\eta)=(\mathcal{F}_s)_\Sigma$;

(d) $R^n\Psi(\mathcal{L}_\eta) = 0$ *for* $n \geq 2$.

Proof. First note that $\mathcal{F}_\eta$ is tamely ramified at the "cusps", i.e. at the points of the subscheme C_η of $(X_\eta - \Sigma_\eta)$ associated to the points of $X_\eta \backslash Y_\eta$. Since C_η is smooth over η, Proposition 2.1.9 of [SGA7], XIII applies and (a) follows.

To prove (b) we note that by definition, we have $\mathrm{R}^0\Psi_Y(\mathcal{F}) = (i_Y)^*(j_Y)_*(\mathcal{F}_\eta)$. But $\mathcal{F}_\eta = (j_Y)^*\mathcal{F}_S$ so we have

$$\mathrm{R}^0\Psi_Y(\mathcal{F}_\eta) = (i_Y)^*(j_Y)_*(j_Y)^*\mathcal{F}_S = (i_Y)^*\mathcal{F}_S = \mathcal{F}_s.$$

Now (b) follows from (a).

To prove (c), let y be an s-point of X_s. It follows from (a) that

$$\big(\mathrm{R}^n\Psi_X(\mathcal{L}_\eta)\big)_y = \big((\alpha_s)_!\mathrm{R}^n\Psi_Y(\mathcal{F}_\eta)\big)_y = \begin{cases} 0 & \text{if } y \in C_s; \\ \\ \big(\mathrm{R}^n\Psi_Y(\mathcal{F}_\eta)\big)_y & \text{if } y \in Y_s. \end{cases}$$

If $y \notin \Sigma$ then y is a smooth point and the acyclicity of smooth morphisms property implies that $\big(\mathrm{R}^n\Psi_Y(\mathcal{F}))\big)_y = 0$. It follows that the sheaves $\mathrm{R}^n\Psi_X(\mathcal{L})$ are skyscraper sheaves supported on Σ.

Now suppose $y \in \Sigma$. Let $Z \xrightarrow{f} Y_S$ be an étale neighborhood of y in Y_S such that the restriction of $\mathcal{F}_S$ to Z is constant, i.e. f is a finite étale cover such that $f^*(\mathcal{F}_S)$ is a constant sheaf on Z. Choose z to be an s-point of Z such that $f(z) = y$. Since f is proper and finite, we have

$$(\mathrm{R}^1\Psi_Y(\mathcal{F}_\eta))_y = (f_s^*(\mathrm{R}^1\Psi_Y(\mathcal{F}_\eta)))_z = (\mathrm{R}^1\Psi_Z(f^*\mathcal{F}_S))_z = H^1(Z_{(z)} \otimes_S s, f^*\mathcal{F}_S).$$

As $f^*\mathcal{F}_S$ is constant, equal to $(f^*\mathcal{F}_S)_z = (\mathcal{F}_S)_y$, we have

$$\big(\mathrm{R}^1\Psi_Y(\mathcal{F}_\eta)\big)_y = H^1(Z_{(z)}, \Lambda_Z) \otimes_\Lambda (\mathcal{F}_s)_y = \big(\mathrm{R}^1\Psi_Y(\Lambda_Y)\big)_y \otimes_\Lambda (\mathcal{F}_s)_y,$$

where Λ_Z, Λ_Y are the constant sheaves $\Lambda = \mathbb{Z}/p^m\mathbb{Z}$ on Z^{et} and Y_S^{et} respectively. From Lemma 1.5 [Il], we have $\big(\mathrm{R}^1\Psi_Y(\Lambda_Y)\big)_y = \big(\mathrm{R}^1\Psi_X(\Lambda_X)\big)_y = \Lambda$. Hence, we have

$$\big(\mathrm{R}^1\Psi_Y(\mathcal{F}_\eta)\big)_y = (\mathcal{F}_s)_y$$

as claimed. This proves (c).

To prove (d), we use the fact that for every $y \in \Sigma$ and $n \geq 1$ we have $\big(\mathrm{R}^n\Psi_Y(\mathcal{F})\big)_y \cong H^n(Y_{(y)} \times_S \eta, \mathcal{F})$, where $Y_{(y)}$ is the spectrum of the strict henselization of Y at y. For dimension considerations, if $n \geq 2$ we have $\big(\mathrm{R}^n\Psi_Y(\mathcal{F})\big)_y = 0$ and (d) follows. □

2.3 The ℓ-New Cohomology Groups

We now fix a positive integer N with $\ell \not| N$. We let $X_0 := X_{N\ell}$ and $X := X_N$. Similarly, we let Y_0, Y be the associated affine modular curves over $\mathbb{Z}_\ell$. We have two finite morphisms

$$\pi_\ell,\ \pi'_\ell : Y_0 \longrightarrow Y$$

where π_ℓ is induced by the forgetful functor and $\pi'_\ell = \pi_\ell \circ w_\ell$ where $w_\ell : Y_0 \longrightarrow Y_0$ is the Atkin-Lehner involution. On the special fibers, we also have closed immersions

$$i, i' : Y_s \hookrightarrow Y_{0,s}$$

where i is induced by the adding level-ℓ structure given by the canonical subgroup at ordinary points, and $i' = w_\ell \circ i$.

Now let $\mathcal{F}$ be a locally constant sheaf of finite abelian groups on Y^{et} and $\mathcal{F}_0 := \pi^*(\mathcal{F})$ be the pullback to Y_0^{et}. We note that the Atkin-Lehner involution induces an isomorphism $\mathcal{F}_0 = \pi_\ell^*(\mathcal{F}) \cong (\pi'_\ell)^*(\mathcal{F})$. For $s \in \{\eta, s\}$ we therefore have a homomorphism

$$\nu_* : H_c^1(Y_*, \mathcal{F}_*) \oplus H_c^1(Y_*, \mathcal{F}_*) \longrightarrow H_c^1(Y_{0,*}, \mathcal{F}_{0,*}) \tag{2.6}$$

induced by the pair of maps π_ℓ^*, $(\pi'_\ell)^*$ and define

$$H_c^1(Y_{0,*}, \mathcal{F})^{\text{new}} := \text{Coker}(\nu_*).$$

Over the special fiber we also have a homomorphism

$$\mu_s : H_c^1(Y_{0,s}, \mathcal{F}_{0,s}) \longrightarrow H_c^1(Y_s, \mathcal{F}_s) \oplus H_c^1(Y_s, \mathcal{F}_s) \tag{2.7}$$

induced by the pair i_*, i'_*. We also define

$$H_c^1(Y_{0,*}, \mathcal{F})_{\text{new}} := \text{Ker}(\mu_*).$$

The composition

$$M_{\mathcal{F}} := \mu_s \circ \nu_s : H_c^1(Y_s, \mathcal{F}_s) \oplus H_c^1(Y_s, \mathcal{F}_s) \longrightarrow H_c^1(Y_s, \mathcal{F}_s) \oplus H_c^1(Y_s, \mathcal{F}_s)$$

is given by the 2×2 matrix

$$M_{\mathcal{F}} = \begin{pmatrix} 1+\ell & T_\ell \\ [\ell]^{-1}T_\ell & 1+\ell \end{pmatrix}$$

over $\text{End}\big(H_c^1(Y_s, \mathcal{F}_s)\big)$.

Now let Σ denote the set of supersingular points in the special fiber $Y_{0,s}$ and let $(\mathcal{F}_0)_\Sigma$ be the skyscraper sheaf defined in the paragraph before Theorem 2.3. The main theorem of this section is the following.

Theorem 2.6. *All of the follwing are true.*

(a) The exact sequence in Theorem 2.3 induces an exact sequence

$$0 \to H^1_c(Y_{0,s}, \mathcal{F}_s)^{\mathrm{new}} \to H^1_c(Y_{0,\eta}, \mathcal{F}_0)^{\mathrm{new}} \to H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \to H^2_c(Y_{0,s}, \mathcal{F}_{0,s}).$$

(b) Moreover, μ_s induces a canonical exact sequence

$$0 \to H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \to H^1_c(Y_{0,s}, \mathcal{F}_{0,s}) \xrightarrow{\mu_s} (H^1_c(Y_s, \mathcal{F}_s))^2 \to 0$$

and a canonical isomorphism

$$\mu_s : H^2_c(Y_{0,s}, \mathcal{F}_{0,s}) \xrightarrow{\sim} H^2_c(Y_s, \mathcal{F}_s)^2.$$

(c) There is a canonical exact sequence

$$Ker(M_\mathcal{F}) \to H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \to H^1_c(Y_{0,s}, \mathcal{F}_s)^{\mathrm{new}} \to Coker(M_\mathcal{F}) \to 0.$$

Proof. Since Y_s is smooth we have $R^i\Psi_Y(\mathcal{F}_\eta) = 0$ for $i \geq 1$ and therefore the vanishing cycles sequence for Y gives isomorphisms $H^1_c(Y_s, \mathcal{F}_s) \cong H^1_c(Y_\eta, \mathcal{F}_\eta)$. Putting everything together we obtain the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \to & (H^1_c(Y_s, \mathcal{F}_s))^2 & \cong & (H^1_c(Y_\eta, \mathcal{F}_\eta))^2 & \to & 0 \\
 & & \downarrow \nu_s & & \downarrow \nu_\eta & & \downarrow \\
0 & \to & H^1_c(Y_{0,s}, \mathcal{F}_{0,s}) & \to & H^1_c(Y_{0,\eta}, \mathcal{F}_{0,\eta}) & \to & H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \\
 & & \downarrow & & \downarrow & & || \\
0 & \to & H^1_c(Y_{0,s}, \mathcal{F}_{0,s})^{\mathrm{new}} & \to & H^1_c(Y_{0,\eta}, \mathcal{F}_{0,\eta})^{\mathrm{new}} & \to & H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

As the first two rows of the diagram are exact, the third row is exact as well. We then use Theorem 2.7, to continue this exact sequence as follows

$$H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \longrightarrow H^2_c(Y_{0,s}, \mathcal{F}_{0,s}) \longrightarrow H^2_c(Y_{0,\eta}, \mathcal{F}_{0,\eta}) \longrightarrow 0.$$

Assertion (a) of the theorem now follows.

On the other hand, for $i \geq 0$, the Meier-Vietoris sequence for $Y_{0,s}$ and its two components gives a long exact sequence in cohomology

$$H^i_c(Y_{0,s}, \mathcal{F}_{0,s}) \longrightarrow (H^i_c(Y_s, \mathcal{F}_s))^2 \longrightarrow H^i(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \longrightarrow$$

$$\longrightarrow H^{i+1}_c(Y_{0,s}, \mathcal{F}_{0,s}) \xrightarrow{\mu_s} (H^{i+1}_c(Y_s, \mathcal{F}_s))^2 \longrightarrow 0.$$

Setting $i = 0$ in this sequence, and using the fact that $H^2\big(Y_s, (\mathcal{F}_s)^*(-1)\big) = 0$, since Y_s is affine open in a smooth proper curve, we conclude by duality that $H_c^0(Y_s, \mathcal{F}_s) = 0$. So the last three terms of this sequence form a short exact sequence proving the first assertion of (b). On the other hand, setting $i = 1$, and using the fact that $H^1(Y_s, (\mathcal{F}_0)_\Sigma) = 0$, we see that μ_s is an isomorphism on H_c^2, proving the second assertion of (b).

To prove (c) we start with the exact sequence of (b) and note that ν_s induces a commutative diagram with short exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}(M_{\mathcal{F}}) & \to & (H_c^1(Y_s, \mathcal{F}_s))^2 & \stackrel{M_{\mathcal{F}}}{\to} & \mathrm{Image}(M_{\mathcal{F}}) & \to & 0. \\ & & \downarrow & & \downarrow \nu_s & & \cap & & \\ 0 & \to & H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) & \to & H_c^1(Y_{0,s}, \mathcal{F}_{0,s}) & \stackrel{\mu_s}{\to} & (H_c^1(Y_s, \mathcal{F}_s))^2 & \to & 0. \end{array}$$

An application of the snake lemma gives us a canonical exact sequence

$$\mathrm{Ker}(M_{\mathcal{F}}) \longrightarrow H^0(Y_{0,s}, (\mathcal{F}_0)_\Sigma) \longrightarrow H_c^1(Y_{0,s}, \mathcal{F}_{0,s})^{new} \longrightarrow \mathrm{Coker}(M_{\mathcal{F}}) \longrightarrow 0$$

and the theorem is proved. $\square$

3 Cohomological Automorphic Forms

Let $\mathbb{G}$ be one of the reductive algebraic groups $\mathbb{GL}_2$ or $\mathbb{B}^\times$. We let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of divisors of degree 0 supported on $\mathbb{P}^1(\mathbb{Q})$ and let $\mathbb{GL}_2(\mathbb{Q})$ act on $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ on the left in the usual way by fractional linear transformations. We then define

$$\mathfrak{S}_{\mathbb{G}} := \begin{cases} \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \otimes \mathbb{Z}[\mathbb{GL}_2(\mathbb{A}_f)] & \text{if } \mathbb{G} = \mathbb{GL}_2, \\ \\ \mathbb{Z} \otimes \mathbb{Z}[\mathbb{B}^\times(\mathbb{A}_f)] & \text{if } \mathbb{G} = \mathbb{B}^\times. \end{cases}$$

We let $\mathbb{G}(\mathbb{Q})$ act diagonally on the left on $\mathfrak{S}_{\mathbb{G}}$ and let $\mathbb{G}(\mathbb{A}_f)$ act on the right through the second factor.

As in §1 we fix a tame level M, choose Q, with $(M, Q) = 1$, and consider the Hecke pair $(\Sigma_{\mathbb{G}}(Q), \mathcal{K}_{\mathbb{G}}(Q))$ and the corresponding Hecke algebra $\mathcal{H} := \mathcal{H}(M, Q)$.

Definition 3.1. Let $\mathbb{G}$ be either $\mathbb{GL}_2$ or $\mathbb{B}^\times$, and F be an abelian group endowed with a right action of $\mathcal{K}_{\mathbb{G}}(Q)$. We let $\mathbb{G}(\mathbb{Q})$ act trivially on F on the left and define

$$\mathcal{S}_{\mathbb{G}}(Q, F) := \mathrm{Hom}_{\mathbb{G}(\mathbb{Q}) \times \mathcal{K}_{\mathbb{G}}(Q)}(\mathfrak{S}_{\mathbb{G}}, F).$$

We will refer to $\mathcal{S}_{\mathbb{G}}(Q, F)$ as the space of F-valued cohomological automorphic forms over $\mathbb{G}$.

In the applications, we will fix a prime p with $(p, M\ell) = 1$, set $Q = \ell$ or $p\ell$, and let F be a projective limit of finite p-primary groups endowed with a continuous (right) action of $\Sigma_{\mathbb{G}}(Q)$. Under these hypotheses we may compute etale cohomology with values in F. In this chapter we shall explain the relationship of the groups $\mathcal{S}_{\mathbb{GL}_2}(Q, F)$ and $\mathcal{S}_{\mathbb{B}^\times}(Q, F)$ to these etale cohomology groups.

3.1 Modular Symbols and Compactly Supported Cohomology

Let F be an abelian group endowed with a right action of $\Sigma := \Sigma(Q)$. We endow the group $\mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), F)$ of additive homomorphisms from $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ to F with a right action of Σ by $\xi \mapsto \xi|\sigma$ where $(\xi|\beta)(D) = \xi(\beta D)|\beta$ for $\xi \in \mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), F)$ and $\beta \in \Sigma$. The group of F-valued modular symbols over $\Gamma := \Gamma_Q := {\mathbb{GL}_2}^+(\mathbb{Q}) \cap \mathcal{K}_{\mathbb{GL}_2}(Q)$ is defined in [AS1] to be the group

$$\mathrm{Symb}_\Gamma(F) := \mathrm{Hom}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), F)^\Gamma.$$

Proposition 3.2. *For $\varphi \in \mathcal{S}_{\mathbb{GL}_2}(Q, F)$ define $\xi_\varphi \in Hom(Div^0(\mathbb{P}^1(\mathbb{Q})), F)$ by $\xi_\varphi(D) = \varphi(D \otimes 1)$. Then the map $\varphi \longmapsto \xi_\varphi$ defines a Hecke equivariant isomorphism*

$$\mathcal{S}_{\mathbb{GL}_2}(Q, F) \xrightarrow{\sim} Symb_\Gamma(F).$$

Moreover, we have isomorphisms

$$H_c^1(Y_{\mathbb{C}}, \widetilde{F}) \cong \mathcal{S}_{\mathbb{GL}_2}(Q, F) \quad \text{and} \quad H_c^2(Y_{\mathbb{C}}, \widetilde{F}) \cong H_0(\Gamma_Q, F)$$

where $Y_{\mathbb{C}}$ is the complex Riemann surface associated to Y_η and $\widetilde{F}$ is the local coefficient system on $Y_{\mathbb{C}}$ associated to the Γ-module F.

Proof. A simple calculation establishes that the map $\varphi \longmapsto \xi_\varphi$ is a Hecke equivariant isomorphism $\mathcal{S}_{\mathbb{GL}_2}(Q, F) \xrightarrow{\sim} \mathrm{Symb}_\Gamma(F)$. Indeed, the inverse map is given by $\xi \longmapsto \varphi_\xi$ where, for $g \in \mathbb{GL}_2(\mathbb{A}_f)$ and $D \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$, we define

$$\varphi_\xi(D \otimes g) = \big(\xi(\gamma D) \mid \gamma g\big) \in F$$

where $\gamma \in {\mathbb{GL}_2}^+(\mathbb{Q})$ is any element for which $\gamma g \in \mathcal{K}_{\mathbb{GL}_2}(Q)$. That the expression on the right is well-defined follows from the invariance of ξ under

$\mathbb{GL}_2(\mathbb{Q}) \cap \mathcal{K}_{\mathbb{GL}_2}(Q)$, and the invariance of φ_ξ under $\mathbb{GL}_2(\mathbb{Q}) \times \mathcal{K}_{\mathbb{GL}_2}(Q)$ is an equally straightforward calculation. This proves the first assertion. The isomorphism $H^1_c(Y_{\mathbb{C}}, \widetilde{F}) \cong \mathrm{Symb}_\Gamma(F)$ is Proposition 4.2 of [AS1] and the isomorphism $H^2_c(Y_{\mathbb{C}}, \widetilde{F}) \cong H_0(\Gamma_Q, F)$ follows from Poincaré duality. □

The following corollary follows from the proposition and Artin's Comparison Theorem.

Corollary 3.3. *Suppose F is a profinite $\mathcal{K}_{\mathbb{GL}_2}(Q)$-module and let $\mathcal{F}_\eta$ be the associated profinite sheaf on Y_η. Then there are canonical Hecke-equivariant isomorphisms*

$$\begin{array}{rcl} H^1_c(Y_\eta, \mathcal{F}_\eta) & \cong & \mathcal{S}_{\mathbb{GL}_2}(Q, F) \\ H^2_c(Y_\eta, \mathcal{F}_\eta) & \cong & H_0(\Gamma, F). \end{array}$$

where I is the augmentation ideal in $\mathbb{Z}[\Gamma]$.

3.2 Skyscraper Sheaves and Quaternionic Automorphic Forms

Now let p be a prime different from ℓ. Chosse $Q \geq 1$ with $(Q, M) = 1$ and for each prime $q \neq \ell$ let $\mathcal{K}_q := \mathcal{K}_q(Q) \subset GL_2(\mathbb{Q}_q)$ be defined as in 1.3. Let

$$\mathcal{K}_B := \mathcal{R}_\ell^\times \times \mathcal{K} \quad \text{where} \quad \mathcal{K} := \mathcal{K}_p \times \mathcal{K}^{(p)} \quad \text{and} \quad \mathcal{K}^{(p)} := \prod_{q \neq p, \ell} \mathcal{K}_q.$$

Fix a finite p-power torsion group F endowed with a continuous action of $\mathcal{K}_p$ and let $\Delta_{0,s}$ act on F via the homomorphism $\Delta_{0,S} \longrightarrow \mathcal{K}_p$ induced by the action of $\mathcal{K}_p$ on the p-adic Tate module of the universal elliptic curve over the modular curve $Y_{0,S}$. We may therefore apply the language of section 2.1 and in particular we obtain a sheaf $\mathcal{F}_{0,S}$ on $Y_{0,S}$ as in proposition 2.1. We let Σ be the set of supersingular points on $Y_{0,s}$ and let $(\mathcal{F}_{0,s})_\Sigma$ be the skyscraper sheaf of $\mathcal{F}_{0,s}$ supported on Σ.

Theorem 3.4. *There is a canonical Hecke equivariant isomorphism*

$$S(\mathcal{K}_B, F) \cong H^0(Y_{0,s}, (\mathcal{F}_{0,s})_\Sigma).$$

Proof. To see this we consider the tower $Y^{(p)}_{0,s} := \varprojlim_n Y_0(p^n)_{\mathbb{F}_\ell}$ of modular curves in characteristic ℓ with full level p^n-level structure. This is a galois tower of etale covers of $Y_{0,s}$ with galois group $\mathcal{K}_p$, and the action of $\Delta_{0,s}$ on this tower is given by the surjective homomorphism $\Delta_{0,s} \longrightarrow \mathcal{K}_p$. The set $\Sigma_\ell^{(p)} \subseteq Y_0^{(p)}$ of (limits of) points lying over elements of Σ inherits an action of $\Delta_{0,s}$ that factors through this homomorphism. It follows that we have an isomorphism

$$H^0(Y_{0,s}, (\mathcal{F}_{0,s})_\Sigma) \cong \big(\mathrm{Maps}(\Sigma_\ell^{(p)}, F)\big)^{\mathcal{K}_p}.$$

To complete the proof, it will therefore suffice to show that there is a canonical $\mathcal{K}_p$-equivariant bijection $\Sigma_\ell^{(p)} \xrightarrow{\sim} B^\times \backslash B^\times / \mathcal{K}_B$.

Indeed, by definition Σ is the set of isomorphism classes of pairs (E, σ) consisting of a supersingular elliptic curve E over $\overline{\mathbb{F}}_\ell$ together with a $\mathcal{K}^{(p)}$-level structure σ on E. If we now let $(E_0, \sigma_0) \in \Sigma$ be fixed and if we assume our chosen maximal order $\mathcal{R}$ in B is isomorphic to the endomorphism ring $\mathrm{End}_{\overline{\mathbb{F}}_\ell}(E_0)$, then the set $\Sigma_\ell^{(p)}$ may be regarded as the set of isomorphism classes of triples (E, σ, ξ) consisting of an element $(E, \sigma) \in \Sigma$ together with an isomorphism $\xi : Ta_p(E_0) \xrightarrow{\sim} Ta_p(E)$.

On the other hand, to any $g \in \widehat{B}^\times$, we may associate the isomorphism class of projective rank one right $\mathcal{R}$-modules represented by the $\mathcal{R}$-module $M_g := g\widehat{\mathcal{R}} \cap B$. We then let E_g be the supersingular elliptic curve given by $E_g := M_g \otimes_{\mathcal{R}} E_0$ (see [Se2]) and note that $M_g = \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(E_0, E_g)$. In particular, g induces an isomorphism from the level structure (σ_0, ξ_0) to a corresponding level structure (σ_g, ξ_g) on E_g. Moreover, the isomorphism class of the triple (E_g, σ_g, ξ_g) is unchanged when g is replaced by any other element of the double coset $B^\times g \mathcal{K}_B^{(p)}$. It follows that the map $g \mapsto (E_g, \sigma_g, \xi_g)$ induces a $\mathcal{K}_p$-equivariant bijection

$$B^\times \backslash \widehat{B}^\times / \mathcal{K}_B^{(p)} \xrightarrow{\sim} \Sigma_\ell^{(p)}.$$

which is, in fact, a bijection (see [G], section 2). Putting everything together we obtain

$$H^0(Y_{0,s}, (\mathcal{F}_{0,s})_\Sigma) \cong \big(\mathrm{Maps}(\Sigma_\ell^{(p)}, F)\big)^{\mathcal{K}_p} \cong \mathrm{Maps}(B^\times \backslash \widehat{B}^\times, F)_p^{\mathcal{K}} \cong \mathcal{S}(\mathcal{K}_B, F)$$

as claimed. □

3.3 Automorphic Geometric Jacquet-Langlands

We keep the conventions of the last section, so that we have a fixed tame level $M > 1$ (which we suppress from the notations) and two distinct primes

p, ℓ with $(M, p\ell) = 1$. We now fix $N \in \mathbb{Z}^+$ with $(N, M\ell) = 1$ and in the following will take $Q = N$ or $Q = N\ell$.

We have two inclusions $i_1, i_2 : \mathcal{K}_{\mathbb{GL}_2}(N\ell) \hookrightarrow \mathcal{K}_{\mathbb{GL}_2}(N)$ where i_1 is the natural inclusion and i_2 is given by conjugation $x \longmapsto \beta_\ell^{-1} x \beta_\ell$ with $\beta_\ell := \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$. Thus for any finite abelian group F endowed with a continuous right action of $\mathcal{K}_p := \mathcal{K}_p(N)$ the map $x \mapsto x|\beta$ for $x \in F$ induces a $\mathcal{K}_{\mathbb{GL}_2}(N\ell)$-morphism $\beta_* : i_1^*(F) \longrightarrow i_2^*(F)$. which in turn induces an isomorphism

$$\rho_\beta : \mathcal{S}_{\mathbb{GL}_2}(N\ell, i_1^*(F)) \longrightarrow \mathcal{S}_{\mathbb{GL}_2}(N\ell, i_2^*(F)).$$

On the other hand, we have trace maps $Tr_k : S_{\mathbb{GL}_2}(N\ell, i_k^*(F)) \longrightarrow S_{\mathbb{GL}_2}(N, F)$. Thus, identifying $S_{\mathbb{GL}_2}(N\ell, F) = S_{\mathbb{GL}_2}(N\ell, i_1^*(F))$, we obtain a map

$$\tau = (\tau_1, \tau_2) : S_{\mathbb{GL}_2}(N\ell, F) \longrightarrow S_{\mathbb{GL}_2}(N, F) \times S_{\mathbb{GL}_2}(N, F)$$

given by $\tau_1 = Tr_1$ and $\tau_2 = Tr_2 \circ \rho_\beta$. We define

$$\mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, F) := \mathrm{Coker}(\tau).$$

Finally, we let $\{F^{(n)}\}_n$ be a projective system of finite p-primary abelian groups endowed with a continuous right action of $\mathcal{K}_p := \mathcal{K}_p(N)$, and let $\{\mathcal{F}_S^{(n)}\}_n$ and $\{\mathcal{F}_{0,S}^{(n)}\}_n$ and be the associated system of p-primary sheaves on Y_S^{et} and $Y_{0,S}$ respectively. Let $F = \varprojlim_n F_n$ be the associated projective limit module, and use the following conventions for computing projective limits of cohomology with values in these sheaves:

$$H^*(Y, \mathcal{F}) := \varprojlim_n H^*(Y, \mathcal{F}^{(n)}) \quad \text{and} \quad H^*(Y_0, \mathcal{F}_0) := \varprojlim_n H^*(Y_0, \mathcal{F}_0^{(n)}).$$

With these conventions, the maps ν_s and μ_s defined in section 2.3 induce maps

$$\nu_* : H^1_c(Y_*, \mathcal{F}_*) \oplus H^1_c(Y_*, \mathcal{F}_*) \quad \longrightarrow \quad H^1_c(Y_{0,*}, \mathcal{F}_{0,*})$$

and

$$\mu_s : H^1_c(Y_{0,s}, \mathcal{F}_{0,s}) \quad \longrightarrow \quad H^1_c(Y_s, \mathcal{F}_s) \oplus H^1_c(Y_s, \mathcal{F}_s).$$

As in section 2.3 we define

$$H^1_c(Y_{0,*}, \mathcal{F})^{\mathrm{new}} := \mathrm{Coker}(\nu_*)$$

and note that the matrix

$$M_{\mathcal{F}} := \mu_s \circ \nu_s : H^1_c(Y_s, \mathcal{F}_s) \oplus H^1_c(Y_s, \mathcal{F}_s) \longrightarrow H^1_c(Y_s, \mathcal{F}_s) \oplus H^1_c(Y_s, \mathcal{F}_s)$$

is given by the 2×2 matrix

$$M_{\mathcal{F}} = \begin{pmatrix} 1+\ell & T_\ell \\ [\ell]^{-1}T_\ell & 1+\ell \end{pmatrix}$$

over $\mathrm{End}\big(H^1_c(Y_s, \mathcal{F}_s)\big)$.

We then have the following reformulation of Theorem 2.6 which we will refer to as the geometric Jacquet-Langlands theorem.

Theorem 3.5. *Let $\{F_n\}_n$ be a projective system of finite p-primary abelian groups with continuous action of Σ_p and let $\mathcal{F}$ and $\mathcal{F}_0$ be the associated systems of sheaves on Y and Y_0 respectively (see above). Let $F := \varprojlim_n F_n$ be the projective limit of the Σ_p-modules F_n. Then we have canonical Hecke equivariant exact sequences*

(a) $0 \to H^1_c(Y_{0,s}, \mathcal{F}_{0,s})^{\mathrm{new}} \to \mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, F) \to \mathcal{S}_{\mathbb{B}^\times}(N, F) \longrightarrow H^2_c(\Gamma, F)^2$;

(b) $0 \to \mathcal{S}_{\mathbb{B}^\times}(N, F) \to H^1_c(Y_{0,s}, \mathcal{F}_{0,s}) \xrightarrow{\mu_s} (H^1_c(Y_s, \mathcal{F}_s))^2 \to 0$;

(c) $Ker(M_{\mathcal{F}}) \to \mathcal{S}_{\mathbb{B}^\times}(N, F) \to H^1_c(Y_{0,s}, \mathcal{F}_{0,s})^{\mathrm{new}} \to Coker(M_{\mathcal{F}}) \to 0$.

Proof. From Theorem 2.6 (a) we have an exact sequence

$$0 \to H^1_c(Y_{0,s}, \mathcal{F}^{(n)}_{0,s})^{\mathrm{new}} \to \mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, F^{(n)}) \to \mathcal{S}_{\mathbb{B}^\times}(N, F^{(n)}) \longrightarrow H^2_c(\Gamma, F^{(n)})^2$$

for each n. Since these are all finite groups the Mittag-Leffler conditions are satisfied. Passing to the projective limit over n, we therefore obtain an exact sequence of projective limits. However, it follows immediately from the definitions that formation of cohomological automorphic forms commutes with projective limits. Hence

$$\varprojlim_n \mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, F^{(n)}) \cong \mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, F) \quad \text{and} \quad \varprojlim_n \mathcal{S}_{\mathbb{B}^\times}(N, F^{(n)}) \cong \mathcal{S}_{\mathbb{B}^\times}(N, F).$$

Likewise, since $H^2_c(\Gamma, F^{(n)})$ can be computed from a finite chain complex, the Mittag-Leffler conditions imply that formation of cohomology commutes with projective limits, so we have

$$\varprojlim_n H^2_c(\Gamma, F^{(n)}) \cong H^2_c(\Gamma, F).$$

This proves (a). The proofs of (b) and (c) are similar, but easier. □

4 Locally Analytic Functions and Distributions

4.1 Basic Definitions and Simple Facts

We let $|\cdot|$ be the usual p-adic absolute value on $\mathbb{C}_p$, i.e. the one normalized so that $|p| = p^{-1}$. More generally, for $d \geq 1$ we let $|\cdot| : \mathbb{C}_p \longrightarrow \mathrm{R}^{\geq 0}$ be the norm defined by

$$|(x_1, \ldots, x_d)| = \max_{i=1,\ldots,d} |x_i|.$$

For a compact subset $X \subseteq \mathbb{Z}_p^d$, and a positive real number $r \in |\mathbb{C}_p^\times| = p^{\mathbb{Q}}$ we let

$$B[X, r] := \left\{ z \in \mathbb{C}_p^d \;\middle|\; \exists x \in X \text{ s.t.} \, |z - x| \leq r \right\}$$

and note that this is an affinoid neigborhood of X in $\mathbb{C}_p^d$.

Let $\mathcal{W}$ denote the weight space, i.e. the rigid analytic space over $\mathbb{Q}_p$ whose K-points are given by: $\mathcal{W}(K) := \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, K^\times)$ for any complete subfield K of $\mathbb{C}_p$. It is easy to see (and well-known) that every point of $\mathcal{W}(K)$ is locally analytic on $\mathbb{Z}_p^\times$. For n a positive integer, we say that a point $\kappa \in \mathcal{W}(K)$ is n-analytic if κ extends to a rigid analytic function on the affinoid $B[\mathbb{Z}_p^\times, p^{-n}]$.

For any K-affinoid subspace $S \subset \mathcal{W}$ we let $\kappa_S : \mathbb{Z}_p^\times \longrightarrow A(S)^\times$ be the canonical character defined by $\kappa_S(a)(s) = a^s$ for $a \in \mathbb{Z}_p^\times$ and $k \in S$. (Here and throughout the paper we let $A(Z) = \mathcal{O}_Z(Z)$ denote the algebra of global K-rigid analytic functions on Z.) In the applications, S will almost always be either a singleton point $\kappa \in \mathcal{W}(K)$ or a K-affinoid subdomain $U \subseteq \mathcal{W}$.

Let $T_0 := \mathbb{Z}_p^\times \times \mathbb{Z}_p$, which we regard as a compact open subset of the space of row vectors $(\mathbb{Z}_p)^2$. We have the following structure on T_0:

a) a natural left action of $\mathbb{Z}_p^\times$ by scalar multiplication;

b) a natural right action of the semigroup

$$\Xi_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \;\middle|\; ad - bc \neq 0, \text{ and} \, (a, c) \in \mathbb{Z}_p^\times \times p\mathbb{Z}_p \right\}$$

and its subgroup

$$\mathrm{Iw}(\mathbb{Z}_p) := \Xi(\mathbb{Z}_p) \cap \mathbb{GL}_2(\mathbb{Z}_p)$$

given by matrix multiplication on the right.

The two actions obviously commute.

For any K-Banach algebra A we let

$$\mathcal{A}(T_0, A) := \left\{ f : T_0 \longrightarrow A \;\middle|\; f \text{ is locally analytic} \right\},$$

and for any integer $n > 0$ we let let $\mathcal{A}[n](T_0, A)$ denote the subspace consisting of functions that extend to A-affinoid functions on $B[T_0, p^{-n}]$. Clearly, for each $n > 0$, $\mathcal{A}[n](T_0, A)$ is a Banach A-module and the inclusion $A[n](T_0, A) \hookrightarrow A[n](T_0, A)$ is a compact (i.e. completely continuous) linear map. Moreover, we have

$$\mathcal{A}(T_0, A) = \varinjlim_{n} \mathcal{A}[n](T_0, A),$$

which we endow with the compact inductive limit topology (see [Sch]).

Definition 4.1. Let S be a K-affinoid subspace of $\mathcal{W}$. Then we make the following definitions:

(1) $\mathcal{A}_S := \left\{ f \in \mathcal{A}(T_0, A(S)) \;\middle|\; \forall a \in \mathbb{Z}_p^\times,\, t \in T_0, \text{ we have } f(at) = \kappa_S(a) f(t) \right\}$. Clearly, $\mathcal{A}_S$ is a closed $A(S)$-submodule of $\mathcal{A}(T_0, A(S))$. We endow it with the induced topology.

(2) $\mathcal{D}_S := \mathrm{Hom}_{A(S)}(\mathcal{A}_S, A(S))$ where here and elsewhere "Hom" denotes *continuous* homomorpisms.

We note that both $\mathcal{A}_S$ and $\mathcal{D}_S$ inherit actions of $\Xi(\mathbb{Z}_p)$ over $A(S)$. We will denote these actions by

$$\begin{array}{ccccc} \Xi(\mathbb{Z}_p) & \times & \mathcal{A}_S & \longrightarrow & \mathcal{A}_S \\ (\gamma & , & f) & \longmapsto & \gamma f \end{array} \quad \textit{and} \quad \begin{array}{ccccc} \mathcal{D}_S & \times & \Xi(\mathbb{Z}_p) & \longrightarrow & \mathcal{D}_S \\ (\mu & , & \gamma) & \longmapsto & \mu|\gamma \end{array}$$

Each of these pairings is continuous in both variables, that the action on $\mathcal{A}_S$ is a left action and the action on $\mathcal{D}_S$ is a right action.

If $S \subseteq \mathcal{W}$ is a K-affinoid subspace and $\kappa \in S(K)$, then we define specialization maps

$$\begin{array}{ccc} \mathcal{A}_S(T_0) & \longrightarrow & \mathcal{A}_\kappa(T_0) \\ f & \longmapsto & f_\kappa \end{array} \quad \textit{and} \quad \begin{array}{ccc} \eta_\kappa : \mathcal{D}_S(T_0) & \longrightarrow & \mathcal{D}_\kappa(T_0) \\ \mu & \longmapsto & \mu_\kappa \end{array} \tag{4.1}$$

where $f_\kappa(x, y) := f(x, y)(\kappa)$ and $\mu_\kappa \in \mathcal{D}_\kappa(T_0)$ is given by $\mu_\kappa : f \in \mathcal{A}_\kappa(T_0) \longmapsto \mu(f_S)(\kappa)$, where $f_S \in \mathcal{A}_S(T_0)$ is given by $f_S(x, y) := \chi_S(x) f(1, y/x)$.

Proposition 4.2. *Let $U \subseteq \mathcal{W}$ be a K-affinoid subdomain, let $\kappa \in U(K)$, and let $t_\kappa \in A(U)$ be a local parameter at κ which has no zeroes in U other than at κ. Then we have canonical exact sequences of $\Xi(\mathbb{Z}_p)$-modules*

$$0 \longrightarrow \mathcal{A}_U(T_0) \xrightarrow{t_\kappa} \mathcal{A}_U(T_p) \longrightarrow \mathcal{A}_\kappa(T_0) \to 0$$

$$0 \longrightarrow \mathcal{D}_U(T_0) \xrightarrow{t_\kappa} \mathcal{D}_U(T_p) \xrightarrow{\eta_\kappa} \mathcal{D}_\kappa(T_0) \to 0.$$

For future reference we record in the next proposition a simple result concerning the structure of the module $\mathcal{D}_U$. Before stating the theorem, it is convenient to first give the following definitions.

Definition 4.3. For $n > 0$ define

$$\mathcal{A}[n] := \mathcal{A}[n](T_0, K) \text{ and } \mathcal{D}[n] := \mathrm{Hom}(\mathcal{A}[n], K).$$

For any affinoid subdomain U of $\mathcal{W}$ we then define $A(U)$-Banach modules

$$\mathcal{A}_U[n] := \mathcal{A}[n](T_0, A(U)) \quad \textit{and} \quad \widetilde{\mathcal{D}}_U[n] := \mathrm{Hom}_{A(U)}(\mathcal{A}_U[n], A(U)).$$

The module $\mathcal{A}_U[n]$ is an orthonormalizable $A(U)$-Banach module, but unfortunately $\widetilde{\mathcal{D}}_U[n]$ is not ON-able as $A(U)$-module. So we consider instead the subspace

$$\mathcal{D}_U[n] := \mathrm{Hom}_K^{cpt}(\mathcal{A}[n](\mathbb{Z}_p), A(U)) \subseteq \widetilde{\mathcal{D}}_U[n]$$

of completely continuous K-linear maps.

We note that $\mathcal{D}_U[n] \cong \mathbb{D}[n]\widehat{\otimes}_K A(U)$ as $A(U)$-module and is therefore orthonormalizable over $A(U)$, since $\mathbb{D}[n]$ is orthonormalizable over K.

Lemma 4.4. *For any K-affinoid subdomain $U \subseteq \mathcal{W}$ and all sufficiently large n (depending on U), $A_U[n]$ is a $\Xi(\mathbb{Z}_p)$-invariant subspace of $\mathcal{A}_U$. Moreover, for n sufficiently large, we have the following assertions.*

(1) *$\Xi(\mathbb{Z}_p)$ acts on $A_U[n]$, and therefore by duality also on $\widetilde{\mathbb{D}}_U[n]$, as a semigroup of operators of norm ≤ 1.*

(2) *$\mathbb{D}_U[n]$ is a $\Xi(\mathbb{Z}_p)$-invariant subspace of $\widetilde{\mathbb{D}}_U[n]$.*

(3) *The canonical map $\mathcal{D}_U \longrightarrow \widetilde{\mathbb{D}}_U[n]$ is $\Xi(\mathbb{Z}_p)$-invariant, and its image is contained in $\mathbb{D}_U[n]$.*

Proof. To appear. $\square$

4.2 Distribution Modules and Finite Projective Limits

In this section we will prove the following theorem.

Theorem 4.5. *For every $n > 1$, there is a projective system $\{F^{(m)}[n]\}_m$ of finite $\Lambda := \mathcal{O}_K[[\mathbb{Z}_p^\times]]$ and a $\Lambda[\mathrm{Iw}(\mathbb{Z}_p]$-isomorphism*

$$\mathcal{D}[n]^\circ \xrightarrow{\sim} \varprojlim_m F^{(m)}[n].$$

Proof. The proof follows immediately from the following simple lemma by taking $A = \mathcal{A}[n]$ and $A_0 = \mathcal{A}[1]$.

□

Lemma 4.6. *Let A be a K-Banach space endowed with a continuous action of a topological group G. Let $\mathbb{D}$ be the space of K-linear functionals on A, endowed with the dual action of G. Suppose, moreover, there is a K-Banach space A_0 with continuous action of G and an inclusion $i : A_0 \hookrightarrow A$ satisfying the following three properties:*

(1) *i is G-equivariant;*

(2) *i is completely continuous and has norm ≤ 1;*

(3) *the image of i is dense in A.*

Then there is a projective system $\{F^{m)}, \phi_m\}_{m\geq 0}$ of finite G-modules $F^{(m)}$ and surjective G-morphisms $\phi_m : F^{(m+1)} \longrightarrow F^{(m)}$ such that

$$\mathbb{D}^\circ \cong \varprojlim_m F^{(m)}$$

Proof. For each $m \geq 0$, consider the image of the composition

$$p^{-m}A_0^\circ \xrightarrow{i} p^{-m}A^\circ \longrightarrow p^{-m}A^\circ/A^\circ.$$

The image is finite by complete continuity. Letting $A^{(m)} := p^{-m}A_0^\circ + A^\circ$, we have an increasing sequence of open subsets of A

$$A^\circ = A^{(0)} \subseteq A^{(1)} \subseteq \cdots \subseteq A^{(n)} \subseteq \cdots \subseteq A.$$

Thus $\bigcup_m A^{(m)}$ is an open subset of A containing $A_0 = \bigcup_m p^{-m}A_0^o$. Since, by assumption A_0 is dense in A we see that

$$A = \bigcup_n A^{(m)}.$$

Now set

$$\mathbb{D}^{(m)} := \left\{ \mu \in \mathbb{D}^\circ \;\middle|\; \mu(\alpha) \in \mathbb{Z}_p \text{ for all } \alpha \in A^{(m)} \right\}.$$

Then $\mathbb{D}^{(n)}$ is a weakly open subset of $\mathbb{D}^\circ$. Indeed, for any $\alpha \in A^{(n)}$, the set $U_\alpha := \{\mu \in \mathbb{D}^\circ \,|\, \mu(\alpha) \in \mathbb{Z}_p\}$ is weakly open in $\mathbb{D}^\circ$. Moreover, $\mathbb{D}^{(n)} = \cap_{i=1}^r U_{\alpha_i}$ for any complete set $\alpha_1, \ldots, \alpha_r \in A^{(m)}$ of representatives for $A^{(m)}$ modulo A°. We let

$$F^{(m)} := \mathbb{D}^\circ / \mathbb{D}^{(m)}$$

endowed with the discrete topology given the induced action of G. We note that the map $\mathbb{D}^\circ/\mathbb{D}^{(m)} \longrightarrow F^{(m)}$ is an isomorphism of finite discrete G-modules.

Thus we have a sequence of inclusions

$$\cdots \subseteq \mathbb{D}^{(m)} \subseteq \cdots \mathbb{D}^{(1)} \subseteq \mathbb{D}^{(0)} = \mathbb{D}^\circ$$

of weakly open subsets of $\mathbb{D}^\circ$. Moreover, $\bigcap_n \mathbb{D}^{(n)} = \{0\}$. Indeed, if $\mu \in \bigcap_n \mathbb{D}^{(n)}$, then $\mu(\alpha) \in \mathbb{Z}_p$ for all $\alpha \in \bigcup_n A^{(n)} = A$, hence $\mu = 0$.

For each $m \geq 0$ the pairing

$$A^{(n)}/A^\circ \times \mathbb{D}^\circ/\mathbb{D}^{(m)} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

is perfect and therefore the canonical map

$$\mathbb{D}^\circ \longrightarrow \varprojlim_m F^{(m)}$$

is a G-equivariant isomorphism. □

4.3 Application to Jacquet-Langlands for Eigencurves

Theorem 4.7. *Fix $n > 1$ and let $\{\mathcal{D}^{(m)} := \mathcal{D}^{(m)][n]}$ be a projective system of finite p-primary abelian groups defined as in the last setion. Let $\mathcal{D}^\circ$ and $\mathcal{D}_0^\circ$ be the associated systems of finite sheaves on Y and Y_0 respectively. Let $\mathcal{D}^\circ := \varprojlim_m F^{(m)}$ be the projective limit. Then we have canonical Hecke equivariant exact sequences*

(a) $0 \to H_c^1(Y_{0,s}, \mathcal{D}_{0,s}^\circ)^{\mathrm{new}} \to \mathcal{S}_{\mathbb{GL}_2}^{new}(N\ell, \mathcal{D}^\circ) \to \mathcal{S}_{\mathbb{B}^\times}(N, \mathcal{D}^\circ) \longrightarrow H_c^2(\Gamma, \mathcal{D}^\circ)^2;$

(b) $0 \to \mathcal{S}_{\mathbb{B}^\times}(N, \mathcal{D}^\circ) \to H^1_c(Y_{0,s}, \mathcal{D}^\circ_{0,s}) \xrightarrow{\mu_s} (H^1_c(Y_s, \mathcal{D}^\circ_s))^2 \to 0$;

(c) $Ker(M_{\mathcal{D}^\circ}) \to \mathcal{S}_{\mathbb{B}^\times}(N, \mathcal{D}^\circ) \to H^1_c(Y_{0,s}, \mathcal{D}^\circ_{0,s})^{\mathrm{new}} \to Coker(M_{\mathcal{D}^\circ}) \to 0$.

Tensoring the map $\mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, \mathcal{D}^\circ) \to \mathcal{S}_{\mathbb{B}^\times}(N, \mathcal{D}^\circ)$ with K we obtain a Hecke equivariant morphism

$$\mathcal{S}^{new}_{\mathbb{GL}_2}(N\ell, \mathcal{D}[n]) \to \mathcal{S}_{\mathbb{B}^\times}(N, \mathcal{D}[n])$$

and the above theorem can be used to prove that this map induces a closed immersion of the quaternionic eigencurve into the new part of the modular eigencurve.

5 Finite slope decompositions

Let as before K denote a finite extension of $\mathbb{Q}_p$ and let $(A, |\cdot|)$ be a Banach K-algebra. Let also (M, u) be a pair consisting of an ON-able Banach A-module M together with a completely continuous, A-linear homomorphism $u : M \longrightarrow M$. We recall a few definitions and results from chapter 4 of [AS2].

Definition 5.1. An element $a \in A^\times$ will be called a multiplicative unit if $|ax| = |a||x|$ for all elements $x \in A$.

Let us remark that if $F(t) \in A[[t]]$ is non-zero then it has a well defined Newton polygon (see §4.2 of [AS2].) Briefly, we first define the valuation $v_A : A - \{0\} \longrightarrow \mathbb{Q}$ by the relation $|x|_A = p^{-v_A(x)}$ for every $x \in A - \{0\}$. Than, the Newton polygon of the non-zero series $F(t) = \displaystyle\sum_{n=0}^{\infty} a_n t^n \in A[[t]]$ is the sub-convex hull in $\mathbb{R}^2$ of the following set:

$$\{(i, v_A(a_i)) \quad | \quad i \in \mathbb{N} \text{ such that } a_i \neq 0\}.$$

Definition 5.2. Let $h \in \mathbb{Q}$ and non-zero elements $P(t) \in A[t]$ and $F(t) \in A[[t]]$.

a) We say that $P(t)$ has *slope* $\leq h$ if the leading coefficient of $P(t)$ is a multiplicative unit, $P(0) = 1$ (i.e. $P(t)$ is a Fredholm polynomial) and if all the slopes of the edges of its Newton polygon are smaller or equal to h.

b) We say that $F(t)$ has *slope* $\leq h$ if $F(0) = 1$ (i.e. $F(t)$ is a Fredholm series) and all the slopes of the edges of its Newton polygon are larger then h.

Remark 5.3. 1) Let us notice that the constant polynomial $P(t) = 1$ has slope $\leq h$ for all $h \in \mathbb{Q}$.

2) The hypothesis on the leading and constant coefficient of the polynomial $P(t)$ (respectively of the constant coefficient of $F(t)$) in definition 5.2 are made in order to insure the following compatibility. Suppose that A is an affinoid algebra over K with affinoid space $X = \mathrm{Spm}(A)$. Let $x \in X$ and if $G(t) = \sum_{n=0}^{\infty} a_n t^n \in A[[t]]$ we set $G(t)_x := \sum_{n=0}^{\infty} a_n(x) t^n \in k(x)[[t]]$.

If $P(t) \in A[t]$ has slope $\leq h$ then for all $x \in X$, $P(t)_x$ has slope $\leq h$ in $k(x)[t]$. If $F(t) \in A[[t]]$ has slope $> h$, then for all $x \in X$ $F(t)_x$ has slope $> h$.

We will use the following notation: if $Q(t) \in A[t]$ is a non-zero polynomial of degree d, then $Q^*(t) := t^d Q(1/t)$.

Let now (M, u) be a pair as at the beginning of this section and $h \in \mathbb{Q}$. Let us define $\mathcal{S} = \mathcal{S}_h \subset A[t]$ to be the set of polynomials $Q^*(t)$, where $Q(t)$ runs over non-zero polynomials of slope $\leq h$. Let us remark that $1 \in \mathcal{S}$.

It is shown in section 4.6 of [AS2] that $\mathcal{S}$ is a multiplicatively closed subset of $A[t]$. Let us denote by $\mathcal{S}_{(M,u)}$ the image of $\mathcal{S}$ in $A[u]$.

Definition 5.4. A slope $\leq h$ decomposition of the pair (M, u) is a direct sum decomposition as $A[u]$-modules

$$M = M^{(h)} \oplus M_h,$$

with the properties:

a) $M^{(h)}$ is finitely generated as an A-module and for every $x \in M^{(h)}$ there is $Q^*(u) \in \mathcal{S}_{(M,u)}$ such that $Q^*(u)x = 0$.

b) For every $Q^*(u) \in \mathcal{S}_{(M,u)}$, the induced homomorphism $Q^*(u) : M_h \longrightarrow M_h$ is an isomorphism.

It is shown in [AS2] that if a slope $\leq h$ decomposition of (M, u) exists then it makes the following diagram of $A[u]$-modules with exact rows commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^{(h)} & \longrightarrow & M & \xrightarrow{j} & \mathcal{S}^{-1}_{(M,u)} M & \longrightarrow & 0 \\
 & & || & & || & & \downarrow\cong & & \\
0 & \longrightarrow & M^{(h)} & \longrightarrow & M & \longrightarrow & M_h & \longrightarrow & 0
\end{array}$$

Here the map j is the canonical one: $j(x) = \dfrac{x}{1}$.

We have the following:

Theorem 5.5. *If $A = K$, then for all $h \in \mathbb{Q}$ the pair (M, u) has a unique slope $\leq h$ decomposition.*

Theorem 5.6. *Let now $U \subset \mathcal{W}$ be an admissible affinoid sub-domain, $h \in \mathbb{Q}$ and $x_0 \in U(K)$. Then there exists an affinoid sub-domain $U_0 \subset U$ such that $x_0 \in U_0(K)$ and such that the pair $(M|_{U_0}, u|_{U_0})$ has a unique slope $\leq h$ decomposition over $A(U_0)$, compatible with the slope $\leq h$ decomposition of (M_{x_0}, u_{x_0}).*

Let us make precise the meaning of "compatible" slope $\leq h$ decompositions in theorem 5.6. Let $M|_{U_0} = M^{(h)} \oplus M_h$ be the slope $\leq h$ decomposition of $(M|_{U_0}, u|_{U_0})$ given by the theorem. Let $t_{x_0} \in A(U)$ denote a uniformizer at x_0 and denote by the same symbol its restriction to U_0. Let $M_{x_0} := M/t_{x_0}M$ and u_{x_0} the reduction of u. Then (M_{x_0}, u_{x_0}) is a pair consisting of an ON-able Banach module over K and a completely continuous K-linear operator. By theorem 5.5 the pair (M_{x_0}, u_{x_0}) has a slope $\leq h$ decomposition $M_{x_0} = M_{x_0}^{(h)} \oplus (M_{x_0})_h$. The compatibility in theorem 5.6 asserts that we have canonical isomorphisms as $K[u_{x_0}]$-modules $(M^{(h)})_{x_0} \cong M_{x_0}^{(h)}$ and $(M_h)_{x_0} \cong (M_{x_0})_h$.

All the details of the proofs of these results are in chapter 4 of [AS2] but let us sketch here outlines of the proofs for the convenience of the reader.

The main ideas used in proving these decompositions are the following. We first define the Fredholm determinant of u, $F(t) = F_{(M,u)}(t) = \det(1 - tu) \in A\{\{t\}\}$. If $h \in \mathbb{Q}$ and $F(t)$ is such a power series (it is entire and $F(0) = 1$) we try to find a slope $\leq h$ factorization of $F(t)$. This is a factorization of the type

$$F(t) = Q(t)S(t),$$

where $Q(t) \in A[t]$ has slope $\leq h$ and $S \in A\{\{t\}\}$ is a series of slope $> h$.

If a slope $\leq h$ factorization of the Fredholm determinant of u exists such that moreover the ideal in $A\{\{t\}\}$ generated by $Q(t)$ and $S(t)$ is the unit ideal we have a Riesz decomposition

$$M \cong M_Q \oplus M_S,$$

which can be shown to be the slope $\leq h$ decomposition of (M, u).

Now, if $A = K$ the program sketched above can be followed without obstruction: every entire, Fredholm series $F(t) \in K\{\{t\}\}$ has a slope $\leq h$ factorization and if $F(t) = F_{(M.u)}(t)$, the corresponding Riesz decomposition of (M, u) is the desired slope $\leq h$ decomposition. This proves theorem 5.5.

For theorem 5.6 let $F(t) = F_{(M,u)}(t) = \sum_{n=0}^{\infty} a_n t^n \in A(U)\{\{t\}\}$ be the Fredholm determinant of u and let $x_0 \in U(K)$. We denote, as in remark 5.3 by $t_{x_0} \in A(U)$ a uniformizer of $A(U)$ at x_0 and let $M_0 := M/t_{x_0}M$ and $u_0 : M_0 \longrightarrow M_0$ the induced $K = A(U)/t_{x_0}A(U)$-linear map. Then M_0 is an ON-able K-Banach module and u_0 is completely continuous. Moreover

$$F_0(t) = F(t)_(x_0) := \sum_{n=0}^{\infty} a_n(x_0)t^n$$

is the Fredholm determinant of u_0. By theorem 5.5 $F_0(t)$ has a slope $\leq h$ factorization $F_0(t) = Q_0(t)S_0(t)$. In general this factorization cannot be lifted to a slope $\leq h$ factorization of $F(t)$ but one can explicitly define an affinoid sub-domain $U_0 \subset U$ containing x_0 such that $F(t)|_{U_0}$ has a slope $\leq h$ factorization $F(t)|_{U_0} = Q(t)S(t)$ over $A(U_0)\{\{t\}\}$. Moreover this factorization has the property that $Q(t)_{x_0} = Q_0(t)$ and $S(t)_{x_0} = S_0(t)$. The slope $\leq h$ factorization of $F(t)|_{U_0}$ gives rise to a Riesz decomposition

$$M|_{U_0} = M_Q \oplus M_S,$$

which turns out to be the desired slope $\leq h$ decomposition of $(M|_{U_0}, u|_{U_0})$. This proves theorem 5.6.

Let now $U \subset \mathcal{W}$ be an admissible affinoid disk, $n \in \mathbb{N}$ and $M_U[n]$ any one of the $A(U)$-Banach modules $H^1_c(\Gamma, \mathbb{D}_U[n])$, $H^1_c(\Gamma, \mathbb{D}_U[n])^{\mathrm{new}-\ell}$, $S_U(\ell, Mp)[n]$. Let $u_n : M_U[n] \longrightarrow M_U[n]$ be the U_p-operator. Let us recall that $M_U[n]$ an ON-able Banach $A(U)$-module and u_n is completely continuous.

Lemma 5.7. *If $U_0 \subset U$ is an admissible affinoid sub-domain such that $(M_U[n]|_{U_0}, u_n|_{U_0})$ has a slope $\leq h$ decomposition. Then $(M_{U_0}[n], u_n)$ has a slope $\leq h$ decomposition.*

Proof. : □

Let us now fix $h \in \mathbb{Q}$, $n \in \mathbb{N}$ and $U \subset \mathcal{W}$ an admissible affinoid open. Let $M_U[n]$ be one of the $A(U)$-modules above and let $u_n : M_U[n] \longrightarrow M_U[n]$ be the U_p-operator. We know that if U is an affinoid disk then $M_U[n]$ is ON-able and u_n is completely continuous. We wish to compare Fredholm determinants and slope $\leq h$ decompositions for $M_U[n]$ and $M_U[n+1]$. We have (see chapter 6 of [AS2])

Proposition 5.8. *a) Suppose that U is an affinoid disk. Then the Fredholm determinants of u_n and u_{n+1} on $M_U[n]$ and $M_U[n+1]$ are equal.*

b) Suppose that both $(M_U[n], u_n)$ and $(M_U[n+1], u_{n+1})$ have slope $\leq h$ decompositions over U (where u_n, u_{n+1} are the U_p-operator for both modules). Then the natural restriction maps $r_n : M_U[n+1] \longrightarrow M_U[n]$ induce isomorphims $M_U[n+1]^{(h)} \cong M_U[n]^{(h)}$.

Proof. a) Let us first recall how the U_p-operator is defined. For $0 \leq a \leq p-1$ let $\beta_a := \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \in M_2(\mathbb{Z}_p)$. If $X \subset \mathbb{Z}_p$ is a subset we define $B[X, p^{-n}] = \{z \in \mathbb{C}_p \mid \text{there exists } x \in X, |z-x| \leq p^{-n}\}$. In section **??** we denoted $U[n] = B[\mathbb{Z}_p, p^{-n}]$. Let us remark that the β_a's shrink $U[n]$, i.e. they give analytic isomorphisms

$$\beta_a : U[n] \cong B[a + p\mathbb{Z}_p, p^{-n-1}],$$

where the action of β_a on $\mathcal{O}_{\mathbb{C}_p}$ is defined in lemma **??**. Therefore let us define by $v : \mathbb{D}_U[n] \longrightarrow \mathbb{D}_U[n+1]$ the $A(U)$-linear operator $v(\mu) = \sum_{a=0}^{p-1} \mu|\beta_a$. The operator $u_n = U_p$ on $M_U[n]$ is then defined by the composition $u_n = r_n \circ v$ and we have a natural commutative diagram (where the diagonal map is v):

$$\begin{array}{ccc} M_U[n] & \xrightarrow{u_n} & M_U[n] \\ r_n \uparrow & \searrow & r_n \uparrow \\ M_U[n+1] & \xrightarrow{u_{n+1}} & M_U[n+1] \end{array}$$

Now the fact that the Fredholm determinants of u_n and u_{n+1} are equal follows from the following linear algebra lemma.

Lemma 5.9. *Let X, Y be ON-able Banach modules over $A(U)$ and $u : X \longrightarrow X$, $u' : Y \longrightarrow Y$, $r : Y \longrightarrow X$ and $v : X \longrightarrow X$ continuous $A(U)$-linear maps such that $u = r \circ v$ and $u' = v \circ r$. Assume that r is completely continuous. Then*

i) Both u and u' are completely continuous.

ii) For every $n \geq 0$ we have $\mathrm{Tr}(u^n) = \mathrm{Tr}((u')^n)$.

iii) $\det(1 - tu) = \det(1 - tu')$.

b) Let us now prove that the restriction map $r_n : M_U[n+1] \longrightarrow M_U[n]$ induces an isomorphism on the slope $\leq h$ submodules. As both $M_U[n]^{(h)}$

and $M_U[n+1]^{(h)}$ are finitely generated $A(U)$-modules, there is a polynomial $Q(t) \in A(U)[t]$ with slope $\leq h$ such that $Q^*(u)M_U[n]^{(h)} = Q^*(u)M_U[n+1]^{(h)} = 0$. Let us recall that as $Q(t)$ has slope $\leq h$ its leading coefficient is a multiplicative unit therefore we may suppose $Q(t)$ monic, i.e. $Q^*(t) = 1 - tP(t)$ for some polynomial $P(t) \in A(U)[t]$. It follows that on both $M_U[n]^{(h)}$ and $M_U[n+1]^{(h)}$ we have $u_n P(u_n) = id$ (respectively $u_{n+1}P(u_{n+1}) = id$), i.e. the restrictions of u_n and u_{n+1} to $M_U[n]^{(h)}$ (and respectively to $M_U[n+1]^{(h)}$) are isomorphisms.

Let us now show the injectivity of $r_n : M_U[n+1]^{(h)} \longrightarrow M_U[n]^{(h)}$. Let $x \in M_U[n+1]^{(h)}$ be such that $r_n(x) = 0$. Therefore $u_{n+1}(x) = v(r_n(x)) = 0$ and because u_{n+1} is an isomorphism it follows that $x = 0$.

For surjectivity, let $y \in M_U[n]^{(h)}$. Let $z \in M_U[n]^{(h)}$ be such that $u_n(z) = y$. Set $x = v(z) \in M_U[n+1]^{(h)}$. We have $r_n(x) = r_n(v(z)) = u_n(z) = y$ and we are done.

$\square$

6 The Jacquet-Langlands correspondence on eigencurves

We'll first define the notion of a *compatible projective system of Banach-modules* and respectively of a *compatible inductive system of Banach-modules* on the weight space $\mathcal{W}$ (the latter we defined in [Ch].) A compatible projective system of Banach modules $\mathbb{M}$ on $\mathcal{W}$ is the following type of data: for each affinoid subdomain $U \subset \mathcal{W}$, we associate a projective system $(\mathbb{M}_U[n])_{n \geq d_U}$ of Banach $A(U)$-modules such that each module $\mathbb{M}_U[n]$ has an action of the abstract Hecke algebra $\mathcal{H}_\mathbb{Z}$ generated over $\mathbb{Z}$ by T_q for $q \not| Mp\ell$ and U_p and the transition maps are euqivariant for this action. Let us denote by $\mathcal{H} := \Lambda \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$, where $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ the Λ-adic universal Hecke algebra. Moreover for every $V \subset U$ admissible affinoid open of $\mathcal{W}$, and for all $n \geq d_U$, we have $A(U)$-linear morphisms which are continuous, compatible with the projective system maps and with the Hecke operators $\mathbb{M}_U[n] \longrightarrow \mathbb{M}_V[n]$. For a sequence $W \subset V \subset U$ of affinoid subdomains the restriction maps defined above satisfy the usual comaptibility relation.

It is clear what a morphism of compatible projective systems of Banach-modules is. Example of such objects and morphisms between them are:

1) $\mathbb{H}$, where $\mathbb{H}_U[n] := H^1_c(\Gamma, \mathbb{D}_U[n])$

2) $\mathbb{S}(B)$, where $\mathbb{S}(B)_U[n] = \mathbb{S}_U(\ell, Mp)[n]$

3) $\mathbb{H}^{\mathrm{new}-\ell}$, where $\mathbb{H}_U^{\mathrm{new}-\ell}[n] = H_c^1(\Gamma, \mathbb{D}_U[n])^{\mathrm{new}-\ell}$.

We have natural morphisms $\mathbb{H} \longrightarrow \mathbb{S}(B)$ and an exact sequence in this category

$$0 \longrightarrow \mathbb{S}(B) \longrightarrow \mathbb{H}^{\mathrm{new}-\ell} \longrightarrow \mathbb{S}(B) \longrightarrow 0.$$

The definition of a compatible inductive system of Banach modules on $\mathcal{W}$ is defined similarly and an example is $\mathbb{S}_\Gamma$ defined as follows: for every admissible affinoid open $U \subset \mathcal{W}$ and $n \geq 0$, $\mathbb{S}_U[n] := (\mathbb{S}_\Gamma)_U[n]$ is the space of p-adic families over $A(U)$ of modular forms of tame level $\Gamma_0(M\ell)$ which are $1/(p+1)p^n$-overconvergent, the map $\mathbb{S}_U[n] \longrightarrow \mathbb{S}_U[n+1]$ being induced by restriction.

Let us denote by $(*)$ the following property of a compatible projective (respectively inductive) system of Banach modules $\mathbb{M}$.

There is an admissible covering $\mathcal{U} := \{U\}$ of $\mathcal{W}$ by admissible affinoid opens with the following properties:

a) $\mathbb{M}_U[n]$ is ON-able $A(U)$-module for all $U \in \mathcal{U}$ and $n \geq d_U$ and the U_p-operator on $\mathbb{M}_U[n]$ is compact.

b) The characteristic power series of U_p on $\mathbb{M}_U[n]$ is independent of $n \geq d_U$. We denote by $P_{\mathbb{M},U}(t)$ this power series.

Suppose from now on that $\mathbb{M}$ satisfies $(*)$. Then the power series $P_{\mathbb{M},U}(t)$ glue on $\mathcal{W}$ to give a power series $P_{\mathbb{M}}(t) \in \Lambda\{\{t\}\}$, where $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$.

Remark 6.1. The compatible systems $\mathbb{H}, \mathbb{D}(B), \mathbb{H}^{\mathrm{new}-\ell}, \mathbb{S}_\Gamma$ satisfy the property $(*)$ for the same admissible covering $\mathcal{U}$. In fact the open sets in $\mathcal{U}$ may be taken to be affinoid disks.

Let us denote by $P_{\mathrm{new}-\ell} \in \Lambda\{\{t\}\}$ the characteristic power series of the U_p-operator acting on the p-adic families of elliptic modular forms of tame level $M\ell$, new al ℓ.

Lemma 6.2. *We have*

$P_{\mathbb{S}(B)} = P_{\mathrm{new}-\ell}$ and $P_{\mathbb{H}^{\mathrm{new}-\ell}}(t) = P_{\mathbb{S}(B)}(t)^2$.

Now we are going to attach in a functorial (contravariant) way to a compatible projective (respectively inductive) system of Banach-modules $\mathbb{M}$ satisfying the condition $(*)$ a rigid analytic variety $\mathcal{C}_{\mathbb{M}}$ together with a natural morphism $\kappa : \mathcal{C}_{\mathbb{M}} \longrightarrow \mathcal{W}$ as follows.

We first define $Z_{\mathbb{M}} \subset \mathcal{W} \times \mathbb{A}^1_{\mathrm{rig}}$ as the rigid analytic variety associated to the noetherian, reduced $\mathbb{Z}_p$-algebra $R_Z := \Lambda\langle t\rangle/P_{\mathbb{M}}(t)\Lambda\langle t\rangle$. In other words the $\mathbb{C}_p$-points of $Z_{\mathbb{M}}$ are pairs $(x, \lambda) \in \mathcal{W}(\mathbb{C}_p) \times \mathbb{C}_p$ such that $(P_{\mathbb{M}})_x(\lambda) = 0$. We have the following diagram of rigid spaces and morphisms:

$$\begin{array}{ccccc} Z_{\mathbb{M}} & \hookrightarrow & \mathcal{W} \times \mathbb{A}^1_{\mathrm{rig}} & \xrightarrow{\mathrm{pr}_2} & \mathbb{A}^1_{\mathrm{rig}} \\ & & \downarrow \mathrm{pr}_1 & & \\ & & \mathcal{W} & & \end{array}$$

associated to the ring homomorphisms: $\Lambda \longrightarrow R_Z \longrightarrow \mathbb{Z}_p\langle t\rangle$ where the second map is induced by the natural augmentation $\Lambda \longrightarrow \mathbb{Z}_p$.

We denote by $\mathcal{T} := \mathcal{T}_{\mathbb{M}}$ the collection of pairs $(U, Q(t))$ where $U \subset \mathcal{W}$ is a connected admissible open affinoid and we have a factorization $P_{\mathbb{M}}(t) = Q(t)S(t) \in 1 + tA(U)\{\{t\}\}$ such that $Q(t) \in A(U)[t]$ with $Q(0) = 1, Q^*(0) \in A(U)^\times$ and $(Q(t), S(t)) = 1$.

Lemma 6.3 ([Co1])**.** *a) There is a natural bijection between $\mathcal{T}$ and the collection $\mathcal{T}_Z := \mathcal{T}_{Z,\mathbb{M}}$ of admissible open affinoids Y of $Z_{\mathbb{M}}$ such that:*

i) $V := \mathrm{pr}_1(Y)$ is an admissible open affinoid of $\mathcal{W}$

ii) Y is a connected component of $\mathrm{pr}_1^{-1}(V) = \mathrm{pr}_1^{-1}(\mathrm{pr}_1(Y))$

iii) $\mathrm{pr}_1|_Y : Y \longrightarrow V$ is a finite morphism.

b) $\mathcal{T}_Z$ is an admissible covering of $Z_{\mathbb{M}}$.

Proof. : We will only recall here how the bijection is defined and send the reader to [Co1] for the details.

Suppose that $(U, Q(t)) \in \mathcal{T}$. Let us define the $A(U)$-algebra $A(Y) := A(U)[t]/(Q(t)A(U)[t]$. Obviously $A(Y)$ is an affinoid algebra over $A(U)$ and let Y be the associated affinoid. The $\mathbb{C}_p$-points of Y are elements $(x, \lambda) \in U(\mathbb{C}_p) \times \mathbb{C}_p$ such that $Q_x(\lambda) = 0$ and so $Y(\mathbb{C}_p) \subset Z_{\mathbb{M}}(\mathbb{C}_p)$. Now the factorization $P_{\mathbb{M}}(t) = Q(t)S(t)$ with the properties above imply that Y is an admissible open affinoid of $Z_{\mathbb{M}}$ and it has the required properties.

Conversely, let $Y \in \mathcal{T}_Z$ and let us denote by $U = \mathrm{pr}_1(Y) \subset \mathcal{W}$ and $Z_Y := Z_{\mathbb{M}} \cap (U \times \mathbb{A}^1_{\mathrm{rig}}) = \mathrm{pr}_1^{-1}(U)$. By property ii) Y is a connected component of Z_Y therefore it is open (and closed) in Z_Y and so it is flat over U. By iii) Y is finite over U, therefore $A(Y)$ is a finite projective $A(U)$-module generated as an $A(U)$-algebra by the image of t. Let $Q(t)$ denote the characteristic polynomial of the $A(U)$-linear map multiplication by t on $A(Y)$. It is monic and $Q(0) \in A(U)^\times$. The canonical surjective homomorphism of $A(U)$-algebras $A(U)[t]/Q(t)A(U)[t] \longrightarrow A(Y)$ is an isomorphism as both

$A(U)$-modules are projective of equal rank. One may now deduce that $Q(t)$ divides $P_{\mathbb{M}}(t)$ in $A(U)\{\{t\}\}$ and that $(Q(t), P_{\mathbb{M}}(t)/Q(t)) = 1$. $\square$

To each $Y \in \mathcal{T}_Z$ as in lemma 6.3 we associate an affinoid $T(Y)$ as follows. Let $(U, Q(t)) \in \mathcal{T}$ be the pair associated to Y by lemma 6.3. The factorization $P_{\mathbb{M}}(t) = Q(t)S(t)$ with $(Q(t), S(t)) = 1$ over $A(U)$ gives, for every $n \geq d_U$ a Riesz decomposition

$$\mathbb{M}_U[n] = \mathbb{M}_U[n]_Q \oplus \mathbb{M}_U[n]_S,$$

compatible with the transition maps $\mathbb{M}_U[n+1] \longrightarrow \mathbb{M}_U[n]$ if $\mathbb{M}$ is a compatible projective system and with $\mathbb{M}_U[n] \longrightarrow \mathbb{M}_U[n+1]$ if $\mathbb{M}$ is a compatible inductive system of Banach modules on $\mathcal{W}$. In fact by proposition 5.8 the above transition maps define isomorphisms $\mathbb{M}_U[n+1]_Q \cong \mathbb{M}_U[n]_Q$ and we'll denote henceforth these modules by $\mathbb{M}_{U,Q}$. Let $\mathcal{H}(Y)$ denote the image of $\mathcal{H} \otimes_\Lambda A(U) \longrightarrow \mathrm{End}_{A(U)}(\mathbb{M}_{U,Q})$, where let us recall $\mathcal{H}$ is the Λ-algebra generated by the Hecke operators. As $\mathbb{M}_{U,Q}$ is a finite projective (in fact free as U is connected) $A(U)$-module, if we endow $\mathrm{End}_{A(U)}(\mathbb{M}_{U,Q})$ with the sup-norm, it becomes a finite free Banach $A(U)$-module. In particular, as $\mathcal{H}(Y)$ is one of its $A(U)$ commutative subalgebras, it is itself finite, free and (topologically) closed i.e. an affinoid $A(U)$-algebra. We denote by $T(Y)$ its associated affinoid. We have natural ring homomorphisms $A(U) \longrightarrow A(Y) = A(U)[t]/Q(t)A(U)[t] \cong A(U)[U_p|\mathbb{M}_{U,Q})] \longrightarrow \mathcal{H}(Y)$ which induce morphisms of affinoids $(**)$ $\quad T(Y) \xrightarrow{\pi_Y} Y \xrightarrow{\mathrm{pr}_1} U$ with composition $\kappa_Y := \mathrm{pr}_1 \circ \pi_Y$.

We have the following

Proposition 6.4 ([Co1]). *a) If $Y_1, Y_2 \in \mathcal{T}_Z$ then $Y_1 \cap Y_2 \in \mathcal{T}_Z$ and the natural map $T(Y_1 \cap Y_2) \longrightarrow T(Y_1)$ is an open immersion.*

b) for any pair $Y, Y' \in \mathcal{T}_Z$ let $T(Y, Y')$ denote the image of $T(Y \cap Y')$ in $T(Y)$ via the open immersion at a) and let $i(Y, Y') : T(Y \cap Y') \longrightarrow T(Y, Y')$ be the natural isomorphism. Let us denote by $\varphi(Y, Y') := i(Y, Y') \circ i(Y', Y)^{-1}$. Then the data $\big((T(Y)_{Y \in \mathcal{T}_Z}, (T(Y, Y'))_{Y, Y' \in \mathcal{T}_Z}, (\varphi(Y, Y'))_{Y, Y' \in \mathcal{T}_Z}\big)$ is a gluing data for the family of affinoids $(T(Y))_{Y \in \mathcal{T}_Z}$ as in [BGR], 9.3.2.

Proof. : We will only sketch the main ideas in a). Let us fix $Y \in \mathcal{T}_Z$, denote $U = \mathrm{pr}_1(Y)$ and let $V \subset U$ be any admissible affinoid open. Let $Y' := \mathrm{pr}_1^{-1}(V) \cap Y \subset Z_{\mathbb{M}}$. Then $Y' \in \mathcal{T}_Z$ and we claim that the canonical morphism $T(Y') \longrightarrow T(Y)$ is an open immersion. To see this let $(U, Q(t)) \in \mathcal{T}$ be the pair associated to Y. Then the pair associated to Y' is $(U, Q'(t))$, where $Q'(t) = Q(t)|_V$. By the uniqueness of Riesz decomposition $\mathbb{M}_{V,Q'} = \mathbb{M}_{U,Q}|_V = \mathbb{M}_{U,Q} \otimes_{A(U)} A(V)$. As $\mathbb{M}_{U,Q}$ is a finite free $A(U)$-module the natural map

$$A(V) \otimes_{A(U)} \mathrm{End}_{A(U)}(\mathbb{M}_{U,Q}) \longrightarrow \mathrm{End}_{A(V)}(\mathbb{M}_{V,Q'})$$

is an isomorphism and we have a natural commutative diagram

$$\begin{array}{ccc} A(V) \otimes_{A(U)} \mathcal{H}(Y) & \hookrightarrow & A(V) \otimes_{A(U)} \mathrm{End}_{A(U)}(\mathbb{M}_{U,Q}) \\ \downarrow & & \downarrow \cong \\ \mathcal{H}(Y') & \hookrightarrow & \mathrm{End}_{A(V)}(\mathbb{M}_{V,Q'}) \end{array}$$

The first horizontal map is injective as $A(U) \longrightarrow A(V)$ is flat and the left vertical arrow is surjective. It follows that the natural map $A(V) \otimes_{A(U)} \mathcal{H}(Y) \longrightarrow \mathcal{H}(Y')$ is an isomorphism which implies that $Y' \longrightarrow Y$ is an open immersion.

Now if $Y_1, Y_2 \in \mathcal{T}_Z$ by the above argument $Y_1 \cap \mathrm{pr}_1^{-1}(\mathrm{pr}_1(Y_2)) \in \mathcal{T}_Z$ so we may reduce to the case $\mathrm{pr}_1(Y_1) = \mathrm{pr}_2(Y_2)$. Then it is clear that $Y_1 \cap Y_2 \in \mathcal{T}_Z$ so may in fact assume further that $Y_2 \subset Y_1$, which is then an open immersion into a connected component. If Y_i is associated to the pair $(U, Q_i(t))$, $i = 1, 2$, then one proves that $Q_1(t)$ divides $Q_2(t)$ in $A(U)[t]$. It follows that $\mathbb{M}_{U,Q_1}$ is a direct factor of $\mathbb{M}_{U,Q_2}$ and so the natural morphism $\mathcal{H}(Y_1) \longrightarrow \mathcal{H}(Y_2)$ induces an isomorphism of $T(Y_2)$ with a connected component of $T(Y_1)$. In particular the map $T(Y_2) \longrightarrow T(Y_1)$ is an open immersion. $\square$

Remark 6.5. Let us remark that if we denote for $Y \in \mathcal{T}_Z$, $T(Y)^{\mathrm{red}}$ the affinoid attached to $\mathcal{H}(Y)/\mathfrak{N}(\mathcal{H}(Y))$, where $\mathfrak{N}(\mathcal{H}(Y))$ is the nilradical of $\mathcal{H}(Y)$, then the family $\big((T(Y)^{\mathrm{red}})_{Y \in \mathcal{T}_Z}, (T(Y, Y')^{\mathrm{red}})_{Y,Y' \in \mathcal{T}_Z}, (\varphi(Y, Y')^{\mathrm{red}})_{Y,Y' \in \mathcal{T}_Z}\big)$ is a gluing data for the family of affinoids $(T(Y)^{\mathrm{red}})_{Y \in \mathcal{T}_Z}$.

Let us denote by $\mathcal{C}_{\mathbb{M}}$ (respectively $\mathcal{C}_{\mathbb{M}}^{\mathrm{red}}$) the rigid analytic space obtained by gluing the family of affinoids $(T(Y))_{Y \in \mathcal{T}_Z}$ (respectively $(T(Y)^{\mathrm{red}})_{Y \in \mathcal{T}_Z}$). We have natural morphisms induced by the diagram $(**)$

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{M}} & \xrightarrow{\pi} & Z_{\mathbb{M}} \\ \downarrow \kappa & & \downarrow \mathrm{pr}_1 \\ \mathcal{W} & = & \mathcal{W} \end{array}$$

The main result of the Appendix, theorem **??** is that the reduced eigencurve $\mathcal{C}_{\mathbb{H}}^{\mathrm{red}}$ is canonically isomorphic to the eigencurve $\mathcal{C}_{\Gamma}^{\mathrm{red}}$, which is the reduced eigencurve whose points parameterize overconvergent elliptic eigenforms of finite slope and of tame level $M\ell$.

We have

Theorem 6.6. *The morphism between the compatible projective systems of Banach-modules $\mathbb{H} \longrightarrow \mathbb{S}(B)$ defines a closed immersion, the Jacquet-Langlands correspondence, $\varphi_{JL} : \mathcal{C}_{\mathbb{S}(B)} \hookrightarrow \mathcal{C}_{\Gamma}$ whose image is the new-ℓ-part of $\mathcal{C}_{\Gamma}$.*

Moreover, let us go back tot he construction of the eigencurve $\mathcal{C}_{\mathbb{H}}$. We had a sheaf of $\mathcal{O}_{\mathcal{W}}$-modules $\mathbb{H}^{\mathrm{fin}}$ on $T_{\mathbb{H}}$ and a sheaf of $\mathcal{O}_{\mathcal{W}}$-algebras on the same Grothendieck tolopogy $\mathcal{A}_{\mathbb{H}}$. Obviously, $\mathbb{H}^{\mathrm{fin}}$ is a sheaf of $\mathcal{A}_{\mathbb{H}}$-modules on $T_{\mathbb{H}}$. Therefore, $\mathbb{H}^{\mathrm{fin}}$ induces a coherent sheaf of $\mathcal{O}_{\mathcal{C}_{\mathbb{H}}}$-modules on $\mathcal{C}_{\mathbb{H}}$, denoted $\mathcal{H}$. Similarly $\left(\mathbb{H}^{\mathrm{new}-\ell}\right)^{\mathrm{fin}}$ defines a coherent sheaf of $\mathcal{O}_{\mathcal{C}_{\mathbb{S}(B)}}$-modules on $\mathcal{C}_{\mathbb{S}(B)}$, denoted $\mathcal{H}^{\mathrm{new}-\ell}$.

Theorem 6.7. *a) The sheaf $\mathcal{H}$ is generically of rank two on $\mathcal{C}_{\mathbb{H}}$ and we have a canonical isomorphism $\varphi_{JL}^*(\mathcal{H}) \cong \mathcal{H}^{\mathrm{new}-\ell}$ as sheaves on $\mathcal{C}_{\mathbb{S}(B)}$.*

b) We have a canonical exact sequence of sheaves on $\mathcal{C}_{\mathbb{S}(B)}$:

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}_{\mathbb{S}(B)}} \longrightarrow \mathcal{H}^{\mathrm{new}-\ell} \longrightarrow \mathcal{O}_{\mathcal{C}_{\mathbb{S}(B)}} \longrightarrow 0.$$

(**Note to ourselves:** The following theorem does not belong here.)

Theorem 6.8. *Let $U \subseteq \mathcal{W}$ be a K-admissible affinoid open and $\kappa \in U(K)$ be an arithmetic point. Then we have a canonical commutative diagram of Hecke modules*

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(\mathcal{K}, \mathcal{D}_U) & \longrightarrow & H^1(\Gamma, \mathcal{D}_U)^{\ell\text{-}new} & \longrightarrow & S(\mathcal{K}, \mathcal{D}_U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S(\mathcal{K}, V_\kappa) & \longrightarrow & H^1(\Gamma, V_\kappa)^{\ell\text{-}new} & \longrightarrow & S(\mathcal{K}, V_\kappa) \end{array}$$

Proof. For each integer $t \geq 1$, let $F_t := V_\kappa^\circ / p^t V_\kappa^\circ$, which we regard as a finite $\Xi^{(d)}$-module. As in Corollary **??**, we let $\mathcal{S}_{n,t}$ denote the directed set of finite $\Xi(\mathbb{Z}_p)$-submodules of $\mathbb{D}_U[n]^\circ / p^t \mathbb{D}_U[n]^\circ$. By Theorem **??** we have, for each triple $(n, t, F_{n,t})$ with $F_{n,t} \in \mathcal{S}_{n,t}$, a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(\mathcal{K}, F_{n,t}) & \longrightarrow & H^1(\Gamma, F_{n,t})^{\ell\text{-}new} & \longrightarrow & S(\mathcal{K}, F_{n,t}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S(\mathcal{K}, F_t) & \longrightarrow & H^1(\Gamma, F_t)^{\ell\text{-}new} & \longrightarrow & S(\mathcal{K}, F_t) \end{array}$$

We note that formation of $S(\mathcal{K},*)$ and $H^1_c(\Gamma,*)$ commute with direct and inverse limits, as well as with $\otimes K$. So, since $\varinjlim$ and $\otimes K$ are exact functors and $\varprojlim$ is left exact, we may use Corollary **??** to pass to the limit of the above diagram over the directed set of all triples $(n,t,F_{n,t})$ to obtain the exact sequence of Theorem 6.8. This completes the proof. □

7 Appendix

by Glenn Stevens

In this appendix we wish to compare two reduced eigencurves: $\mathcal{C}^{\mathrm{red}}_{\mathbb{H}}$ and $\mathcal{C}^{\mathrm{red}}_{\Gamma}$. Let us recall the projective system of Banach module $\mathbb{H}$, where $\mathbb{H}_U[n] = H^1_c(\Gamma,\mathbb{D}_U[n])$, which produces the eigencurve $\mathcal{C}^{\mathrm{red}}_{\mathbb{H}}$. In fact we prefer to work here with the projective system of Banach modules $\mathbb{H}^-$ defined by $\mathbb{H}^-_U[n] = H^1_c(\Gamma,\mathbb{D}_U[n])^{\iota=-1}$, where ι is the natural involution acting on $H^1_c(\Gamma,\mathbb{D}_U[n])$. The Hecke equivariant morphism of compatible systems of Banach modules $\mathbb{H}^-\hookrightarrow\mathbb{H}$ induces isomorphisms of rigid spaces $\mathcal{C}_{\mathbb{H}}\cong\mathcal{C}_{\mathbb{H}^-}$ and $\mathcal{C}^{\mathrm{red}}_{\mathbb{H}}\cong\mathcal{C}^{\mathrm{red}}_{\mathbb{H}^-}$.

In section 6 we also mentioned the compatible system of inductive Banach modules $\mathbb{S}_\Gamma$ where $(\mathbb{S}_\Gamma)_U[n]$ is the $A(U)$-Banach module of $1/(p+1)p^n$-overconvergent p-adic families of modular forms of tame level $\Gamma_0(M\ell)$, which produces $\mathcal{C}^{\mathrm{red}}_{\Gamma}$.

Proposition 7.1. *Let us fix $w\in\mathcal{H}$ an operator which acts completely continuously on every $\mathbb{H}^-_U[n]$ and $\mathbb{S}_U[n]$ for $U\in\mathcal{U}$, $n\geq d_U$. For example $w=vU_p$ for any $v\in\mathcal{H}$. Then the Fredholm determinants $P_{w,\mathbb{S},U,n}(t) := \det(1-tw|\mathbb{S}_U[n])$ and $P_{w,\mathbb{H}^-,U,n} := \det(1-tw|\mathbb{H}^-_U[n])$ are independent of n.*

As a consequence of proposition 7.1, the power series denoted $P_{w,\mathbb{S},U}(t)$ and $P_{w,\mathbb{H}^-,U}(t)$ glue for various U's to give Fredholm series $P_{w,\mathbb{S}}(t), P_{w,\mathbb{H}^-}(t)\in 1+t\Lambda\{\{t\}\}$.

Proposition 7.2. *Let $w\in\mathcal{H}$ be as in proposition 7.1. Then $P_{w,\mathbb{S}}(t) = P_{w,\mathbb{H}^-}(t)$ as power series in $\Lambda\{\{t\}\}$.*

The Fredholm series of w on $\mathbb{S}$ or on $\mathbb{H}^-$ as in the proposition 7.2 will be denoted $P_w(t)$. In particular the Fredholm series of U_p are equal. These were denoted in section 6 by $P_{\mathbb{S}}(t) = P_{\mathbb{H}^-}(t)$ and will be denoted henceforth by $P(t)$. Let us remark that the two spectral curves $Z_{\mathbb{S}}$ and $Z_{\mathbb{H}^-}$ are canonically isomorphic (they only depend on the Fredholm series $P(t)$ of U_p) and will be denoted Z_P.

Let us now fix $\kappa \in \mathcal{W}(K) - \{0\}$ and $n \geq d_\kappa$. Let us denote $\mathbb{H}^-_\kappa[n] := H^1_c(\Gamma, \mathbb{D}_\kappa[n])^{\iota=-1}$ and by $\mathbb{S}_\kappa[n]$ the K-Banach space of $1/(p+1)p^n$-overconvergent modular forms of tame level $\Gamma_0(M\ell)$ and weight $\kappa+2$. If $w \in \mathcal{H}$ is a Hecke operator as in proposition 7.1, then it acts completely continuously on $\mathbb{H}^-_\kappa[n]$ and $\mathbb{S}_\kappa[n]$. Moreover the Fredholm series of U_p on $\mathbb{H}^-_\kappa[n]$ and $\mathbb{S}_\kappa[n]$ are equal and equal to $P_\kappa(t) = \sum_{n=0}^{\infty} a_n(\kappa)t^n$ in $P(t) = \sum_{n=0}^{\infty} a_n t^n \in \Lambda\{\{t\}\}$.

Suppose that we have a factorization $P_\kappa(t) = Q(t)S(t) \in 1 + tK\{\{t\}\}$ where $Q(t) \in K[t]$ satisfies $Q(0) = 1$, and $Q(t), S(t)) = 1$. Let $\mathbb{H}^-_\kappa[n] = \mathbb{H}^-_{\kappa,Q} \oplus (\mathbb{H}^-_\kappa[n])_S$ and $\mathbb{S}_\kappa[n] = \mathbb{S}_{\kappa,Q} \oplus (\mathbb{S}_\kappa[n])_S$ be the associated Riesz decompositions, where let us recall that the $Q(t)$-components are independent of n.

Proposition 7.3. *In the above hypothesis and notations, the characteristic polynomials of w acting on $\mathbb{H}^-_{\kappa,Q}$ and $\mathbb{S}_{\kappa,Q}$ are equal.*

Let now $(U, Q(t)) \in \mathcal{T} = \mathcal{T}_{\mathbb{H}^-} = \mathcal{T}_{\mathbb{S}}$ be a pair as in section 6. Let us recall that $U \subset \mathcal{W}$ is a connected admissible affinoid open and $Q(t) \in A(U)[t]$ is such that $Q(0) = 1$, $Q^*(0) \in A(U)^\times$ and we have a factorization $P(t) = Q(t)S(t) \in 1 + tA(U)\{\{t\}\}$ with $(Q(t), S(t)) = 1$. Let $\mathbb{H}^-_{U,Q}$ and $\mathbb{S}_{U,Q}$ be the $Q(t)$-factors of $\mathbb{H}^-_U[n]$ and respectively $\mathbb{S}_U[n]$ for $n >> 0$ (which are independent of n) and let $w \in \mathcal{H}$ be a Hecke operator as in proposition 7.1.

Proposition 7.4. *a) $\mathbb{H}^-_{U,Q}$ and $\mathbb{S}_{U,Q}$ are finite free $A(U)$-modules of rank equal to the degree of $Q(t)$.*

b) The characteristic polynomials of w acting on $\mathbb{H}^-_{U,Q}$ and $\mathbb{S}_{U,Q}$ are the same.

Let as above $Y \in \mathcal{T}_Z = \mathcal{T}_{Z,\mathbb{H}^-} = \mathcal{T}_{Z,\mathbb{S}}$ be the admissible open affinoid of Z_P associated to the pair $(U, Q(t)) \in \mathcal{T}$ above. We denote by $\mathcal{H}(Y)_{\mathbb{H}^-}$ and $\mathcal{H}(Y)_{\mathbb{S}}$ the images of $\mathcal{H} \otimes_\Lambda A(U)$ in $\mathrm{End}_{A(U)}(\mathbb{H}^-_{U,Q})$ respectively in $\mathrm{End}_{A(U)}(\mathbb{S}_{U,Q})$.

Proposition 7.5. *We have a canonical isomorphism as $A(U)$ algebras*

$$\psi_Y : \mathcal{H}(Y)_{\mathbb{H}^-}/\mathfrak{N}(\mathcal{H}(Y)_{\mathbb{H}^-}) \cong \mathcal{H}(Y)_{\mathbb{S}}/\mathfrak{N}(\mathcal{H}(Y)_{\mathbb{S}})$$

which commutes with the natural morphisms from $A(Y)$.

Theorem 7.6. *The family of isomorphisms* $(\psi_Y)_{Y\in\mathcal{T}_Z}$ *in proposition 7.5 induces an isomorphism of rigid spaces* $\psi : \mathcal{C}_\Gamma^{\mathrm{red}} \longrightarrow \mathcal{C}_{\mathbb{H}^-}^{\mathrm{red}}$ *which commutes with the morphisms to* $\mathcal{W}$ *and* Z_P *respectively.*

8 Miscellaneous things to be used in the above.

Corollary 8.1. *a) Let* $\kappa = (k, \epsilon) \in \mathcal{W}$ *be an arithmetic weight where* $k \geq 2$ *and if* $k = 2$ *we assume that* ϵ *is a non-trivial character. Let us denote by* $S_\kappa(\Gamma_0(M\ell p^d))^{\mathrm{new}-\ell}$ *the space of classical "new at* ℓ*" cuspidal eigenforms for* $\Gamma_0(M\ell p^d)$ *of weight-character* κ*. Then we have an isomorphism as Hecke modules* $S_\kappa(\Gamma_0(M\ell p^d)^{\mathrm{new}-\ell} \cong S_\kappa(\ell, Mp^d)$*.*

b) Let $U \subset \mathcal{W}$ *be an affinoid subdomain,* $d = 1, n \geq 1$*. Proposition* **??** *implies that we have an exact sequence of* $A(U)$*-modules, Hecke equivariant*

$$0 \longrightarrow \mathbb{S}_U(\ell, Mp)[n] \longrightarrow H^1_c(\Gamma, \mathbb{D}_U[n])^{\mathrm{new}-\ell} \longrightarrow \mathbb{S}_U(\ell, Mp)[n]$$

compatible with specializations at arithmetic weights $\kappa \in U$*.*

Proof. a) Using the representation V_κ of $\mathrm{Iw}^{(d)}$ as in section §3 and proposition **??** we obtain an exact sequence, Hecke equivariant

$$0 \longrightarrow S_\kappa(\ell, Mp^d) \longrightarrow H^1_c(\Gamma, V_\kappa)^{\mathrm{new}-\ell} \xrightarrow{\varphi_U} S_\kappa(\ell, Mp^d) \longrightarrow 0.$$

The exactness on the right is a consequence of the condition on κ. By classical Eichler-Shimura, $H^1_c(\Gamma, V_\kappa)^{\mathrm{new}-\ell} \cong (S_\kappa(\Gamma_0(M\ell p^d))^{\mathrm{new}-\ell})^2$ and so a) follows.

b) We write $\mathbb{D}_U[n]$ as a projective limit of inductive limit of finite representations F as in proposition **??**, apply the proposition to each representation and take the limits. □

Let U, d, n be as in the corollary 8.1 and let us denote by $M_U[n] := H^1_c(\Gamma, \mathbb{D}_U[n])^{\mathrm{new}-\ell}/\mathbb{S}_U(\ell, Mp)$. We'd like to study the injective $A(U)$-linear map $\varphi_U : M_U[n] \longrightarrow \mathbb{S}_U(\ell, Mp)$.

Lemma 8.2. *Let* K *be a finite extension of* $\mathbb{Q}_p$ *with ring of integers* $\mathcal{O}_K$ *and residue field* k *and let us denote by* π *a uniformizer. Let* $A := K\langle T\rangle$ *be the Tate algebra of dimension* 1 *over* K *and let* M *denote a Banach* A*-module which is ON-able i.e. orthonormalizable. Let* N *be an* A*-submodule of* M *such that the natural map* $\overline{N} := N^\circ/\pi N^\circ \longrightarrow \overline{M} := M^\circ/\pi M^\circ$ *is injective.*

Then

a) N is a Banach A-module which is ON-able.

b) N is a direct summand of M in the category of ON-able Banach A-modules.

Proof. Let $\overline{A} = A^\circ/\pi A^\circ \cong k[T]$. We recall that in order to prove a) it would be enough to show that $\overline{N}$ is a free $\overline{A}$ module. But as M is ON-able, $\overline{M}$ is a free $\overline{A}$-module and as the latter is a PID, every submodule of $\overline{M}$ is free.

b) Choose an $\overline{A}$-basis of $\overline{N}$ which extends to a basis of $\overline{M}$. Lift these to orthonormal basis B, B' of N respectively M such that $B' \cap N = B$. Let $B'' = B' - B$. Then the unique A-submodule of M with orthonormal basis B'' is a complement of N. $\square$

Proposition 8.3. *Let U, d, n be as in corollary 8.1. If $U \subset \mathcal{W}$ is an affinoid disk then the modules $H^1_c(\Gamma, \mathbb{D}_U[n]), M_U[n], \mathbb{S}_U(\ell, Mp)$ are $A(U)$-Banach modules which are ON-able.*

Proof. First let us recall that by lemma 1.2 $\mathbb{D}_U[n]$ is an $A(U)$-Banach module which is ON-able.

We may write $H^1_c(\Gamma, \mathbb{D}_U[n]) \cong \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), \mathbb{D}_U[n])$. As $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is a finitely generated $\mathbb{Z}[\Gamma]$-module, let $D_1, D_2, ..., D_t$ be a set of generators. Then we have an inclusion as $A(U)$-modules

$$H^1_c(\Gamma, \mathbb{D}_U[n]) \hookrightarrow \prod_{i=1}^{t} \mathbb{D}_U[n] \text{ given by } \alpha \longrightarrow (\alpha(D_i))_{i=1,t}.$$

Without loss of generality we may assume that U is the affinoid disk of radius 1 centered at the origin and apply lemma 8.2.

We may proceed similarly with $\mathbb{S}_U(\ell, Mp)[n]$ by recalling that the double coset space $\Gamma\backslash\mathbf{PGL}_2(\mathbb{Q}_\ell)/\mathcal{K}_\ell$ is finite.

Finally, as now $H^1_c(\Gamma, \mathbb{D}_U[n])$ is known to be ON-able, its $A(U)$-submodule $H^1_c(\Gamma, \mathbb{D}_U[n])^{\mathrm{new}-\ell}$ is ON-able and a direct summand, therefore the quotient $M_U[n]$ is also ON-able. $\square$

Proposition 8.4. *Let U, d, n be as in proposition 8.3, i.e. $U \subset \mathcal{W}$ is an affinoid disk. Then the morphism $\varphi : M_U[n] \longrightarrow \mathbb{S}_U(\ell, Mp)[n]$ defined above is an isomorphism.*

Proof. Let us consider an arithmetic weight $\kappa = (k, \epsilon) \in U$ satisfying the condition of corollary 8.1 (i.e. such that $k \geq 2$ and if $k = 2$ then ϵ is a non-trivial character.) Let e be such that p^e is the conductor of ϵ and suppose that $e \leq n$. Then we have the following commutative diagram

$$\begin{array}{ccc} M_U[n] & \xrightarrow{\varphi_U} & \mathbb{S}_U(\ell, Mp)[n] \\ \downarrow \rho_\kappa & & \downarrow \rho_\kappa \\ H^1_c(\Gamma^{(e)}, V_\kappa)^{\text{new}-\ell}/S_\kappa(\ell, Mp^e) & \cong & S_\kappa(\ell, Mp^e) \end{array}$$

The lower horizontal map is an isomorphism by corollary 8.1. So now we have an $A(U)$-linear morphism φ_U between two ON-able modules such that specializations $\varphi_{U,\kappa}$ are isomorphisms for a dense subset of weights in U. It follows that φ_U is an isomorphism. **need more details here** $\square$

8.0.1 more modular stuff

Let p, ℓ, M be positive integers as in the introduction, i.e. p, ℓ are distinct primes and $(M, p) = (M, \ell) = 1$ and let us fix $n \geq 0$. We will view the arithmetic group $\Gamma_0(Mp\ell)$ as a subgroup of $\mathrm{Iw}(\mathbb{Z}_p)$. So for every K-affinoid subdomain $U \subset \mathcal{W}$, the group $\Gamma_0(Mp\ell)$ acts naturally on $\mathcal{D}_U$ and we may consider the $A(U)$-module

$$H^1_c(\Gamma, \mathcal{D}_U).$$

The elements of this module will be called distribution valued cohomological modular forms over $\Gamma_0(Mp\ell)$. The Hecke operators T_q for $q \not| Mp\ell$ and U_p act on $H^1_c(\Gamma_0(Mp\ell), \mathcal{D}_U)$ and the U_p-operator acts completely continuously.

Now let $\kappa \in \mathcal{W}(K)$ be an arithmetic point of signature (k, ϵ). By this we mean $k \in \mathbb{Z}^{\geq 0}$ is a non-negative integer, $\epsilon : \mathbb{Z}_p^\times \longrightarrow K^\times$ is a finite order character, and $\kappa \in \mathcal{W}(K)$ is given by $\kappa(t) = t^k \epsilon(t)$ for $t \in \mathbb{Z}_p^\times$. Under these conditions we denote by V_κ the finite dimensional K-vector space of homogeneous polynomials of degree k in $K[X, Y]$. If $d \geq 1$ and ϵ is defined modulo p^d, then we define the pair $(\mathrm{Iw}^{(d)}, \Xi^{(d)})$ by

$$\Xi^{(d)} := \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Xi(\mathbb{Z}_p) \;\middle|\; p^d \text{ divides } c \right\} \quad \text{and} \quad \mathrm{Iw}^{(d)} := \Xi^{(d)} \cap \mathrm{Iw}(\mathbb{Z}_p).$$

We let the semigroup $\Xi^{(d)}$ act on the right on V_κ by $(P|\gamma)(X, Y) = \epsilon(\gamma) P((X, Y)\gamma^*)$ for $\gamma \in \Xi^{(d)}$, where $\epsilon(\gamma)$ is the value of ϵ on the upper left corner of γ and $\gamma^* = \det(\gamma)\gamma^{-1}$. In particular, the arithmetic group $\Gamma_0(Mp^d\ell)$ acts on V_κ and we may consider the compactly supported cohomology

$$H^1_c(\Gamma_0(Mp^d\ell), V_\kappa),$$

which is a finite dimensional K-vector space. We will call the elements of this space classical cohomological modular forms of weight $k+2$, level $\Gamma_0(Mp^d\ell)$, and character ϵ.

We note that the function $T_0 \longrightarrow V_\kappa$ given by $(x,y) \longmapsto \kappa(x)(Y-zX)^k$ with $z = y/x$ is locally analytic and homogeneous of degree κ, so we have a canonical continuous linear map

$$\begin{array}{ccc} \mathcal{D}_\kappa & \longrightarrow & V_\kappa \\ \mu & \longmapsto & \int_{T_0} \kappa(x)(Y-zX)^k \mu(x,y) \end{array}$$

which is easily seen to be equivariant for the action of $\Xi^{(d)}$. For any affinoid subspace $S \subseteq \mathcal{W}$ and any arithmetic point $\kappa \in S$ we then let

$$\eta_\kappa^{alg} : \mathcal{D}_S \longrightarrow V_\kappa$$

be defined by the composition $\mathcal{D}_S \xrightarrow{\eta_\kappa} \mathcal{D}_\kappa \longrightarrow V_\kappa$. In particular we have the following proposition.

Proposition 8.5. *Let $U \subseteq \mathcal{W}$ be an affinoid subdomain and $\kappa \in U$ be an arithmetic point. Then the specialization map $\eta_\kappa^{alg} : \mathcal{D}_U \longrightarrow V_\kappa$ induces a canonical Hecke equivariant morphism*

$$\rho_\kappa : H^1_c(\Gamma_0(Mp\ell), \mathcal{D}_U) \longrightarrow H^1_c(\Gamma_0(Mp^d\ell), V_\kappa).$$

8.0.2 more quaternionic stuff

We will be interested in the sequel in two types of $\Xi^{(d)}$-representations. Namely for $d = 1$ and $V = \mathcal{D}_U$ we denote

$$\mathbb{S}_U(\ell, Mp) := S(\mathcal{K}(\ell, Mp), \mathcal{D}_U),$$

for $U \subset \mathcal{W}$ an affinoid subdomain and call the elements of this space distribution valued automorphic form on B of level $\mathcal{K}(\ell, Mp)$.

If $\kappa \in \mathcal{W}$ is a point of signature (k, ϵ) where $k \in \mathbb{Z}^{\geq 0}$ and ϵ is a finite order character defined modulo p^d $(d \geq 1)$. We denote by

$$S_\kappa(\ell, Mp^d) := S(\mathcal{K}(\ell, Mp^d), V_\kappa)$$

and call its elements classical automorphic forms of weight $k+2$, level $\mathcal{K}(\ell, Mp^d)$ and character ϵ.

We have the specialization map

$$\rho_\kappa : \mathbb{S}_U(\ell, Mp) \longrightarrow S_\kappa(\ell, Mp^d)$$

induced by the $\Xi^{(d)}$-equivariant map $\eta_\kappa^{alg} : \mathcal{D}_U \longrightarrow V_\kappa$ defined in the last section. It follows from the definitions that ρ_κ is Hecke-equivariant.

8.1 Some computations in sheaf cohomology

We obtain the obvious

Corollary 8.6. *a)* $R^0\Psi_X(\mathcal{L}) \cong (\alpha_s)_!(\mathcal{F}_s)$.

b) $R^1\Psi_X(\mathcal{L})$ *is a sheaf supported at* Σ_s *and we have*

$$R^1\Psi_X(\mathcal{L}) = \oplus_{y\in\Sigma_s}(i_y)_*(R^1\Psi_X\mathcal{L})_y = \oplus_{y\in\Sigma_s}(i_y)_*(R^1\Psi_Y(\mathcal{F}))_y,$$

where for every $y \in \Sigma_s$, i_y *is the composition* $y \longrightarrow \Sigma_s \subset X_s$.

c) $R^i\Psi_X(\mathcal{L}) = 0$ *for* $i \geq 2$.

Proposition 8.7. *a) For every* $y \in \Sigma_s$, $(R^1\Psi_Y(\mathcal{F}))_y \cong (\mathcal{F}_s)_y$

b) $(\mathcal{F}_s)_y \cong F$ *as* $\pi_1(Y_s, y) \cong \pi_1(Y, y) \cong \pi_1(Y, \overline{\eta})$*-representations.*

Proof. : a) For every $y \in \Sigma_s$ we have $(\mathrm{R}^1\Psi_Y(\mathcal{F}))_y \cong H^1(Y_{(y)} \times_S \eta, \mathcal{F})$, where $Y_{(y)}$ is the spectrum of the strict henselization of Y at y. Let $Z \xrightarrow{f} Y_S$ be a finite étale cover such that $f^*(\tilde{\mathcal{F}})$ is a constant sheaf on Z and let z be a geometric point of Z such that $f(z) = y$. Let us also remark that f is proper. We have

$$(\mathrm{R}^1\Psi_Y(\mathcal{F}))_y = (f_s^*(\mathrm{R}^1\Psi_Y(\mathcal{F})))_z = (\mathrm{R}^1\Psi_Z(f^*\tilde{\mathcal{F}}))_z = H^1(Z_{(z)} \otimes_S \eta, f^*\tilde{\mathcal{F}}).$$

As $f^*\mathcal{F}$ is constant, equal to $(f^*\tilde{\mathcal{F}})_z = \tilde{\mathcal{F}}_y$, if we denote by Λ the constant sheaf $\mathbb{Z}/p^m\mathbb{Z}$ on Y (and on Z and X) we have

$$(\mathrm{R}^1\Psi_Y(\mathcal{F}))_y = H^1(Z_{(z)} \times_S \eta, \Lambda) \otimes \tilde{\mathcal{F}}_y = (\mathrm{R}^1\Psi_Y(\Lambda))_y \otimes \tilde{\mathcal{F}}_y.$$

As before because on X, $C \cap \Sigma = \phi$, for every $y \in \Sigma_s$ we have $\mathrm{R}^1\Psi_Y(\Lambda)_y = \mathrm{R}^1\Psi_X(\Lambda)_y = \mathbb{Z}/p^m\mathbb{Z}$. The last equality follows from [Il] and we chose to ignore a Tate twist by -1. This proves a)

b) By lemma **??**, as $\tilde{\mathcal{F}}$ is locally constant we have for $y \in \Sigma_s$

$$(\mathcal{F}_s)_y \cong (\tilde{\mathcal{F}})_y \cong (\tilde{\mathcal{F}})_{\overline{\eta}} \cong \mathcal{F}_{\overline{\eta}} = F,$$

as $\pi_1(Y_s, y) \cong \pi_1(Y, y) \cong \pi_1(Y, \overline{\eta})$-representations. □

Let us now describe the sheaves $\mathrm{R}^i\Psi(\mathcal{L}^{(d)}), i \geq 0$. Let us consider the commutative diagram of curves and morphisms

$$\begin{array}{ccc} X_S^{(d)} & \xrightarrow{f_{S,d}} & X_S \\ \uparrow j_d & & \uparrow j \\ X_\eta^{(d)} & \xrightarrow{f_{\eta,d}} & X_\eta \end{array}$$

Lemma 8.8. *For every $i \geq 0$ we have $R^i\Psi_X((f_{\eta,d})_*\mathcal{L}^{(d)}) = (f_{S,d})_*(R^i\Psi_{X^{(d)}}\mathcal{L}^{(d)})$.*

Proof. We have $j \circ f_{\eta,d} = f_{S,d} \circ j^{(d)}$, therefore for all $n \geq 0$ we have $\mathrm{R}^n(j \circ f_{\eta,d}) = \mathrm{R}^n(f_{S,d} \circ j^{(d)})$. The Lerray spectral sequences computing these functors

$$\mathrm{R}^a j_* \circ \mathrm{R}^b(f_{\eta,d})_* \Longrightarrow \mathrm{R}^{a+b}(j \circ f_{\eta,d})_*$$

and

$$\mathrm{R}^a(f_{S,d})_* \circ \mathrm{R}^b j_*^{(d)} \Longrightarrow \mathrm{R}^{a+b}(f_{S,d} \circ j^{(d)})_*$$

degenerate at E_2 because $\mathrm{R}^k(f_{\eta,d})_* = 0$ and $\mathrm{R}^k(f_{S,d})_* = 0$ for $k \geq 1$ as $f_{\eta,d}$ and $f_{\eta,d}$ are finite morphisms. Therefore we have $\mathrm{R}^n j_* \circ (f_{\eta,d}) = (f_{S,d})_* \circ \mathrm{R}^n j_*^{(d)}$ which implies that $\mathrm{R}^n\Psi_X((f_{\eta,d})_*\mathcal{L}^{(d)}) = (f_{S,d})_*(\mathrm{R}^n\Psi_{X^{(d)}}\mathcal{L}^{(d)})$. $\square$

Corollary 8.9. *We have natural isomorphisms*

$$H^i(X_s, R^n\Psi_X((f_{\eta,d})_*\mathcal{L}^{(d)})) \cong H^i(X_s^{(d)}, R^n\Psi_{X^{(d)}}(\mathcal{L}^{(d)}),$$

for all $i, n \geq 0$. Thus we have

a) $H^i(X_s, R^0\Psi_X(f_{\eta,d})_*\mathcal{L}^{(d)}) \cong H_c^i(Y_s^{(d)}, \mathcal{F}^{(d)})$.

b) $H^0(X_s, R^1\Psi_X(f_{\eta,*})_*\mathcal{L}^{(d)}) \cong H^0(Y_s^{(d)}, R^1\Psi_{X^{(d)}}\mathcal{F}^{(d)})$.

8.2 Geometric Jacquet-Langlands

Let notations be as in section 2. Let us recall the diagram of section 2. In view of lemma **??** and corollary 8.9 it may be written as:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_c^1(Y_s, \mathcal{F}_s) & \longrightarrow & H_c^1(Y_\eta, \mathcal{F}) & \longrightarrow & H^0(Y_s, \mathrm{R}^1\Psi_X(\mathcal{F})) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_c^1(Y_s^{(d)}, \mathcal{F}_s^{(d)}) & \longrightarrow & H_c^1(Y_\eta^{(d)}, \mathcal{F}^{(d)}) & \longrightarrow & H^0(Y_s^{(d)}, \mathrm{R}^1\Psi_{X^{(d)}}(\mathcal{F}^{(d)}))
\end{array}$$

By identifying $\mathbb{C} \cong \bar{\mathbb{Q}}_\ell$ we may write $H_c^1(Y_\eta, \mathcal{F}) = H_c^1(Y_\mathbb{C}, \mathcal{F}_\mathbb{C}) = H_c^1(\Gamma, F)$ and $H_c^1(Y_\eta^{(d)}, \mathcal{F}^{(d)}) = H_c^1(Y_\mathbb{C}^{(d)}, \mathcal{F}_\mathbb{C}^{(d)}) = H_c^1(\Gamma^{(d)}, F^{(d)})$ where $\Gamma = \Gamma_0(Mp\ell)$ and $\Gamma^{(d)} = \Gamma_0(Mp^d\ell)$ acting on on F, respectively on $F^{(d)}$ via their embedding in $\mathrm{Iw}(\mathbb{Z}_p)$ and $\mathrm{Iw}^{(d)}$ respectively.

On the other hand, we may use Proposition 8.7 and again Corollary 8.9 in order to describe the terms of the diagram. Let us recall from the section 4 the quaternion algebra B and the open compact subgroups $\mathcal{K} := \mathcal{K}(\ell, Mp)$ and $\mathcal{K}^{(d)} := \mathcal{K}(\ell, Mp^d)$ of $\hat{B}^\times$.

Lemma 8.10 ([Ca]). *a) We have natural isomorphisms (compatible for various M's)*

$$\Sigma_s \cong B(\mathbb{Q})^\times \backslash \hat{B}^\times / \mathcal{K}(\ell, Mp) \text{ and } \Sigma_s^{(d)} \cong B(\mathbb{Q})^\times \backslash \hat{B}^\times / \mathcal{K}(\ell, Mp^d).$$

b) Under the isomorphisms above we have isomorphisms of sheaves

$$R^1\Psi_X(\mathcal{F})|_{\Sigma_s} \cong \left(B(\mathbb{Q})^\times \backslash \hat{B}^\times \times F\right) / \mathcal{K}(\ell, Mp)$$

and

$$R^1\Psi_{X^{(d)}}(\mathcal{F}^{(d)})|_{\Sigma_s^{(d)}} \cong \left(B(\mathbb{Q})^\times \backslash \hat{B}^\times \times F^{(d)}\right) / \mathcal{K}(\ell, Mp^d)$$

c)

$$H^0(Y_s, R^1\Psi_X(\mathcal{F})) \cong Maps_{\mathcal{K}}(B(\mathbb{Q})^\times \backslash \hat{B}^\times, F) = S(\mathcal{K}, F)$$

and

$$H^0(Y_s^{(d)}, R^1\Psi_{X^{(d)}}(\mathcal{F}^{(d)})) \cong Maps_{\mathcal{K}^{(d)}}(B(\mathbb{Q})^\times \backslash \hat{B}^\times, F^{(d)}) = S(\mathcal{K}^{(d)}, F^{(d)})$$

In view of lemma 8.10 the above diagram can be written as

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_c(Y_s, \mathcal{F}_s) & \longrightarrow & H^1_c(\Gamma, F) & \longrightarrow & S(\mathcal{K}, F) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_c(Y_s^{(d)}, \mathcal{F}_s^{(d)}) & \longrightarrow & H^1_c(\Gamma^{(d)}, F^{(d)}) & \longrightarrow & S(\mathcal{K}^{(d)}, F^{(d)}) \end{array}$$

We can now state and prove the main result of this section.

Theorem 8.11. *Let $U \subset \mathcal{W}$ be a K-affinoid sub-domain and choose an arithmetic point $\kappa \in U(K)$ of signature (k, ϵ). Then the maps $\varphi_{\mathcal{K},F}$ and the specialization maps ρ_κ determine a commutaive diagram, which is equivariant for the action of the Hecke operators:*

$$\begin{array}{ccc} H^1_c(\Gamma, \mathcal{D}_U) & \xrightarrow{\varphi_{\mathcal{K},U}} & \mathbb{S}_U(\ell, pM) \\ \downarrow \rho_\kappa & & \downarrow \rho_\kappa \\ H^1_c(\Gamma^{(d)}, V_\kappa) & \xrightarrow{\varphi_{\mathcal{K}^{(d)},\kappa}} & S_\kappa(\ell, Mp^d) \end{array}$$

Proof. As in Corollary **??** we may write $\mathcal{D}_U$ in the form

$$\mathcal{D}_U \cong \varprojlim_n \left(\left[\varprojlim_t \varinjlim_{\mathcal{S}_{n,t}} F_{n,t} \right] \otimes_{\mathcal{O}_K} K \right).$$

where $\mathcal{S}_{n,t}$ is the set of finite $\Xi(\mathbb{Z}_p)$-submodules of $\mathbb{D}_U[n]^\circ/p^t$. For each $F_{n,t} \in \mathcal{S}_{n,t}$ we let $\mathcal{F}_{n,t}$ and $\mathcal{F}_{n,t,s} := (\mathcal{F}_{n,t})_s$ denote the corresponding sheaves on Y_η and Y_s respectively . We also let $F_t^{(d)} := V_\kappa^\circ/p^t V_\kappa^\circ$ and $\mathcal{F}_t^{(d)}$ and $\mathcal{F}_{t,s}^{(d)} := (\mathcal{F}_t^{(d)})_s$ be the corresponding sheaves on $Y_\eta^{(d)}$ and $Y_s^{(d)}$ respectively.

Then from Lemma 8.10 we have, for each triple $(n, t, F_{n,t})$ with $F_{n,t} \in \mathcal{S}_{n,t}$, a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_c^1(Y_s, \mathcal{F}_{n,t,s}) & \longrightarrow & H_c^1(\Gamma, F_{n,t}) & \longrightarrow & S(\mathcal{K}, F_{n,t}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_c^1(Y_s^{(d)}, \mathcal{F}_{t,s}^{(d)}) & \longrightarrow & H_c^1(\Gamma^{(d)}, F_t^{(d)}) & \longrightarrow & S(\mathcal{K}^{(d)}, F_t^{(d)}) \end{array}$$

Formation of inductive limits is an exact functor and projective limits are left exact and since formation of $H_c^1(\Gamma, *), S(\mathcal{K}, *)$ and $H_c^1(\Gamma^{(d)}, *), S(\mathcal{K}^{(d)}, *)$ all commute with both inductive and projective limits we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}_s & \longrightarrow & H_c^1(\Gamma, \mathcal{D}_U) & \longrightarrow & S(\mathcal{K}, \mathcal{D}_U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{L}_s^{(d)} & \longrightarrow & H_c^1(\Gamma^{(d)}, V_\kappa) & \longrightarrow & S(\mathcal{K}^{(d)}, V_\kappa) \end{array}$$

where we set

$$\begin{aligned} \mathcal{L}_s &:= \varprojlim_n \left(\left[\varprojlim_t \varinjlim_{\mathcal{S}_{n,t}} H_c^1(Y_s, \mathcal{F}_{n,t,s}) \right] \otimes_{\mathcal{O}_K} K \right) \\ \mathcal{L}_s^{(d)} &:= \varprojlim_n \left(\left[\varprojlim_t \varinjlim_{\mathcal{S}_{n,t}} H_c^1(Y_s^{(d)}, \mathcal{F}_{n,t,s}^{(d)}) \right] \otimes_{\mathcal{O}_K} K \right) \end{aligned}$$

This completes the proof. □

8.3 Modular Sheaves

In this chapter, we define a category of sheaves that will be useful to us for translating the modular data of the previous section into a sheaf theoretic context.

Definition 8.12. Let Y_S be a smooth curve over $S := Spec(\overline{\mathbb{Z}}_\ell)$. We define $\mathcal{E}(Y_S)$ to be the category whose objects and morphisms are given as follows.

(a) The objects of $\mathcal{E}(Y_S)$ are pairs (E/U) where $U \in Y_S^{et}$ and E/U is an elliptic curve over U.

(b) A morphism $(E/U) \longrightarrow (E'/U')$ is a pair (α, ϕ) where $\phi : U \longrightarrow U'$ is an etale morphism and $\alpha : E \longrightarrow E'_U := E' \times_\phi U$ is an isogeny of elliptic curves over U.

Definition 8.13. We also define the category $\mathcal{S}(Y_S)$ to be the category whose objects and morphisms are given as follows.

(a) The objects of $\mathcal{S}(Y_S)$ are pairs $(\mathcal{F}, U)$ where $U \in Y_S^{et}$ and $\mathcal{F}$ is a sheaf of abelian groups on U^{et}.

(b) A morphism $(\mathcal{F}, U) \longrightarrow (\mathcal{F}', U')$ is a pair (η, ϕ) where $\phi : U \longrightarrow U'$ is an etale morphism and $\eta : \mathcal{F} \longrightarrow \phi^*\mathcal{F}'$ is a morphism of sheaves on U^{et}.

Definition 8.14. A modular sheaf on Y_S is a functor $\mathcal{F} : \mathcal{E}(Y_S) \longrightarrow \mathcal{S}(Y_S)$ satisfying the following properties.

(a) Projection of $\mathcal{F}$ to the second factor is the identity functor on Y^{et}.

(b) $\mathcal{F}$ commutes with base change. More precisely, if (E, U) is an object of $\mathcal{E}(Y_S)$ and $V \xrightarrow{\phi} U$ is etale, then for $E_V := E \times_U V$ we have $\mathcal{F}(E_V/V) = \phi^*\mathcal{F}(E/U)$.

8.4 Hecke Operators

Let $Y/S = Y_0(N)/S$ and let E/Y be the universal elliptic curve over Y with level N-structure. Let $Y_0(N\ell) := Y_0(N\ell)/S$ and let E_0/Y_0 be the base change of E to Y_0. For each prime $q \not| N\ell$ we let $\mathcal{F}_\eta := \mathcal{F}(E_\eta/Y_\eta)$ and define

$$T_q : H^1(Y_\eta, \mathcal{F}_\eta) \longrightarrow H^1(Y_\eta, \mathcal{F}_\eta)$$

as follows. Let E_q/Y_q be the universal elliptic curve over Y_q with level Nq-structure. We have two etale morphism $\pi_1, \pi_2 : Y_q \longrightarrow Y$, where π_1 is induced by the forgetful functor and $\pi_2 = \pi_1 \circ w_q$ where w_q is the Atkin Lehner operator.

Now let $\mathcal{F}$ be a modular sheaf on Y. Since E_q/Y_q is given as base change $E_q = E_{Y_q}$ with respect to π_1 we have

$$\mathcal{F}_q := \mathcal{F}(E_q/Y_q) = \pi_1^*\mathcal{F}_\eta.$$

On the other hand, we have a canonical morphism $(E_q, Y_q) \longrightarrow (E_q, Y_q)$ lying over w_q. So by functoriality we have a canonical isomorphism of sheaves on Y_q

$$\mathcal{F}_q \longrightarrow w_q^* \mathcal{F}_q$$

and therefore we have a canonical isomorphism $\pi_1^* \mathcal{F}_q \cong \pi_2^* \mathcal{F}$. We then define the Hecke operator $T_q : H^1(Y_\eta, \mathcal{F}_\eta) \longrightarrow H^1(Y_\eta, \mathcal{F}_\eta)$ by the commutativity of the following diagram

$$\begin{tikzcd} H^*(Y_q, \pi_1^* \mathcal{F}) \arrow[r, phantom, "\cong"] & H^*(Y_q, \pi_2^* \mathcal{F}) \arrow[d] \\ H^*(Y, \mathcal{F}) \arrow[u] \arrow[r, "T_q"] & H^*(Y, \mathcal{F}) \end{tikzcd}$$

(**Stop Reading Here:**)

Now fix geometric generic points $\overline{s} \longrightarrow Y_s$ and $\overline{\eta} \longrightarrow Y_\eta$ on Y_s and Y_η, respectively and define

$$\Delta_s := \pi_1(Y_s, \overline{s}) \quad \text{and} \quad \Delta_\eta := \pi_1(Y_\eta, \overline{\eta}).$$

Denoting by $\overline{s}, \overline{\eta}$ the images of these points in Y_S, and letting $\Delta_S := \pi_1(Y_S, \overline{\eta})$ we may choose a (non-canonical) isomorphism

$$\pi_1(Y_S, \overline{s}) \cong \Delta_S. \tag{8.1}$$

Henceforth, we shall identify these two groups via this isomorphism. The commutative diagram

$$\begin{tikzcd} & \overline{s} \arrow[dl] \arrow[dr] & & \overline{\eta} \arrow[dl] \arrow[dr] & \\ Y_s \arrow[rr] & & Y_S & & Y_\eta \arrow[ll] \end{tikzcd}$$

then induces canonical continuous group homomorphisms

$$\Delta_s \longrightarrow \Delta_S \longleftarrow \Delta_\eta. \tag{8.2}$$

Finally, we let F is any finite abelian p-group endowed with a continuous action of Δ_S (on the right), then using 8.2 we may also regard F as having continuous actions of Δ_s and Δ_η and let

$$\mathcal{F}_s,\ \mathcal{F}_S,\ \text{and } \mathcal{F}_\eta \tag{8.3}$$

be the associated lisse sheaves on the etale sites Y_s^{et}, Y_S^{et}, and Y_η^{et}, respectively. We note that

$$\mathcal{F}_s = i_Y^* \mathcal{F}_S \quad \text{and} \quad \mathcal{F}_S = y_{Y,*}\mathcal{F}_\eta.$$

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