7

Continuous Functions

7.1 Continuity

Changes can occur gradually, or they may happen suddenly—for the better or for worse.

Definition: Function $f(x)$ is continuous at $x_0$ if and only if the function exists and the limit exists and the equality $\lim_{x \to x_0} f(x) = f(x_0)$ holds. In other words, function $f(x)$ is continuous at $x = x_0$ if the values of the function immediately to the right and immediately to the left of $x_0$ are both equal to $f(x_0)$

$$f(x_0 - 0) = f(x_0) = f(x_0 + 0).$$

Otherwise the function is discontinuous.

Examples and counterexamples:

1. The function $y = x^2$ is continuous everywhere since $\lim_{x \to x_0} x^2 = x_0^2$ whatever $x_0$.
2. The function $y = \sqrt{x}$ is continuous at all $x > 0$, since $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$ if $x_0 > 0$. Notice that at $x = 0$, $\sqrt{x}$ has a right limit only, $\lim_{x \to 0^+} \sqrt{x} = 0$.
3. The function $y = 1/x$ is continuous at all $x \neq 0$ since $\lim_{x \to x_0} 1/x = 1/x_0$ as long as $x \neq 0$.
4. The function

$$f(x) = \begin{cases} 
   x, & 0 \leq x < 1 \\
   1, & x \geq 1 
\end{cases}$$