TOPOLOGICAL QUANTUM FIELD THEORY AND ALGEBRAIC STRUCTURES

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ABSTRACT. These notes are from lectures given at the *Quantum field theory and noncommutative geometry workshop* at Tohoku University in Sendai, Japan from November 24 – 30, 2003. We give a survey of operads and their relationship to topological quantum field theories (TQFT). We give simple examples of operads, particularly those arising as moduli spaces of decorated oriented 2-spheres, describe the notion of algebras over them, and we study their "higher loop" generalizations. We then focus upon the example of the moduli space of stable curves and its relationship to cohomological field theories, in the sense of Kontsevich-Manin. The latter can be regarded as a generalization of a TQFT which is relevant to quantum cohomology and to higher KdV integrable hierarchies.

1. INTRODUCTION

These lectures provide a survey, from the perspective of someone interested in $(1+1)$-dimensional topological quantum field theories (TQFTs) and their generalizations, in which we explain how the notion of an operad and algebras over them can be used to analyze such theories. We have by no means attempted to provide a comprehensive account of this very rich subject. On the contrary, the subject can often appear to be almost too rich, overwhelming the uninitiated with long, unmotivated definitions and constructions. Instead, we have attempted, when possible, to give a "hands-on" approach to several key points relevant to TQFTs through a study of simple examples, many of which admit interesting and nontrivial generalizations.

Many operadic notions involve constructions which should be already familiar to mathematical physicists. One such feature is the natural appearance of trees in the theory of operads which suggests an appealing interpretation in terms of tree-level Feynman diagrams. These trees turn out to be a universal feature of operads and algebras over them. Allowing higher loop (or genus) diagrams correspond to a generalization of the notion of an operad, called a modular operad, due to Getzler-Kapranov [GeKa2]. This interpretation in terms of Feynman diagrams becomes particularly compelling in the study of cohomological field theories, in the sense of Kontsevich-Manin [KoMa], where the underlying (modular) operad is the moduli space of stable curves and the graphs appear as dual graphs associated to a stable curve. Here, it becomes useful to introduce a potential function from which one obtains the Feynman rules in the usual fashion. The relations in the underlying operad

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imply that the potential function must satisfy certain differential equations. Sometimes, these differential equations are powerful enough to determine this potential function. These techniques are used in the study of Gromov-Witten invariants (or quantum cohomology) of a smooth, projective variety and in the study of the spin-CohFT.

We hope that the interested reader will use this article as a springboard from which to pursue the subject further on his own. We refer the reader to [BaKi] to read further about TQFTs and their cousins. We refer the reader to [MSS] to read further about operads, homotopy algebras, and their generalizations. Finally, we refer the reader to [Ma] to read further about cohomological field theories and quantum cohomology.

The paper is structured as follows. In section 2, we review the notion of a topological quantum field theory as described by Atiyah as a functor from a geometric cobordism category to the category of vector spaces. We will be primarily interested in the case of a $(1+1)$ dimensional TQFT and we recall this is nothing more than a Frobenius algebra, *i.e.* a commutative, associative algebra with an identity element and an invariant metric.

In the section 3, we recall the definition of an operad and algebras over them. We describe the endomorphism operad and various operads in the category of topological spaces such as the little intervals and the (framed) little disks operad. We then study the operad of their homology groups and describe the algebraic structures for which they are responsible. We then explain the consequences for a refinement of a TQFT called a topological conformal field theory (TCFT).

In the section 4, we study the moduli space of stable curves from an operadic perspective. This can be thought of as a compactification of the moduli space of Riemann surfaces with marked points. The compactification is realized by allowing the Riemann surfaces to degenerate by acquiring nodes. We can "glue" such surfaces by attaching them along two marked points to obtain a node. This becomes the model example of a modular operad. Restricting to genus zero only and regarding one marked point to be outgoing, one obtains a (usual) operad in the category of complex algebraic varieties. A cohomological field theory (CohFT) in the sense of Kontsevich-Manin is nothing more than an algebra over this modular operad. In particular, a CohFT contains a Frobenius algebra by considering only the suboperad $H_0(\overline{\mathcal{M}}_{0,n})$ regarded as a (cyclic) operad. Therefore, a CohFT can be regarded as a generalization of a TQFT. The genus zero part of a CohFT admits a beautiful description in terms of potential functions associated to tree-level Feynman diagrams.

In the section 5, we provide two constructions of CohFTs. The first is the Gromov-Witten invariants of a smooth, projective variety and involves the moduli space of stable maps. The second is called a spin-CohFT and involves the moduli space of higher spin curves. Constructions of CohFTs do not appear to be very easy to come by at the moment.

Remark 1.1. Henceforth, we assume that all ground rings are over $\mathbb{C}$ unless otherwise stated.

2. Topological quantum field theory

Our starting point is the notion of a $(d+1)$-dimensional topological quantum field theory (or TQFT) due Atiyah which, roughly speaking, axiomatizes the locality (or gluing) properties associated to the Feynman path integral in the cases where the observables in the associated quantum field theory depend only upon discrete geometric (hence, "topological") data. The idea is to view a TQFT as a functor from a category of geometric objects depending on only discrete geometric data to the category of vector spaces.

This definition provides a model for more complicated quantum field theories such as conformal field theories [Se] or cohomological field theories [KoMa] which are obtained by enriching the geometric and linear categories.

Roughly speaking, operads appear in a $(d+1)$-dimensional TQFT as a certain subcategory of the geometric category consisting of those objects with n inputs and 1 output. An algebra over this operad is then obtained by restriction.

Remark 2.1. Throughout the remainder of this section, we will assume that all manifolds are oriented and that all homeomorphisms between them preserve orientation. Furthermore, if N is an oriented manifold then N^* is the same underlying manifold N but with its orientation reversed.

Definition 2.2. A *$(d+1)$-dimensional topological quantum field theory (or TQFT)* is a collection of the following data:

- **State Spaces:** To each closed d-dimensional manifold N, we assign a finite dimensional vector space $Z(N)$.
- **Correlators:** To each $(d+1)$-dimensional manifold, M, we assign a vector $Z(M) \in Z(\partial M)$.

There are also the following functorial isomorphisms:

- **Naturality 1:** Any homeomorphism $f : N \to N'$ of d-manifolds induces an isomorphism $f_* : Z(N) \to Z(N')$.
- **Duality:** An isomorphism $Z(N^*) \to Z(N)^*$.
- **Empty Set:** An isomorphism $Z(\phi) \to \mathbb{C}$
- **Multiplicativity:** An isomorphism $Z(N \sqcup N') \to Z(N) \otimes Z(N')$.

These must satisfy the following axioms.

- **Naturality 2:** If $f : M \to M'$ is a homeomorphism of $(d+1)$-manifolds then
$$(f_{\partial M})_*(Z(M)) = Z(M').$$
where $f_{\partial M}$ denotes the restriction of f to ∂M.
- **Gluing:** Let M be a $(d+1)$-manifold such that $\partial M = N_1 \sqcup N_2 \sqcup N_3$ and let $f : N_1 \to N_2^*$ be a homeomorphism. Let M' be the result of gluing together M using f then the composition
$$Z(N_1) \otimes Z(N_2) \otimes Z(N_3) \to Z(N_2)^* \otimes Z(N_2) \otimes Z(N_3) \to Z(N_3)$$
is equal to $Z(M')$ where the first map is induced by f_* and the second is contraction.
- **Identity Map:** Let I be the oriented interval and N be a d-manifold such that $\partial I \times N = N \sqcup N^*$ then we require that $Z(I \times N)$ is equal to the image of the identity map on $Z(N)$ in $Z(N^*) \otimes Z(N) \simeq Z(N)^* \otimes Z(N)$.

We will primarily be interested in the case where $d = 1$ for the purposes of these lectures where things are quite simple.

Definition 2.3. A *Frobenius algebra (with unit)* is a finite-dimensional commutative, associative algebra (over $\mathbb{C}$) with an identity element and an invariant metric.

In other words, it is a tuple $(\mathcal{H}, \eta, \cdot, \mathbf{1})$ where $\mathcal{H}$ is a finite dimensional vector space with a metric η together with a multiplication $\cdot$ with identity element $\mathbf{1}$ such that

$$\eta(a \cdot b, c) = \eta(a, b \cdot c) \tag{1}$$

for all a, b, c in $\mathcal{H}$.

Theorem 2.4. $(1+1)$*-dimensional topological quantum field theories are in one to one correspondence with Frobenius algebras.*

Proof. Suppose we are given a $(d+1)$-dimensional TQFT. Let us now construct its associated Frobenius algebra.

Choose, once and for all, a particular circle from the set of all closed, connected 1-manifolds and call it S^1. Any closed 1-manifold N is isomorphic to the finite disjoint union of circles. By the multiplicativity , duality, and the 1st naturality axiom, $Z(S)$ for any closed 1-manifold S is determined by $\mathcal{H} := Z(S^1)$.

Any 2-manifold M has a finite number of connected components. A connected must be a genus g surface (possibly) with boundary. The key observation is that any such surface can be obtained by gluing together a finite number of disks, cylinders, or pairs of pants along their boundaries. (A disk is a sphere with one boundary component, a cylinder is a sphere with two boundary components and a pair of pants is a sphere with three boundary components.) Choose a particular disk , cylinder , and pair of pants, once and for all, and call them respectively D, C, and P. In each case, let us assume that the orientation assigned to their boundaries agrees with that induced from the surface.

One can view $Z(D)$ as a linear map $\mathbb{C} \to \mathcal{H}$. This map is determined by the image of the unit element in $\mathbb{C}$. Call this image $\mathbf{1}$ in $\mathcal{H}$. It will be the unit element.

Let C' be the exactly the same as C but where its two boundary components have been assigned an orientation opposite that induced from the surface then denote the map $Z(C') : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ by η. This will be the metric.

Let P' be exactly the same as P but where two boundary components are assigned an orientation opposite from the one induced from P while the remaining one is assigned an orientation induced from P so that we have $Z(P') : \mathcal{H}^{\otimes 2} \to \mathcal{H}$. This will be the multiplication.

η is commutative, by the 1st naturality axiom, because there is an isomorphism taking C' to itself which takes one boundary component to the other. The multiplication is commutative for similar reasons.

$\mathbf{1}$ is the unit element in $\mathcal{H}$ because gluing a D to one of the boundary components of C' whose orientation is the opposite of the induced orientation yields a cylinder. One boundary component of the cylinder agrees with the induced orientation and the other does not. Call this cylinder C''. $Z(C'')$ is, by the identity map axiom, equal to the identity map $\mathcal{H} \to \mathcal{H}$. Therefore by the gluing axiom, it follows that $\mathbf{1}$ is the unit element of $\mathcal{H}$.

Let $\eta^{-1} := Z(C) : \mathbb{C} \to \mathcal{H}^{\otimes 2}$. We can see that η^{-1} is the inverse of η as follows. Pick a boundary component of C and of C' and then glue them together. Again, we obtain C'' but $Z(C'')$ is the identity map on $\mathcal{H}$ by the identity axiom. Therefore, η^{-1} is the inverse of η.

Associativity of the product follows by considering a sphere with 4 boundary components, 3 components whose orientations agree with the induced one and the last component whose orientation disagrees with the induced one. Such a surface may be obtained (up to an isomorphism) by gluing together P' in several different ways. This yields the associativity of the multiplication.

Equation (1), the invariance of the metric, follows from a similar argument except that now one glues together a C' with P' instead.

Therefore, $(\mathcal{H}, \eta, \cdot)$ is a Frobenius algebra. It remains to prove that there are no other relations other than those that we have described above. This is the most subtle part of this proof and we the refer the interested reader to [BaKi] for details.

The converse also follows from these arguments. $\square$

Suppose we now consider $(1+1)$-dimensional TQFTs but where we restrict to only 2-manifolds, M, which are spheres with $(n+1)$ boundary components where $n \geq 0$ such that n of the boundary components have orientation disagreeing with the induced orientations and the remaining one has an orientation agreeing with the induced orientation so that $Z(M) : \mathcal{H}^{\otimes n} \to \mathcal{H}$. This is essentially the structure map of an algebra over the operad of the moduli space of oriented genus zero surfaces with boundary.

3. Operads and algebras over them

3.1. **Some useful examples.** Operads appear quite naturally in various settings. Rather than diving straight into its definition, which can be somewhat daunting for the uninitiated, let us begin with some key examples.

Example 3.1 (The Endomorphism Operad)**.** Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ be a $\mathbb{Z}_2$-graded vector space. Elements in $\mathcal{H}_0$ are called the *even elements of* $\mathcal{H}$ while the elements in $\mathcal{H}_1$ are called the *odd elements of* $\mathcal{H}$.

Consider the collection $\mathrm{End}^{\mathcal{H}} := \{\, \mathrm{End}^{\mathcal{H}}(n)\,\}_{n\geq 1}$ where $\mathrm{End}^{\mathcal{H}}(n) := \mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathcal{H})$. An element f in $\mathrm{End}^{\mathcal{H}}(n)$ is an n-ary operation on $\mathcal{H}$, *i.e.* f accepts n elements of $\mathcal{H}$ as an input and yields one element of $\mathcal{H}$ for an output.

$\mathrm{End}^{\mathcal{H}}(n)$ has an action of S_n, the symmetric group on n letters, which permutes the n factors of the tensor product, *i.e.* for all σ in S_n and homogeneous elements $v_1, \dots, v_n$ in $\mathcal{H}$,

$$(\sigma f)(v_1, \dots, v_n) := \pm f(v_{\sigma(1)}, \dots, v_{\sigma(n)}). \tag{2}$$

where the sign is the usual one from the category of $\mathbb{Z}_2$-graded vector spaces.

Given $f \in \mathrm{End}^{\mathcal{H}}(n)$ and $f' \in \mathrm{End}^{\mathcal{H}}(n')$, we can construct an element $f \circ_i f' \in \mathrm{End}^{\mathcal{H}}(n + n' - 1)$ by composing the output of f' into the i-th input of f, *i.e.* for all $i = 1, \dots, n$,

$$\begin{aligned}(f \circ_i f')&(v_1 \otimes \ldots \otimes v_{n+n'-1}) \\ &:= \pm f(v_1 \otimes \ldots \otimes v_{i-1} \otimes f'(v_i \otimes \ldots v_{i+n'-1}) \otimes v_{i+n'} \otimes \cdots \otimes v_{n+n'-1})\end{aligned} \tag{3}$$

for all $v_1, \dots, v_{n+n'-1}$ in $\mathcal{H}$.

Composing three multilinear operations should be associative, in the appropriate sense. More precisely, suppose we have homogeneous elements f in $\mathrm{End}^{\mathcal{H}}(n)$, f' in $\mathrm{End}^{\mathcal{H}}(n')$, and f'' in $\mathrm{End}^{\mathcal{H}}(n'')$ then there are two types of compositions that one could perform to obtain an element in $\mathrm{End}^{\mathcal{H}}(n + n' - 1)$. In the first case, we can successively plug in the output of f'' into f' and plug in the output of f' into f. In the second case, we can plug in both of the outputs of f' and f'' into f but at different locations. In either case, the result is independent of the order in which the compositions are performed. In other words,

Associativity 1: if $1 \leq i < j \leq n$, $n', n'' \geq 1$, and $n \geq 2$ then

$$(f \circ_i f') \circ_{j+n'-1} f'' = (-1)^{|f'||f''|}(f \circ_j f'') \circ_i f' \tag{4}$$

Associativity 2: if $n, n' \geq 1$, $n'' \geq 1$, and $i = 1, \dots, n$ and $j = 1, \dots, n'$ then

$$(f \circ_i f') \circ_{i+j-1} f'' = f \circ_i (f' \circ_j f''). \tag{5}$$

where $|f'|$ and $|f''|$ denotes the $\mathbb{Z}_2$-gradings of f' and f'', respectively.

Furthermore, the composition maps are equivariant under the action of the permutation groups. Finally, there is a unit element I in $\mathrm{End}^{\mathcal{H}}(1)$, which is the identity map, which satisfies for all f in $\mathrm{End}^{\mathcal{H}}(n)$ and $i = 1, \dots, n$,

Unit:

$$\mathrm{I} \circ_1 f = f = f \circ_i \mathrm{I}. \tag{6}$$

Thus $\mathrm{End}^{\mathcal{H}}$ together with its compositions, the action of the symmetric group, and the unit element I yields an example of an operad with unit and is called *the endomorphism operad of* $\mathcal{H}$. This is a model example of an operad in the category of $\mathbb{Z}_2$-graded vector spaces.

Observe that $\mathrm{End}^{\mathcal{H}}(1) = \mathrm{End}(\mathcal{H})$, the space of (usual) endomorphisms of $\mathcal{H}$, and the composition map $\mathrm{End}^{\mathcal{H}}(1) \otimes \mathrm{End}^{\mathcal{H}}(1) \to \mathrm{End}^{\mathcal{H}}(1)$ taking $f \otimes f' \mapsto f \circ_1 f'$ is nothing more than the usual associative composition on $\mathrm{End}(\mathcal{H})$ with unit element I.

The above definition can be immediately generalized to the category of differential graded vector spaces (or dg vector spaces). Suppose that $(\mathcal{H}, d)$ is a dg vector space, *i.e.* $(\mathcal{H}, d)$ consists of a $\mathbb{Z}$-graded vector space $\mathcal{H}$ whose i-th graded subspace is denoted by $\mathcal{H}_i$, linear maps $d : \mathcal{H}_i \to \mathcal{H}_{i-1}$ for all i such that $d^2 = 0$. The endomorphism operad of the complex $(\mathcal{H}, d)$, $\mathrm{End}^{(\mathcal{H},d)}$, is a dg vector space whose p-th graded subspace consists of those elements in $\mathrm{Hom}(\mathcal{H}^{\otimes n}, \mathcal{H})$ with $\mathbb{Z}$-grading p. The compositions, the action of the symmetric groups, and the unit element are defined in the same way as before. The compositions preserve the $\mathbb{Z}$-grading, S_n preserves the $\mathbb{Z}$-grading, and I has $\mathbb{Z}$-grading 0. We will denote $\mathrm{End}^{(\mathcal{H},d)}$ by $\mathrm{End}^{\mathcal{H}}$ when there is no possibility of confusion.

Since the compositions and symmetric groups commute with the differentials and I is closed, by taking homology, one obtains a well-defined operad $\mathrm{End}^{H_\bullet(\mathcal{H})}$ in the category of $\mathbb{Z}$-graded vector spaces.

The next example is an operad in the category of topological spaces. Typically, such operads arise as configuration (or moduli) spaces of geometric objects which can be glued together. This will be situation in which we are most interested.

Example 3.2 (The Framed Little Disks Operad)**.** Consider the collection $\mathcal{F} := \{\mathcal{F}(n)\}_{n\geq 1}$. Let Δ denote the unit disk about the origin in $\mathbb{C}$. Let $F(n)$ consist of configurations of n little disks embedded inside of Δ via a composition of a translation, rotation, and dilation. In other words, $F(n)$ consists of all n-tuples $(x_1, \dots, x_n)$ consisting of $x_i : \Delta \to \Delta$ where $x_i(z) = a_i z + b_i$, a_i is nonzero element in $\mathbb{C}$ and b_i belongs to $\mathbb{C}$ for all $i = 1, \dots, n$. Furthermore, we require that for all $i \neq j$, $x_i(\Delta)$ and $x_j(\Delta)$ can only intersect at most along their boundaries. We can depict an element of $F(n)$ as a unit ("big") disk Δ with n little disks drawn inside it whose interiors are pairwise disjoint, together with a marked point along the boundary of each little disk. The marked point on the i-th little disk is the image of 1 in $\mathbb{C}$ under x_i for all $i = 1, \dots, n$. It is understood that the big disk has a marked point at 1 in $\mathbb{C}$. $\mathcal{F}$ is a collection of topological spaces.

The symmetric group S_n acts upon $F(n)$ by permuting the order of the n little disks. The composition $F(n) \times F(n') \to F(n+n'-1)$ taking $(f, f') \mapsto f \circ_i f'$ for all $i = 1, \dots, n$ is obtained by shrinking f' until its big disk is the same size as the i-th little disk of f and then gluing it into the i-th little disk of f after rotating the shrunken disk so that the marked points coincide.

These compositions are continuous and satisfy the associativity conditions, *i.e.* for all f in $\mathcal{F}(n)$, f' in $\mathcal{F}(n')$ and f'' in $\mathcal{F}(n'')$, Equation (4) and Equation(5) after ignoring the factor of $(-1)^{|f'||f''|}$ in Equation (4). Associativity boils down to the fact that the result of gluing three objects together in a proscribed fashion is independent of the order in which the gluings are performed.

The compositions are equivariant under the action of the permutation groups. There is a unit element in I in $\mathcal{F}(1)$ which consists of the unit map $x_1(z) = z$ which satisfies Equation (6).

Observe, again, that the composition $\circ_1$ on $\mathcal{F}(1)$ is associative, *i.e.* $\mathcal{F}(1)$ is a semigroup with unit.

The resulting operad, $\mathcal{F}$, is called the *framed little disks operad* and it is an operad in the category of complex manifolds with corners.

Armed with these examples, let us proceed with the definition. An operad is a notion which exists in any symmetric monoidal category, *e.g.* the category of topological spaces, dg vector spaces, etc.

Definition 3.3. An *operad (with unit)* $\mathcal{O} = \{\, \mathcal{O}(n) \,\}_{n \geq 1}$ is a collection of objects in a symmetric monoidal category where each $\mathcal{O}(n)$ is endowed with a right action of S_n, the permutation group on n letters, and a collection of morphisms for $n \geq 1$ and $1 \leq i \leq n$, $\mathcal{O}(n) \times \mathcal{O}(n') \to \mathcal{O}(n+n'-1)$ given by $(f, f') \mapsto f \circ_i f'$ satisfying the following properties.

- **Equivariance:** The compositions are equivariant under the action of the permutation groups.
- **Associativity:** For all $f \in \mathcal{O}(n)$, $f' \in \mathcal{O}(n')$, and $f'' \in \mathcal{O}(n'')$, the associativity conditions, Equation (4) and Equation (5), are satisfied (where the factor of $(-1)^{|f'||f''|}$ should be ignored if $\mathcal{O}$ is not an operad of (graded) vector spaces).
- **Unit:** There exists a element I in $\mathcal{O}(1)$ called the *unit* such that Equation (6) holds for all $i = 1, \dots, n$ and f in $\mathcal{O}(n)$.

Remark 3.4. It is implicit in the above that if $\mathcal{O}$ is an operad of (graded) vector spaces then the compositions must respect the $\mathbb{Z}$-grading and the action of the permutation must preserve the $\mathbb{Z}$-grading.

Remark 3.5. An alternate description of the composition maps favored by algebraic topologists is obtained by iterating k composition maps to obtain the morphism $\gamma : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$.

Remark 3.6. Under the composition map $\circ_1$, it follows that $\mathcal{O}(1)$ is an semigroup with unit I. If $\mathcal{O}$ is an operad with unit in the category of dg vector spaces then $\mathcal{O}(1)$ is a differential graded associative algebra with unit.

Remark 3.7. Some authors extend the definition of an operad $\mathcal{O}$ to include a component $\mathcal{O}(0)$ as well. In the endomorphism operad, $\mathrm{End}^{\mathcal{H}}(0) := \mathcal{H}$.

Let $\mathcal{O}$ be an operad in the category of topological spaces then it automatically gives rise to two associated operads. First of all, one can define the operad of its singular chain complexes $(C_\bullet(\mathcal{O}), \partial)$. This is an operad in the category of differential graded vector spaces where $C_\bullet(\mathcal{O}) := \{ C_\bullet(\mathcal{O}(n)) \}_{n\geq 1}$, $C_\bullet(\mathcal{O}(n))$ is the space of singular chains on $\mathcal{O}(n)$ with differentials $\partial : C_{p+1}(\mathcal{O}(n)) \to C_p(\mathcal{O}(n))$ for all $p \geq 0$ and the compositions $C_p(\mathcal{O}(n)) \otimes C_{p'}(\mathcal{O}(n')) \to C_{p+p'}(\mathcal{O}(n+n'-1))$ taking $f \otimes f' \mapsto f \circ_i f'$ for all $i = 1, \ldots, n$ are induced by the corresponding compositions in $\mathcal{O}$. The S_n action on $C_\bullet(\mathcal{O}(n))$ and unit I in $C_0(\mathcal{O}(1))$ are induced. Secondly, by taking the homology, one obtains the induced operad $H_\bullet(\mathcal{O}) := \{ H_\bullet(\mathcal{O}(n)) \}_{n\geq 1}$ in the category of $\mathbb{Z}$-graded vector spaces.

3.2. **Morphisms of operads.** We now describe morphisms between operads and provide some concrete examples.

Definition 3.8. Let $\mathcal{O}$ and $\mathcal{O}'$ be operads. A *morphism of operads* is a collection of maps $\mu : \mathcal{O}(n) \to \mathcal{O}'(n)$ for all $n \geq 1$ taking $f \mapsto \mu_f$ such that

Equivariance: For all σ in S_n and f in $\mathcal{O}(n)$,
$$\mu_{f\sigma} = (\mu_f) \circ \sigma$$
where $\circ$ denotes the composition of μ_f and the action of σ on $\mathcal{O}'(n)$.

Compositions: For all f in $\mathcal{O}(n)$ and f' in $\mathcal{O}(n')$ and $i = 1, \ldots, n$,
$$\mu_{f \circ_i f'} = \mu_f \circ_i \mu_{f'}.$$

Unit: If I is the unit in $\mathcal{O}$ and I$'$ is the unit in $\mathcal{O}'$ then
$$\mu_{\mathrm{I}} = \mathrm{I}'.$$

A morphism of operads $\mu : \mathcal{O} \to \mathcal{O}'$ is said to be an *isomorphism* if it has an inverse morphism.

In particular, if $\mu : \mathcal{O} \to \mathcal{O}'$ is a morphism of operads then its restriction morphism $\mu : \mathcal{O}(1) \to \mathcal{O}'(1)$ is a homomorphism of semigroups with unit.

Example 3.9 (The Little Disks Operad). Let $\mathcal{D} := \{ \mathcal{D}(n) \}_{n\geq 1}$ where $\mathcal{D}(n)$ consists of tuples $(x_1, \ldots, x_n)$ in $\mathcal{F}(n)$ such that $x_i(z) = a_i z + b_i$ where $|a_i| = 1$ for all $i = 1, \ldots, n$. This means that each little disk must embed into the big disk by a composition of only a translation and dilation. Pictorially, it consists of only those framed little disks such that each little disk has a boundary with a marked point that is located at the right most point on that disk.

Since $\mathcal{D}$ is closed under the action of the permutation groups and compositions, and contains the unit element, $\mathcal{D}$ is a suboperad of $\mathcal{F}$, *i.e.* the inclusion maps $\mathcal{D} \to \mathcal{F}$ is an injective morphism of operads.

Example 3.10 (The Segal Operad). Let $\mathcal{P} := \{ \mathcal{P}(n) \}_{n\geq 1}$ be the moduli space of Riemann spheres with $(n+1)$ holomorphically embedded disks. In other words, an element in $\mathcal{P}(n)$ consists of $\mathbb{C}P^1$ together with a tuple $(z_1, \ldots, z_{n+1})$ where $z_i : \Delta \to \mathbb{C}P^1$ are holomorphic embeddings of the unit disk Δ for all $i = 1, \ldots, (n+1)$ whose pairwise images can only overlap along their boundaries. Two such configurations are isomorphic if they are related by an automorphism of $\mathbb{C}P^1$. Recall that the group of automorphisms of $\mathbb{C}P^1$ is isomorphic to $\mathrm{PGL}(2, \mathbb{C}) \simeq \mathrm{PSL}(2, \mathbb{C})$ where latter acts on $\mathbb{C}P^1$ in a standard chart as
$$z \mapsto \frac{az+b}{cz+d}$$
where a, b, c, d belong to $\mathbb{C}$ such that $ad - bc = 1$.

S_n acts upon $\mathcal{P}(n)$ by permuting the first n disks, leaving the $(n+1)$-st disk alone. The composition $\mathcal{P}(n) \times \mathcal{P}(n') \to \mathcal{P}(n+n'-1)$ taking $(C, C') \mapsto C \circ_i C'$ is obtained by cutting out the i-th disk of C and the $(n'+1)$st disk of C' and then gluing along their boundaries using $z \mapsto \frac{1}{z}$ for all $i = 1, \ldots, n$. We call the resulting operad $\mathcal{P}$ the *Segal operad*. It can be regarded as an infinite dimensional version of a complex manifold with corners. In particular, it is an operad in the category of topological spaces.

We will now construct an injective morphism of operads $\nu : \mathcal{F} \to \mathcal{P}$. Let $(x_1, \ldots, x_n)$ denote an element of $\mathcal{F}(n)$. Choose on $\mathbb{C}P^1$ one of its standard charts and call it w. One can thus regard $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$. Identify the big disk of $\mathcal{F}(n)$ with the unit disk about 0 in $\mathbb{C}P_1$. Let $z_i := x_i$ for all $i = 1, \ldots, n$. Let Δ' denote the closure of the complement of this (big) unit disk in $\mathbb{C}P_1$. Let $z_{n+1} : \Delta \to \mathbb{C}P_1$ be the embedding whose image is Δ' induced by the chart $\frac{1}{w}$ about the point ∞ in $\mathbb{C}P_1$. The tuple $(z_1, \ldots, z_{n+1})$ yields a well-defined element of $\mathcal{P}(n)$ which preserves the action of the permutation groups, compositions, and the unit. Therefore, $\nu : \mathcal{F} \to \mathcal{P}$ is a morphism of operads.

Suppose there are two elements in $\mathcal{F}(n)$, $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$, such that their images under ν in $\mathcal{P}(n)$ are equal. Let $(z_1, \ldots, z_{n+1})$ and $(z'_1, \ldots, z'_{n+1})$ denote the holomorphically embedded disks in $\mathbb{C}P^1$ associated to $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$, respectively. Notice that $z_{n+1} = z'_{n+1}$. There must be an element of $\mathrm{PGL}(2, \mathbb{C})$ which takes $(z_1, \ldots, z_{n+1})$ to $(z'_1, \ldots, z'_{n+1})$. However, the only element in $\mathrm{PGL}(2, \mathbb{C})$ which fixes the embedding z_{n+1} is the identity element. Therefore, μ is injective.

Of course, μ cannot be surjective since $\mathcal{F}(n)$ is finite dimensional while $\mathcal{P}(n)$ is infinite dimensional.

3.3. **Algebras over operads.** Let A be an associative algebra with unit. A representation of A is a vector space $\mathcal{H}$ together with a homomorphism of associative algebras $A \to \mathrm{End}(\mathcal{H})$ preserving the unit elements. If $\mathcal{O}$ is an operad, say in the category of vector spaces, then $\mathcal{O}(1)$ is an associative algebra with unit. Does there exist a notion of a "representation" of the operad $\mathcal{O}$ on a vector space $\mathcal{H}$ which when restricted to $\mathcal{O}(1)$ reduces to the usual representation of the associative algebra $\mathcal{O}(1)$? The answer is yes, however, the standard terminology used is that $\mathcal{H}$ is an algebra over the operad $\mathcal{O}$ or $\mathcal{H}$ is a $\mathcal{O}$-algebra rather than $\mathcal{H}$ is a representation of $\mathcal{O}$. It is through this notion that the relationship with topological field theories becomes most evident

Definition 3.11. Let $(\mathcal{H}, d)$ be a dg vector space, $\mathcal{O}$ be an operad in the category of dg vector spaces, together with a morphism of operads $\mu : \mathcal{O} \to \mathrm{End}^{\mathcal{H}}$.

Or, let $\mathcal{O}$ be an operad in the category of topological spaces, $\mathcal{H}$ be a topological vector space, and $\mu : \mathcal{O} \to \mathrm{End}^{\mathcal{H}}$ be a morphism of operads.

In either case, we say that $\mathcal{H}$ is an *$\mathcal{O}$-algebra* or is an *algebra over $\mathcal{O}$*.

Remark 3.12. If $\mathcal{O}$ is an operad in the category of smooth (resp. complex) manifolds then one may wish to impose that the structure morphism $\mu : \mathcal{O} \to \mathrm{End}^{\mathcal{H}}$ be smooth (resp. holomorphic).

Example 3.13 (The Little Intervals Operad and the Associative Operad)**.** We now introduce a version of the little disks operad but where instead of two dimensional disks, we consider a closed interval. Let $J := [0, 1]$ be the unit interval oriented with an arrow pointing towards the right.

Let $\mathcal{I} := \{\mathcal{I}(n)\}_{n\geq 1}$ where $\mathcal{I}(n)$ consists of configurations of n subintervals in the interval J, *i.e.* $\mathcal{I}(n)$ consists of n-tuples $(x_1, \dots, x_n)$ where $x_i : J \to J$ such that $x_i(z) := a_i z + b_i$ where a_i is a positive real number and b_i belongs to $\mathbb{R}$. We also require that for all $i \neq j$, $x_i(J)$ and $x_j(J)$ can only intersect at most along their boundaries. The composition map $\mathcal{I}(n) \times \mathcal{I}(n') \to \mathcal{I}(n + n' - 1)$ taking $(C, C') \mapsto C \circ_i C'$ arises by inserting the interval C' into the i-th interval of C in an orientation preserving fashion. S_n acts on $\mathcal{I}(n)$ by reordering the subintervals. The unit I in $\mathcal{I}(1)$ is the identity map $x_1 : J \to J$. $\mathcal{I}$ is an operad in the category of topological spaces and is called the *little intervals operad*.

Let's apply the homology functor and consider the operad $H_\bullet(\mathcal{I})$ in the category of $\mathbb{Z}$-graded vector spaces. Can we characterize algebras over this operad?

Proposition 3.14. *$\mathcal{H}$ is algebra over $H_\bullet(\mathcal{I})$ iff $\mathcal{H}$ is an associative algebra.*

Proof. The space $\mathcal{I}(n)$ contains a connected component $\mathcal{I}'(n)$ whose n subintervals have an ordering which increases as one goes from left to right. S_n acts simply, transitively on the space of connected components of $\mathcal{I}(n)$. Therefore, one can identify the set of connected components of $\mathcal{I}(n)$ with the elements in S_n as an S_n set where S_n acts on itself by right multiplication.

The connected component $\mathcal{I}'(n)$ is contractible. (Indeed, it is homotopic to the configuration of n ordered points in the oriented interval since each subinterval can be shrunk to its midpoint and the latter is contractible.) Therefore, $H_p(\mathcal{I}(n))$ vanishes unless $p = 0$ and $H_0(\mathcal{I}(n))$ is canonically isomorphic (as a S_n-module) to the right regular representation $\mathbb{C}[S_n]$.

Since the compositions $\circ_i$ preserve the subcollection $\{\mathcal{I}'(n)\}_{n\geq 1}$ for all $i = 1, \dots, n$, if $\mathbf{1}_n$ denotes the identity element in S_n then under the identification of $H_0(\mathcal{I}(n))$ with $\mathbb{C}[S_n]$, we have

$$\mathbf{1}_n \circ_i \mathbf{1}_{n'} = \mathbf{1}_{n+n'-1} \tag{7}$$

for all $i = 1, \dots, n$.

$H_0(\mathcal{I}(1))$ contains only multiples of the unit element $\mathrm{I} = \mathbf{1}_1$. The image of I must map to the identity map on $\mathcal{H}$.

Given a morphism $\mu : H_0(\mathcal{I}) \to \mathrm{End}^{\mathcal{H}}$, let $c := \mathbf{1}_2$ be the class in $H_0(\mathcal{I}(2)) \simeq \mathbb{C}[S_2]$ which is represented by a point in $\mathcal{I}'(2)$. Call $m := \mu_c : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ the multiplication operation. Equation (7) implies, in particular, that

$$c \circ_1 c = c \circ_2 c \tag{8}$$

in $H_0(\mathcal{I}'(3))$. Consequently, $m \circ_1 m = \mu_{c\circ_1 c} = \mu_{c \circ_2 c} = m \circ_2 m$. Or, in other words, m is an associative multiplication on $\mathcal{H}$.

$H_0(\mathcal{I}(2))$ contains another class $b\sigma$, where σ is the transposition in S_2, obtained from b by transposing the ordering of the two intervals. The associated binary operation $\mu_{b\sigma} = m \circ \sigma$ is the opposite multiplication which takes $v_1 \otimes v_2 \mapsto (-1)^{|v_1||v_2|} m(v_2, v_1)$. However, it is not another basic operation in the sense that it is completely determined by m and the action of the permutation group.

Since all other classes in $H_0(\mathcal{I}(n))$ for $n \geq 3$ can be generated by iterated compositions of the class b in $H_0(\mathcal{I}(2))$ together with the action of the permutation groups, there are no n-ary operations which cannot be obtained from the multiplication m under successive compositions and the permutation group actions.

Finally, it follows from Equation (7) that there are no relations in $H_0(\mathcal{I}(n))$ aside from those generated by Equation (8).

To prove the converse, given an associative algebra $\mathcal{H}$, we need to construct a morphism of operads $\mu : H_\bullet(\mathcal{O}) \to \mathrm{End}^{\mathcal{H}}$. We define μ_c for $c = \mathbf{1}_2$ to be the given multiplication on $\mathcal{H}$ and then extend μ to all of $H_\bullet(\mathcal{O})$ in a way compatible with the compositions using Equation (7) and by using the equivariance of μ under the permutation group. □

In light of the previous proposition, the operad $H_\bullet(\mathcal{I})$ can be called the *associative operad* since algebras over it are nothing more than associative algebras.

The importance of operads is that they are gadgets which parameterize algebraic structures of a given type just as the associative operad, above, governs associative algebras. Common algebraic structures (for example, associative algebras, commutative associative algebras, Lie algebras, Poisson algebras, A_∞ algebras, etc) consisting of a collection of n-ary operations for $n \geq 1$ have an underlying operad responsible for them.

Definition 3.15. Let $\mathcal{H}$ be a $\mathbb{Z}$-graded vector space together with two binary operations, a *multiplication* $a \otimes b \mapsto a \cdot b$ of degree 0 and a *bracket* $a \otimes b \mapsto [a, b]$ of degree 1, such that for all homogeneous elements a, b, c in $\mathcal{H}$,

Commutativity of the Product: $a \cdot b = (-1)^{|a||b|} b \cdot a$,
Associativity of the Product: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
"Skew"-Symmetry of the Bracket: $[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$,
Jacobi Identity: $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]]$, and
Leibniz Rule: $[a, b \cdot c] = [a, b]c + (-1)^{|b|(|a|-1)} b[a, c]$.

Such a triple $(\mathcal{H}, \cdot, [,])$ is called a *Gerstenhaber algebra*.

A Gerstenhaber algebra is basically a kind of Poisson (super)algebra but where the bracket has degree 1. We will now find the operad responsible for it.

Theorem 3.16 (F. Cohen [Co], V. I. Arnold [Ar])**.** *$\mathcal{H}$ is an algebra over $H_\bullet(\mathcal{D})$ if and only if $\mathcal{H}$ is a Gerstenhaber algebra. Thus, $H_\bullet(\mathcal{D})$ is called the Gerstenhaber operad.*

Proof. The operad $\mathcal{D}(n)$ is homotopy equivalent to $C_n^{\mathbb{C}}$, the configuration of n points in $\mathbb{C}$, which consists of pairwise distinct n-tuples $(p_1, \dots, p_n)$ in $\mathbb{C}$. The equivalence is obtained by shrinking each little disk to its center point.

Let $\mu : H_\bullet(\mathcal{D}) \to \mathrm{End}^{\mathcal{H}}$ be a morphism of operads. $D(1)$ is contractible thus it contains only multiples of the unit I. $D(2)$ is homotopic to S^1. Therefore, $H_\bullet(\mathcal{D}(2)) \simeq H_\bullet(S^1)$. Let β denote the class of a point in $H_0(D(2))$. Let γ denote the class in $H_1(\mathcal{D}(2))$ corresponding to the fundamental class of S^1. One can view γ as being represented by an embedded circle in $D(2)$ where the second disk completes one counterclockwise revolution around the first disk. Thus, μ_β is a binary operation of degree 0, which we'll denote by $a \otimes b \mapsto a \cdot b$, and μ_γ is a binary operation of degree 1. Denote the map $a \otimes b \mapsto [a, b]$ by $\mu_\gamma(a \otimes b)(-1)^{|a|}$ for all homogeneous a, b in $\mathcal{H}$.

Since S^1 is connected, $\beta = \beta\sigma$ where σ is the transposition in S_2. Therefore, $\mu_\beta = \mu_{\beta\sigma} = \mu_\beta \circ \sigma$ or, equivalently, for all homogeneous a, b in $\mathcal{H}$, $a \cdot b = (-1)^{|a||b|} b \cdot a$. Similarly, $\gamma\sigma = \gamma$ implies $[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$.

The Leibniz rule follows from an identity between compositions of β and γ in three different ways in $H_1(\mathcal{D}(3))$ while the Jacobi identity follows from an identity between compositions of β in three different ways in $H_2(\mathcal{D}(3))$.

To prove that there are no other basic operations and no other basic relations it is useful to use a presentation of the cohomology groups of configuration spaces due to Arnold [Ar]. □

Remark 3.17. $H_\bullet(\mathcal{D})$ has several important suboperads, one responsible for the multiplication only, and the other responsible for the Lie bracket only.

The suboperad $H_0(\mathcal{D})$ of $H_\bullet(\mathcal{D})$ is the operad responsible for commutative, associative algebras. Thus it is may be called the *commutative operad.* Observe that $H_0(\mathcal{D}(n)) \simeq \mathbb{C}$ for all n.

The suboperad $\mathcal{L} := \{\, \mathcal{L}(n) \,\}_{n\geq 1}$ of $H_\bullet(\mathcal{D})$ where $\mathcal{L}(n) := H_{n-1}(\mathcal{D}(n))$ is the operad responsible for the Lie algebra structure where the Lie bracket has degree 1. However, after shifting the grading of $\mathcal{H}$ by 1, one may regard the bracket as usual Lie algebra with a degree zero bracket. Hence, $\mathcal{L}$ is often called the *Lie operad.*

3.4. **Topological conformal field theory.** Let us begin with a variant of topological field theory called a conformal field theory whose axiomatization is due to Segal [Se].

Definition 3.18. A *($g = 0$, $c = 0$) conformal field theory (CFT)* is a topological vector space $\mathcal{H}$ together with a smooth morphism of operads $\mu : \mathcal{P} \to \mathrm{End}^{\mathcal{H}}$. $\mathcal{H}$ is called the *state space.*

The $g = 0$ corresponds to the fact that $\mathcal{P}$ only features Riemann surfaces of genus 0. The $c = 0$ means that μ is a morphism of operads. In a general CFT, μ would only preserve the compositions up to a certain projective factor. In particular, restricting to the semigroup $\mathcal{P}(1)$, $\mathcal{H}$ would be a projective representation of the semigroup $\mathcal{P}(1)$. This is the essentially the geometric origin of the central extension of the Virasoro algebra which is usually denoted by c.

We also remark that Segal's definition of a CFT contains more features than those presented above such as an inner product, decompositions into left ("holomorphic") and right movers ("antiholomorphic") and so forth but the above will suffice for our purposes.

It is not easy to characterize the algebraic structure on $\mathcal{H}$. However, there is a result by Huang which shows that a vertex operator algebra is a CFT where the μ are holomorphic. Indeed, to study the connection with vertex algebras, one must enlarge the operad $\mathcal{P}$ to a larger space, $\mathcal{P}'$, in which the holomorphically embedded disks are now allowed to pairwise overlap provided that their centers do not coincide. The compositions are then analytically continued from $\mathcal{P}$ to a maximal subspace of $\mathcal{P}'$. The result is that $\mathcal{P}'$ is no longer an operad but a partial operad (or perhaps "operadoid"*) which means that compositions are not always defined. Vertex operator algebras are essentially an algebra over $\mathcal{P}'$ (up to a projective factor). We refer to [Hu] and the references therein for more details to this approach.

A purely algebraic construction of the BV algebra structure in terms of the closely related algebraic framework of topological chiral algebras appears in [LZ].

While a general CFT is a rather complicated object, there is a variant of a CFT called a topological conformal field theory that has a sector which is easier to handle.

*We are pleased to be able to use include this term in light of some of the other topics presented at this conference.

Definition 3.19. A *$(g = 0)$ topological conformal field theory (TCFT)* is a topological dg vector space $\mathcal{H}$ together with a morphism of operads $\mu : C_\bullet(\mathcal{P}) \to \mathrm{End}^{\mathcal{H}}$.

Again, we have simplified the definition of a TCFT [Se2] as one usually assumes in addition, that the morphism μ arises by integrating smooth chains on $\mathcal{P}(n)$ over differential forms on $\mathcal{P}(n)$ with values in $\mathrm{End}^{\mathcal{H}}(n)$. In this situation, a TCFT is a special kind of CFT.

Remark 3.20. In the physics literature, the differential $d : \mathcal{H}_{p+1} \to \mathcal{H}_p$ associated to a TCFT is sometimes called a BRST operator and the grading p of $\mathcal{H}_p$ is (the negative of) the ghost number.

Definition 3.21. Let $\mathcal{H}$ be a Gerstenhaber algebra together with an operation $\Delta : \mathcal{H} \to \mathcal{H}$ of degree 1 such that $\Delta^2 = 0$ and

$$[a, b] = (-1)^{|a|}(\Delta(a \cdot b) - (\Delta a) \cdot b - (-1)^{|a|} a \cdot (\Delta b)) \tag{9}$$

for all homogeneous a and b in $\mathcal{H}$ then we say that $\mathcal{H}$ is a *Batalin-Vilkovisky (or BV) algebra.*

The following theorem and lemma are due to Getzler [Ge].

Theorem 3.22. *Let $(\mathcal{H}, d)$ be a $(g = 0)$ TCFT then $H_\bullet(\mathcal{H})$ is a BV algebra.*

Proof. The proof follows from the following Lemma and the observation that the morphism of operads $\mathcal{F} \to \mathcal{P}$ is a homotopy equivalence. □

Lemma 3.23. *$\mathcal{H}$ is a $H_\bullet(\mathcal{F})$-algebra if and only if it is a BV algebra.*

Proof. Since $\mathcal{F}(n)$ is a trivial $(S^1)^n$-bundle over $\mathcal{D}(n)$, its homology groups can be readily computed.

Let $\mu : H_\bullet(\mathcal{F}) \to \mathrm{End}^{\mathcal{H}}$ be a morphism of operads. Since $\mathcal{D} \to \mathcal{F}$ is a morphism of operads, it induces a morphism $H_\bullet(\mathcal{D}) \to H_\bullet(\mathcal{F})$. Thus, $\mathcal{H}$ is a $H_\bullet(\mathcal{D})$-algebra which, by Theorem 3.16, implies that $\mathcal{H}$ is a Gerstenhaber algebra.

Observe that $\mathcal{F}(1)$ is homotopic to (an oriented) S^1 corresponding to the direction of the marked point on the boundary of the little disk. Let c be the element in $H_1(\mathcal{F}(1))$ corresponding to the fundamental class of S^1. Call $\Delta := \mu_c$.

It is clear that $c \circ_1 c = 0$ since $H_2(S^1) = 0$, therefore, $\Delta^2 = 0$. Equation (9) can be verified directly.

The computation of the homology groups implies that there are no other basic operations or relations. □

One can also prove the existence of homotopy algebraic structures associated to a TCFT. For example, Kimura-Stasheff-Voronov [KSV] proved that a certain relative subcomplex of a TCFT whose structure morphisms arise from integration of smooth differential forms yields a homotopy Lie (or L_∞) algebra. The operad responsible for this structure arises from a certain "real" compactification of the moduli space of Riemann spheres with n marked points. This structure was also been observed in the physics literature by Zweibach [Zw] in the context of closed string field theory.

In the previous examples, we treated the $(n + 1)$st embedded disk of $\mathcal{P}(n)$ differently from the first n embedded disks. The $(n + 1)$st disk is a kind of "output" while the first n disks are "inputs." However, from a geometric point of view, it is reasonable to put all such disks on the same footing, *i.e.* allow the permutation group S_{n+1} to act upon $\mathcal{P}(n)$ by permuting all disks and to allow sewings between arbitrary pairs of disks on two different spheres. Formalizing this structure gives rise to the notion a cyclic operad [GeKa]. Algebras over cyclic operads typically correspond to algebraic structures (like associative algebras or Lie algebras) with an invariant metric.

By allowing moduli spaces of higher genus Riemann surfaces as well and allowing sewings (including sewing two disks on the same surface to obtain a new surface of higher genus), one obtains an example of a modular operad [GeKa2]. Rather than describing such technical notions in complete generality, we will now focus upon the model example of a modular operad, namely the moduli space of stable curves, and study algebras over them. Such algebras are often referred to as a cohomological field theory in the sense of Kontsevich-Manin.

4. The moduli space of stable curves and cohomological field theories

4.1. The cyclic operad of stable trees.

Let us begin with a useful toy model.

Definition 4.1. Let $[n]$ denote the set $\{1, \ldots, n\}$ for all $n \geq 1$.

Definition 4.2. Let I be a finite set such that $|I| \geq 4$. A *stable partition of I* is a partition of I, $I_+ \sqcup I_- = I$, such that $|I_\pm| \geq 2$.

Definition 4.3. Let I be a finite set such that $|I| \geq 3$. A *stable I-tree*, Γ, is a connected graph containing no circuits (*i.e.* a tree) together with a bijection from I to its set of tails $T(\Gamma)$ whose vertices are of valence at least 3. Let $T(\Gamma)$, $V(\Gamma)$, and $E(\Gamma)$, respectively, denote the set of tails, the set of vertices, and the set of edges of Γ. A *stable n-tree* when $n \geq 3$ is a stable $[n]$-tree. The *n-corolla*, δ_n, is the unique stable n-tree with exactly one vertex.

Let $\mathcal{T} := \{ \mathcal{T}_n \}_{n \geq 2}$ where $\mathcal{T}_n$ is the free $\mathbb{C}$-span of the set of stable n-trees. T_n is an S_n-module by permutation of the labels on its tails.

Let $n \geq 4$ be an integer and consider a partition $N \sqcup N' := [n]$ where $|N| = n$ and $|N'| = n'$ such that $n, n' \geq 2$. Let $N = \{ i_1, \ldots, i_n \}$ and $N' = \{ j_1, \ldots, j_{n'} \}$ where $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_{n'}$.

Define the composition maps $\mathcal{T}_{n+1} \times \mathcal{T}_{n'+1} \to \mathcal{T}_{n+n'}$ taking $(\Gamma, \Gamma') \mapsto \Gamma \circ_{(N,N')} \Gamma'$ defined by gluing the $(n+1)$st tail of Γ to the 1st tail of Γ' and then relabeling the remaining tails so that the k-th tail of Γ is labeled by i_k for all $k = 1, \ldots, n$ and the p-th tail of Γ' is labeled by j_{p-1} for all $p = 2, \ldots, n'+1$.

It is probably easiest to visualize the composition by reversing the arrow of time. That is, given a stable n''-tree, Γ'', choose an edge e in Γ'' and cut it. Choose one of the resulting trees and call it Γ and let Γ' denote the other. Let $n+1$ denote the number of tails of Γ and $n'+1$ denote the number of tails of Γ'. Let t and t' denote the new tails on Γ and Γ' respectively obtained by cutting e. Let N denote the set of labels on Γ and N' denote the set of labels on Γ'. Clearly, N and N' form a stable partition of $[n+n']$ where $|N| = n$ and $|N'| = n'$. Let $N = \{i_1, \ldots, i_n\}$ and $N' = \{j_1, \ldots, j_{n'}\}$ where $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_{n'}$. Relabel Γ so that the tail with the label i_k is labeled instead by k for all $k = 1, \ldots, n$ and then label the tail t by $n+1$. Similarly, relabel Γ' so that the tail with label j_p is labeled instead with $p+1$ for all $p = 1 \cdots, n'$ and then label the tail t' by 1. With these new labels for Γ' and Γ'', we have $\Gamma \circ_{(N,N')} \Gamma' = \Gamma''$.

Associativity can then be understood, under time reversal, as taking a tree Γ'', and choosing two distinct edges e and e'. Cutting Γ'' at e followed by cutting the result at e' and repeating the process in the last paragraph, one obtains 3 stable trees which when glued back together yield Γ''. On the other hand, cutting Γ'' along e' followed by cutting the result at e and repeating the process above, one obtains 3 stable trees which when glued together in the other order yields Γ''.

These composition maps are equivariant under the action of the permutation groups.

$\mathcal{T}$ contains the structure of an operad.[†] Let $\mathcal{T}(n) := \mathcal{T}_{n+1}$ for all $n \geq 2$. The operadic composition $\mathcal{T}(n) \times \mathcal{T}(n') \to \mathcal{T}(n+n'-1)$ taking $(\Gamma, \Gamma') \mapsto \Gamma \circ_i \Gamma'$ for all $i = 1, \ldots, n$ is defined by $\Gamma \circ_i \Gamma' = \Gamma' \circ_{(N', N)} \Gamma$ where

$$N = [n] \qquad \text{and} \qquad N' = \{\, n+1, \ldots, n+n' \}. \tag{10}$$

Example 4.4. Let $(\mathcal{H}, \eta)$ be a finite dimensional vector space $\mathcal{H}$ with a symmetric, nondegenerate, bilinear form. Let $\mathcal{E}nd_n^{\mathcal{H}} := (\mathcal{H}^*)^{\otimes(n)}$ where $\mathcal{H}^*$ denotes the dual vector space.

Given f in $\mathcal{E}nd_{n+1}^{\mathcal{H}}$ and f' in $\mathcal{E}nd_{n'+1}^{\mathcal{H}}$ and a stable partition of $N \sqcup N'$ of $[n+n']$ where $N = \{\, i_1, \ldots, i_n \,\}$ and $N' = \{\, j_1, \ldots, j_{n'} \,\}$ such that $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_{n'}$ then

$$(f \circ_{(N,N')} f')(v_1, \ldots, v_{n+n'}) := f(v_{i_1}, \cdots, v_{i_n}, e_\alpha)\eta^{\alpha\beta} f'(e_\beta, v_{j_1}, \cdots, v_{j_{n'}})$$

for all $v_1, \ldots, v_{n+n'}$ where $\{e_\alpha\}$ is a basis for $\mathcal{H}$, $\eta^{\alpha\beta}$ is the inverse metric in this basis, and the summation convention has been used. Furthermore, $\mathcal{E}nd_n^{\mathcal{H}}$ has a natural action of S_n.

One can recover the usual endomorphism operad of $\mathcal{H}$ as follows. Using the metric to identify $\mathcal{E}nd_{n+1}^{\mathcal{H}}$ with $\mathrm{End}^{\mathcal{H}}(n)$, one can recover the S_n action on the latter by restriction to the subgroup S_n in S_{n+1}. The operadic compositions in $\mathrm{End}^{\mathcal{H}}$ may be recovered by considering the partitions in Equation (10).

4.2. **Tree level Feynman diagrams.** Let us consider the analog of an algebra over an operad in this setting. Let $(\mathcal{H}, \eta)$ be a finite dimensional vector space with a symmetric, nondegenerate bilinear form. Consider a morphism $\mu : \mathcal{T} \to \mathcal{E}nd^{\mathcal{H}}$ preserving the compositions and equivariant under the action of the permutation groups. This means that we must associate to each n-tree in $\mathcal{T}_n$ an element in $(\mathcal{H}^*)^{\otimes n}$ compatible with compositions and the permutation groups.

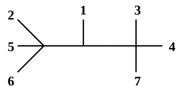
Specifying μ is equivalent to a choice of an element $\Phi^{(n)}$ in the n-fold symmetric product $S^n\mathcal{H}^*$ for all $n \geq 3$. This is because μ_{δ_n}, where δ_n is the n-corolla, is an element in the n-fold symmetric product $S^n\mathcal{H}^*$. It is symmetric because the tree δ_n is invariant under S_n. Define $\mu_{\delta_n} := \Phi^{(n)}$. Since any tree in $\mathcal{T}$ is obtained by gluing together corollas, μ is completely determined. Furthermore, $\Phi^{(n)}$ can be arbitrary for all $n \geq 3$ because $\mathcal{T}$ is freely generated, with respect to the composition, by the set of corollas.

This is a situation very familiar to physicists. If Γ is an n-tree in $\mathcal{T}(n)$ then associate to each valence n vertex a totally symmetric tensor $\Phi^{(n)}$ ("an interaction vertex") and to each internal edge the inverse metric ("a propagator") contracting in the manner indicated by the graph. In other words, specifying μ is nothing more than a choice of Feynman rules on $\mathcal{H}$. For example, let $\{e_\alpha\}$ be a basis for $\mathcal{H}$, $\{e^\alpha\}$ denote its dual basis in $\mathcal{H}^*$, $\eta_{\alpha\beta}$ be the metric, $\eta^{\alpha\beta}$ be the inverse metric, and $\Phi_{\alpha_1, \ldots, \alpha_n}$ denote the symmetric n tensor $\Phi^{(n)}$ with respect to this basis.

[†]Notice that since $\mathcal{T}(1)$ does not exist, this is an operad without unit. However, if one desires, one may adjoin a unit by hand.

For example, the following stable 7-tree

(11)



has the associated tensor in $(\mathcal{H}^*)^{\otimes 7}$

$$\Phi_{\alpha_2\alpha_5\alpha_6\beta}\eta^{\beta\gamma}\Phi_{\gamma\alpha_1\delta}\eta^{\delta\epsilon}\Phi_{\epsilon\alpha_3\alpha_4\alpha_7}e^{\alpha_1}\otimes e^{\alpha_2}\otimes\cdots\otimes e^{\alpha_7} \tag{12}$$

where the summation convention has been used.

These Feynman rules are captured by a "Lagrangian" whose "kinetic term" is

$$\frac{1}{2}\eta(x,x) \tag{13}$$

and whose "potential function" is

$$\Phi(x) := \sum_{n=3}^{\infty}\frac{1}{n!}\Phi^{(n)}(x,x,\ldots,x) \tag{14}$$

where x belongs to $\mathcal{H}$ and $\Phi(x)$ is regarded as a formal power series on $\mathcal{H}$. However, in the definition of μ, only tree level terms in the usual perturbative expansion are relevant. We'll have use for the higher loop graphs later.

Let us now consider an important special case. Let $\mathcal{T}' := \{\mathcal{T}'_n\}_{n\geq 2}$ denote the subspace in $\mathcal{T}_n$ spanned by trivalent, stable n-trees. This subspace is closed under compositions and the permutation group actions. A morphism $\mu : \mathcal{T}' \to \mathcal{E}nd^{\mathcal{H}}$ is equivalent to Feynman rules associated only to trivalent graphs. Thus, μ is completely characterized, since we keep the same kinetic term, by the (cubic) potential function $\Phi(x) = \frac{1}{3!}\Phi^{(3)}(x,x,x)$.

4.3. **The moduli space of stable curves.** Let $\mathcal{M}_{g,n}$ denote the moduli space of smooth, stable curves of genus g with n marked points. This consists of tuples $(C;p_1,\ldots,p_n)$ where C is a smooth, connected, complex algebraic curve of genus g together with pairwise distinct marked points $p_1,\ldots,p_n$ on C where any two such configurations $(C;p_1,\ldots,p_n)$ and $(C';p'_1,\ldots,p'_n)$ are isomorphic if and only if there is an isomorphism between the curves $\phi : C \to C'$ such that $\phi(p_i) = p'_i$ for all $i = 1,\ldots,n$. (If one prefers, one may also regard this as the moduli space of Riemann surfaces of genus g with n marked points.) Furthermore, we require that the pointed curve $(C;p_1,\ldots,p_n)$ be stable, meaning that it has no infinitesimal automorphisms. This means that $2g-2+n > 0$ must hold or, equivalently, if $g = 0$ then $n \geq 3$, if $g = 1$ then $n \geq 1$, and if $g \geq 2$ then n is arbitrary.

$\mathcal{M}_{g,n}$ is a smooth, complex orbifold of complex dimension $3g-3+n$. While these spaces are noncompact, $\mathcal{M}_{g,n}$ has a natural compactification $\overline{\mathcal{M}}_{g,n}$ due to Deligne-Mumford called the *moduli space of stable curves of genus g with n marked points*. It consists of tuples $(C;p_1,\ldots,p_n)$ where C is a complex curve with (at worst) nodal singularities, the pairwise distinct points $p_1,\ldots,p_n$ cannot be at a node, and $(C;p_1,\ldots,p_n)$ must have no infinitesimal automorphisms. $\overline{\mathcal{M}}_{g,n}$ is a smooth, connected, compact, complex orbifold of complex dimension $3g-3+n$ which contains $\mathcal{M}_{g,n}$ as a dense open subset. This means that a nodal curve with n marked points is stable if and only if each irreducible component of genus 0 has at least 3 special points and each irreducible component of genus 1 has at least 1 special point where by special point we mean either a node or a marked point.

Associated to each partition (N, N') of $[n+n']$, where $|N| = n$ and $|N'| = n'$, compatible with stability, there is a composition morphism $\circ_{(N,N')} : \overline{\mathcal{M}}_{g,n+1} \times \overline{\mathcal{M}}_{g',n'+1} \to \overline{\mathcal{M}}_{g+g',n+n'}$ which takes the curve C of genus g, the curve C' of genus g' and attaches the $(n+1)$st marked point of C with the 1st point of C' thereby creating a node. The remaining $(n+n')$ marked points are then reordered using the partition (N, N') as before. In addition, for each distinct pair i and j in $[n+2]$, we obtain another gluing morphism $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ which glues the curve together along its i-th and j-th marked points to create a node. The collection $\{\overline{\mathcal{M}}_{g,n}\}$ together with these gluing morphisms (which are equivariant under the action of the permutation groups) is the model example of a *modular operad* in the category of smooth, complex orbifolds.

Let us now associate to each stable curve of genus g with n marked points a graph called its *dual graph* as follows. Collapse each irreducible component of genus g to a point to obtain a vertex and label the vertex with g. For each special point on that component, draw a half edge from the vertex. If two irreducible components meet at a node then connect their corresponding half edges together to form and edge. Finally, for each remaining half-edge (or tail) we give it the same label in $[n]$ as its associated marked point. The resulting connected graph is called a *stable graph of genus g with n tails*. An n-corolla of genus g, $\delta_{g,n}$, is a stable graph of genus g with n tails with exactly one vertex and it corresponds to a smooth curve of genus g with n marked points subject to the stability condition that if $g = 0$ then $n \geq 3$ and if $g = 1$ then $n \geq 1$.

Tracing through the definitions, a stable graph of genus g with n tails is a graph, together with a bijection between $[n]$ and its set of tails, which is either $\delta_{g,n}$ or it can be obtained by from gluing together a finite number of stable corollas such that the *genus of* Γ, $g(\Gamma)$ defined by $g(\Gamma) := b_1(\Gamma) + \sum_{v \in V(\Gamma)} g(v)$, is equal to g where b_1 denotes the 1st Betti number, $V(\Gamma)$ denotes the set of vertices, and $g(v)$ denotes the genus label at vertex v.

A stable n-tree can be regarded as a genus 0 stable graph with n tails by assigning each vertex a genus of 0. Conversely, any stable graph of genus 0 with n tails is a stable n-tree by erasing the genus 0 label at each vertex. We shall henceforth make this identification.

To each stable graph, one can associate an orbifold. Given a stable graph Γ of genus g with n tails, we can associate an orbifold $\overline{\mathcal{M}}_\Gamma$ consisting of the closure of the locus of curves in $\overline{\mathcal{M}}_{g,n}$ whose dual graph is Γ. $\overline{\mathcal{M}}_\Gamma$ is a compact orbifold whose complex codimension is equal $|E(\Gamma)|$ where $E(\Gamma)$ is the set of (internal) edges of the graph. In particular, $\overline{\mathcal{M}}_{\delta_{g,n}} = \overline{\mathcal{M}}_{g,n}$. Furthermore, the inclusion map $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ is a regular embedding.

4.4. $g = 0$ **Cohomological field theories and Feynman diagrams.** Our goal in this subsection is to consider only genus 0 moduli spaces of stable curves, take their homology groups, and to characterize algebras over them.

Let us restrict to composition morphisms associated to a stable partitions (N, N') of $[n+n']$ where $|N| = n$ and $|N'| = n'$ taking $\overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,n'+1} \to \overline{\mathcal{M}}_{0,n+n'}$. This induces composition morphisms $H_\bullet(\overline{\mathcal{M}}_{0,n+1}) \times H_\bullet(\overline{\mathcal{M}}_{0,n'+1}) \to H_\bullet(\overline{\mathcal{M}}_{0,n+n'})$.

Definition 4.5. A $g = 0$ *cohomological field theory (CohFT)* is a vector space with metric $(\mathcal{H}, \eta)$ together with maps for all $n \geq 3$, $\mu : H_\bullet(\overline{\mathcal{M}}_{0,n}) \to \mathcal{H}^{*\otimes n}$, which preserve compositions and are S_n equivariant.

Remark 4.6. Here, we are regarding $\mathcal{H}^{*\otimes n}$ as $\mathcal{E}nd_n^{\mathcal{H}}$.

Remark 4.7. There is also a flat identity element in the definition of a CohFT which is a generalization of the identity element in a Frobenius algebra but we will ignore it for the purposes of these lectures.

Remark 4.8. Henceforth, unless otherwise stated, we will assume that $\mathcal{H}$ is an ungraded vector space in order to avoid being distracted by signs.

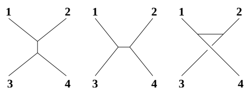
Let Γ be a stable n-tree. Given any stable n-tree Γ, we can associate the homology class $[\overline{\mathcal{M}}_\Gamma]$ in $H_p(\overline{\mathcal{M}}_{0,n})$ where $p = 2(n-3-|E(\Gamma)|)$. This induces a map $f : \mathcal{T}_n \to H_\bullet(\overline{\mathcal{M}}_{0,n})$ for all $n \geq 3$ which is S_n equivariant and preserves compositions. Given any $g = 0$ CohFT $\mu : H_\bullet(\overline{\mathcal{M}}_{0,n}) \to \mathcal{H}^{*\otimes n}$, composition with f yields a morphism $\nu : \mathcal{T}_n \to \mathcal{H}^{*\otimes n}$. By the analysis in Subsection 4.2, ν is completely determined by tree level Feynman diagrams associated to a potential function $\Phi(x)$ of the form in Equation (14). However, it's not true that an arbitrary potential function $\Phi(x)$ can arise from the morphism ν. This is because the morphism $\mathcal{T}_n \to H_\bullet(\overline{\mathcal{M}}_{0,n})$ is not injective, in general, *i.e.* there could be two different linear combination of n-trees whose associated cycles are homologous. This means that the result of applying the Feynman rules to these linear combinations of graphs must yield the same element in $\mathcal{H}^{*\otimes n}$. This imposes constraints on the potential function Φ. What are these constraints, exactly?

In order to get an idea of how to think about these relations, let's consider the special case of trivalent trees. If Γ belongs to $\mathcal{T}'_n$ then $[\overline{\mathcal{M}}_\Gamma]$ belongs to $H_0(\overline{\mathcal{M}}_{0,n})$ since amongst all stable n-trees, a trivalent n-tree has the maximal number of (internal) edges. Therefore, f restricts to a morphism $f' : \mathcal{T}'_n \to H_0(\overline{\mathcal{M}}_{0,n})$ for all $n \geq 3$. We recall that since $\overline{\mathcal{M}}_{g,n}$ is connected, the kernel of $f' : \mathcal{T}'_n \to \mathcal{H}^{*\otimes n}$, call it $\mathcal{R}_n$ for the space of relations, is quite large in general. However, we seek a more diagrammatic characterization of $\mathcal{R}_n$.

There is only one stable 3-tree, the corolla δ_3. Also, $\overline{\mathcal{M}}_{0,3}$ is a point since one can always use the automorphism group, $\mathrm{PGL}(2,\mathbb{C})$, of $\mathbb{C}P_1$ to move the 3 marked points to $0, 1, \infty$, for example. Therefore, $f' : \mathcal{T}'_3 \to H_0(\overline{\mathcal{M}}_{0,3})$ mapping δ_3 to $[\overline{\mathcal{M}}_{\delta_3}] = [\overline{\mathcal{M}}_{0,3}]$ is an isomorphism.

The open moduli space $\mathcal{M}_{0,4}$ is a $\mathbb{C}P_1$ together with distinct marked points (p_1, p_2, p_3, p_4). Using $\mathrm{PGL}(2,\mathbb{C})$, one can move p_1 to 0, p_2 to 1, p_3 to ∞ and p_4 can be anywhere else. Therefore, $\mathcal{M}_{0,4} \simeq \mathbb{C}P_1 - \{0, 1, \infty\}$. The compactification $\overline{\mathcal{M}}_{0,4}$ introduces nodal curves corresponding to the limits where the 4-th point approaches either the 1st, 2nd or 3rd marked points. These three nodal curves have the following dual graphs.

(15)



Call the above graphs S, T, and U, respectively. Note that U has only two vertices but is drawn in this suggestive fashion for the sequel.

Adding in these three points, we see that $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{C}P_1$. Since $\mathbb{C}P_1$ is connected, we have the equality

$$[\overline{\mathcal{M}}_S] = [\overline{\mathcal{M}}_T] = [\overline{\mathcal{M}}_U] \tag{16}$$

in $H_0(\overline{\mathcal{M}}_{0,4})$. [‡]

Furthermore, the identity in Equation (16) generates all elements in the kernel of $f' : \mathcal{T}'_n \to H_0(\overline{\mathcal{M}}_{0,n})$ for all $n \geq 4$ in the following sense. Let Γ be a trivalent, stable n-tree where $n \geq 4$. Choose a subgraph of Γ which can be identified with S and then replace it by either a T or a U graph. We call this operation a move on Γ. We can now define an equivalence relation on the space of trivalent n-trees, namely, two stable trivalent n-trees Γ and Γ' are said to be equivalent if and only if there exists a sequence of moves taking Γ to Γ'. It is easy to check that any two trivalent n-trees are equivalent to each other. Therefore, the basic relation in Equation (16) generates the entire kernel of f'.

If $\mu' : H_0(\overline{\mathcal{M}}_{0,n}) \to \mathcal{H}^{*\otimes n}$ is a morphism preserving compositions and permutation group actions then its composition with f', $\nu' : \mathcal{T}'_n \to \mathcal{H}^{*\otimes n}$ for all $n \geq 2$, corresponds to precisely those Feynman rules which satisfy the identity

$$\nu_S = \nu_T = \nu_U. \tag{17}$$

Let $\Phi^{(3)}$ in $S^3\mathcal{H}^*$ be the 3-tensor associated to each trivalent vertex then one can define an operation $m : \mathcal{H}^{\otimes 2} \to \mathcal{H}$ via

$$\eta(m(a,b),c) = \Phi^{(3)}(a,b,c)$$

for all a, b, c in $\mathcal{H}$. Equation (17) is equivalent to saying that $(\mathcal{H}, \eta, m)$ is a commutative, associative algebra with invariant inner product, *i.e.* a Frobenius algebra although possibly without an identity element.

We have just proven the following.

Proposition 4.9. *The morphism $\mu' : H_0(\overline{\mathcal{M}}_{0,n}) \to \mathcal{H}^{*\otimes n}$ is equivalent to endowing $(\mathcal{H}, \eta)$ with the structure of a Frobenius algebra (possibly without an identity element).*

An equivalent way of saying the same thing is to require that the potential function $\Phi(x) := \frac{1}{3!}\Phi^{(3)}(x,x,x)$ satisfy the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation for all a, b, c, d:

$$(\partial_a\partial_b\partial_e\Phi)\,\eta^{ef}\,(\partial_f\partial_c\partial_d\Phi) = (\partial_b\partial_c\partial_e\Phi)\,\eta^{ef}\,(\partial_f\partial_a\partial_d\Phi), \tag{18}$$

where $\eta_{ab} := \eta(e_a, e_b)$, η^{ab} is in inverse matrix to the metric η_{ab}, ∂_a is derivative with respect to x^a, and the summation convention has been used.

It is this last formulation that turns out to be the most useful in the general case when we allow trees with vertices of arbitrary valences. This meant the potential function is no longer cubic.

Theorem 4.10 (Kontsevich-Manin)**.** *Let $(\mathcal{H}, \eta)$ be a $g = 0$ CohFT. Let Φ be its associated potential function, as defined by Equation (14), then it must satisfy Equation (18), the WDVV equation. Conversely, any function Φ of the form in Equation (14) satisfying the WDVV equation is the potential function of some CohFT.*

Proof. The idea of this proof is to observe that there are forgetful morphisms $\pi : \overline{\mathcal{M}}_{0,n+4} \to \overline{\mathcal{M}}_{0,4}$ which "forgets" n points. If two algebraic cycles on $\overline{\mathcal{M}}_{0,4}$ yield the same homology class then their flat pullbacks via π must also be in the same homology class of $\overline{\mathcal{M}}_{0,n+4}$. Therefore, pulling back the cycles $\overline{\mathcal{M}}_S$, $\overline{\mathcal{M}}_T$, and $\overline{\mathcal{M}}_U$, one obtains relations in $H_\bullet(\overline{\mathcal{M}}_{0,n+4})$. Furthermore, the pullback of $\overline{\mathcal{M}}_\Gamma$ for $\Gamma =$

[‡]To a physicist, the choice of names STU for these graphs should be familiar given the previous equality.

S, T, U can be written explicitly in terms of a linear combination of $(n+4)$-trees. These relations are encoded in the WDVV equation. See [Ma] for details. □

Remark 4.11. If $\mathcal{H}, \eta$ is a graded vector space with degree 0 metric η then the same theorem holds except that a factor of $(-1)^{|x_a|(|x_b|+|x_c|)}$ should be inserted to the immediate right of the equal sign.

Remark 4.12. The triple $(\mathcal{H}, \eta, \Phi)$ together with the flat identity element, which we have suppressed, forms an example of a formal Frobenius manifold, a generalization of a Frobenius algebra. This is a formal version of the notion of a Frobenius manifold due to Dubrovin [Du].

Theorem 4.10 is important because in many cases, the potential function is sufficiently constrained by the WDVV equation so that it can be solved.

4.5. **Cohomological Field Theories.** To summarize so far, the collection $\{H_0(\overline{\mathcal{M}}_{0,n})\}$ is responsible for Frobenius algebras. The collection $\{H_\bullet(\overline{\mathcal{M}}_{0,n})\}$ is responsible for (formal) Frobenius manifolds. Let us now study CohFTs in all genera.

Definition 4.13. A *cohomological field theory (CohFT)*, $(\mathcal{H}, \eta, \Lambda)$, is a finite dimensional vector space $\mathcal{H}$ with a nondegenerate, symmetric, bilinear form η together with a collection $\Lambda := \{\Lambda_{g,n}\}$ where for all (g,n), $\Lambda_{g,n}$ belong to $H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{H}^{*\otimes n}$. Let $\{e_a\}$ be a basis for $\mathcal{H}$ and let $\eta_{ab} = \eta(e_a, e_b)$ and η^{ab} be the inverse. The following conditions must be satisfied for all $v_1, \ldots, v_n$ in $\mathcal{H}$:

Invariance: $\Lambda_{g,n}$ is invariant under the diagonal action of S_n on the space $H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{H}^{*\otimes n}$.

Splitting: For any g', g'', $g' + g'' = g$, and any partition $N' \sqcup N'' = 1 \ldots, n$ such that $|N'| = n'$ and $|N''| = n''$, let $\rho_{(N',N'')} : \overline{\mathcal{M}}_{g',n'+1} \times \overline{\mathcal{M}}_{g'',n''+1} \to \overline{\mathcal{M}}_{g'+g'',n'+n''}$ where ρ is equal to the gluing morphism $\circ_{(N',N'')}$ then for all $v_1, \ldots, v_n$ in $\mathcal{H}$,

(19)
$$(\rho^*_{(N',N'')}\Lambda_{g,n})(v_1, \ldots, v_n) = \Lambda_{g',n'+1}(v_{i_1}, \ldots, v_{i'_n}, e_a)\eta^{ab}\Lambda_{g'',n''+1}(e_b, v_{j_1}, \ldots, v_{j''_n})$$

where $N' = \{i_1, \ldots, i_{n'}\}$ and $N'' = \{j_1, \ldots, j_{n''}\}$ such that $i_1 < i_2 < \cdots < i_{n'}$ and $j_1 < j_2 < \cdots < j_{n''}$ and the summation convention has been used.

Self-Sewing: Let $\rho_{n+1,n+2} : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ be the gluing morphism which glues together the $(n+1)$st and $(n+2)$nd marked points then for all $v_1, \ldots, v_n$ in $\mathcal{H}$,

$$(\rho^*_{n+1,n+2}\Lambda_{g+1,n})(v_1, \ldots, v_n) = \Lambda_{g,n+2}(v_1, \ldots, v_n, e_a, e_b)\eta^{ab}. \tag{20}$$

The classes $\Lambda_{g,n}$ are called the *genus g, n-point (cohomological) correlators of the CohFT*.

Remark 4.14. The above definition is dual to the earlier notion of CohFT, in some sense. Given a CohFT as above, one can define $\mu : H_\bullet(\overline{\mathcal{M}}_{g,n}) \to \mathcal{H}^{*\otimes n}$ via $c \mapsto \mu_c$. Therefore, by restricting to $H_\bullet(\overline{\mathcal{M}}_{0,n})$ for all n, one recovers the definition of a $g = 0$ CohFT which was introduced earlier.

Let us define a generalization of the previous notion of potential function.

Definition 4.15. Let $(\mathcal{H}, \eta, \Lambda)$ be a CohFT. The *(small phase space) potential function of the CohFT*, Φ, belongs to $\lambda^{-2}\mathbb{C}[[\mathcal{H}, \lambda^2]]$ where λ is a formal variable. It is defined by $\Phi(x) := \sum_{g=0}^{\infty} \lambda^{2g-2}\Phi_g(x)$, $\Phi_g(x) := \sum_n \frac{1}{n!}\Phi^{(g,n)}(x)$, and $\Phi^{(g,n)}(x) := \int_{[\overline{\mathcal{M}}_{g,n}]} \Lambda_{g,n}(x, x, \ldots, x)$.

Remark 4.16. The genus zero part of Φ, $\Phi_0(x)$, agrees with $\Phi(x)$ from Equation (14). This is just a matter of unraveling definitions. More generally, $\Phi^{(g,n)}(x)$ provides the Feynman rules associated to a genus g vertex of valence n.

The formal parameter λ may be regarded as a kind of coupling constant associated to the genus expansion. λ^2 plays a role analogous to that of $\hbar$ in the usual loop expansion in the usual Feynman diagrams.

It would be nice to have an analog of Theorem 4.10 in general genus now that we have defined a potential function which contains information about the higher genus moduli spaces. Unfortunately, it's need not be true that a CohFT (in all genera) is completely determined by its potential function $\Phi(x)$.

This is for the following reason. Let us replace $\mathcal{T}_n$ by $\mathcal{T}_{g,n}$, the free linear space of stable graphs of genus g with n tails. Of course, $\mathcal{T}_n = \mathcal{T}_{0,n}$. A morphism $\mathcal{T}_{g,n} \to \mathcal{H}^{*\otimes n}$ is still determined by Feynman rules except that now there are more corollas and loops are now allowed.

One can also still define the morphism $\mathcal{T}_{g,n} \to H_\bullet(\overline{\mathcal{M}}_{g,n})$ in analogy with the previous case. Unfortunately, in general, there will be classes for $g \geq 1$ $H_\bullet(\overline{\mathcal{M}}_{g,n})$ which need not lie in the image of the morphism $\mathcal{T}_{g,n} \to H_\bullet(\overline{\mathcal{M}}_{g,n})$. For example, $H_\bullet(\overline{\mathcal{M}}_{g,n})$ may have odd dimensional cohomology classes when $g \geq 1$. Since our algebraic cycles $\overline{\mathcal{M}}_\Gamma$ are complex, their associated homology classes are necessarily even dimensional. There will also be even dimensional classes in $H_\bullet(\overline{\mathcal{M}}_{g,n})$ in general which do not lie in the image. This means that there will be operations on $\mathcal{H}$ which do not come from applying the Feynman rules to any linear combination of stable graphs. Therefore, there are additional basic operations in a CohFT which are not captured by the Feynman rules in the present form.

Another problem is that the groups $H_\bullet(\overline{\mathcal{M}}_{g,n})$ are unknown in general although in low genera, some facts are known. For example, there is an analog of the WDVV equation but in genus 1 due to Getzler [Ge2].

5. Construction of cohomological field theories

Because of our incomplete knowledge about $H_\bullet(\overline{\mathcal{M}}_{g,n})$, examples of CohFTs (in all genera) appear to be difficult to come by. To my knowledge, there are only two main sources of examples at the moment. One comes from the Gromov-Witten invariants and the second comes from spin CohFTs. However, they share many common features. They both involve moduli spaces of curves with additional data. Forgetting the additional decorations, one obtains the moduli space of stable curves. Construction of a CohFT involves pushing down canonical cohomology classes on these decorated moduli spaces via this forgetful morphism.

5.1. **Gromov-Witten Invariants and Quantum Cohomology.** Let V be a smooth, projective variety. There is an associated CohFT such that $\mathcal{H} = H^\bullet(V)$ and η is the Poincaré pairing. We need only the classes $\Lambda_{g,n}$ to define the CohFT.

Remark 5.1. In order to notational clutter due to signs, let us assume that $\mathcal{H} := H^{2\bullet}(V)$, the even dimensional classes on V.

Definition 5.2. The decorated moduli space in this case is $\overline{\mathcal{M}}_{g,n}(V)$, the *moduli space of stable maps into* V. It consists of tuples $(f : C \to V; p_1, \dots, p_n)$ up to isomorphism where C is a genus g (at worst) nodal curve with pairwise distinct marked points $p_1, \dots, p_n$ on C away from its nodes, and $f : C \to V$ is a morphism. (One can think of f as a holomorphic map.) An isomorphism between

$(f : C \to V; p_1, \dots, p_n)$ to $(f' : C' \to V'; p'_1, \dots, p'_n)$ is an isomorphism of the underlying curves which takes p_i to p'_i for all i and which takes f to f'. The tuple $(f : C \to V; p_1, \dots, p_n)$ is said to be *stable* if it has no infinitesimal automorphisms. If β belongs to $H_2(V;\mathbb{Z})$ then $\overline{\mathcal{M}}_{g,n}(V,\beta)$ consists of those tuples $(f : C \to V; p_1, \dots, p_n)$ in $\overline{\mathcal{M}}_{g,n}(V)$ such that class $[f(C)] = \beta$.

Because of the different definition of stability, $\overline{\mathcal{M}}_{g,n}(V)$ is usually defined even when $2g - 2 + n$ is not positive.

$\overline{\mathcal{M}}_{g,n}(V)$ is a kind of orbifold (a Deligne-Mumford stack). There are *evaluation maps* $ev_i : \overline{\mathcal{M}}_{g,n}(V) \to V$ taking $(f : C \to V; p_1, \dots, p_n) \mapsto f(p_i)$ for all $i = 1, \dots, n$.

There is also a morphism $\mathrm{st} : \overline{\mathcal{M}}_{g,n}(V) \to \overline{\mathcal{M}}_{g,n}$ which "forgets" the stable map f, *i.e.* it forgets f and then collapses any irreducible components of the curve which consequently have become unstable. In particular, when V is a point then st is an isomorphism.

The following theorem follows from the work of many people. Please see [Ma] and the references therein.

Theorem 5.3. *For all $v_1, \dots, v_n$ in $\mathcal{H}$, and g, n, let*

$$\Lambda_{g,n}(v_1, \dots, v_n) := \mathrm{st}_*((\cup_{i=1}^{n} ev_i^* v_i) \cap [\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}), \tag{21}$$

and $\Lambda := \{\Lambda_{g,n}\}$ then $(\mathcal{H}, \eta, \Lambda)$ is a CohFT.

Remark 5.4. To be precise, the ground ring of $\mathcal{H}$ should be enlarged to include the formal parameter q^β where β belongs to $H_2(V;\mathbb{Z})$. We suppress this in order to avoid notational clutter.

Notice that we did not define $[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}$. It is called the *virtual fundamental class of* $\overline{\mathcal{M}}_{g,n}(V)$ and its definition is very technical. However, the idea is that $\overline{\mathcal{M}}_{g,n}(V)$ has an expected dimension which comes from deformation theory. The virtual fundamental class is a homology class of dimension equal to the expected dimension and it plays the role of the fundamental class for the purposes of this construction. There are cases when the virtual fundamental class is the usual fundamental class, *e.g.* when $g = 0$ and V is a projective space or flag variety.

The associated potential function Φ has a genus g, n-ary piece, $\Phi^{(g,n)}$. When $\Phi^{(g,n)}$ is regarded as an element of the symmetric product $S^n(\mathcal{H}^*)$ then it is called *the genus g, n-point Gromov-Witten invariants of V.*

Remark 5.5. The definition of the potential function can be extended to all (g, n) through the formula $\Phi^{(g,n)}(x) := \frac{1}{n!}\int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} ev_1^* x \cup \dots \cup ev_n^* x$ for all x in $\mathcal{H}$.

In particular, if we restrict to only genus 0 then the above boils down to a potential function Φ_0 satisfying the WDVV equation. If we restrict to only to the cubic terms in Φ_0 then this contains the information about the Frobenius algebra structure. $(\mathcal{H}, \eta)$ together with the multiplication (called the *quantum multiplication*) of this Frobenius algebra structure is called the *quantum cohomology of V*. It is called the quantum cohomology of V because the cubic terms in Φ_0 are integrals over $\overline{\mathcal{M}}_{0,3}(V) = \sqcup_\beta \overline{\mathcal{M}}_{0,3}(V,\beta)$ but if we ignore all contributions except for those from $\overline{\mathcal{M}}_{0,3}(V, 0)$ then the resulting Frobenius algebra $(\mathcal{H}, \eta)$ has a multiplication which reduces to the usual cup product. Therefore, the quantum multiplication is a deformation of the usual cup product and one regards the usual cup product as a kind of classical multiplication.

5.2. **Spin Cohomological Field Theories.** In this section, we will review the construction of spin CohFTs. There is a spin CohFT associated to every integer $r \geq 2$.

The role of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(V)$ is played by the *moduli space of r-spin curves* $\overline{\mathcal{M}}^{1/r}_{g,n}$ where $r \geq 2$ is an integer. The moduli space $\overline{\mathcal{M}}^{1/r}_{g,n} = \sqcup_{\mathbf{m}} \overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{m})$ where the disjoint union is over $\mathbf{m} := (m_1, \dots, m_n)$, $m_i = 0, \dots r-1$, and $i = 1, \dots, n$.

We will now give a quick overview of r-spin curves. We refer the interested reader to [JKV] for details. For now, let us assume that r is prime where things are simpler.

Let $(C; p_1, \dots, p_n)$ be a smooth stable curve of genus g with n marked points. Let $\mathbf{m} = (m_1, \dots, m_n)$ where $m_i = 0, \dots, r-1$ for all i. Let $\omega(-\sum_{i=1}^n m_i p_i) \to C$ denote the holomorphic line bundle $\omega(-\sum_{i=1}^n m_i p_i)$ where $\omega(-\sum_{i=1}^n m_i p_i) := \omega \otimes \mathcal{O}(-\sum_{i=1}^n m_i p_i)$, $\mathcal{O}$ is the structure sheaf, and ω is the dualizing sheaf. (In this smooth case, one can think of ω as the holomorphic cotangent bundle of C.) An *r-spin structure on $(C; p_1, \dots, p_n)$ of type $\mathbf{m}$* consists of a line bundle $\mathcal{L} \to C$ together with an isomorphism $b : \mathcal{L}^{\otimes r} \to \omega(-\sum_{i=1}^n m_i p_i)$. In other words, $\mathcal{L}$ is an r-th root of $\omega(-\sum_{i=1}^n m_i p_i)$. For reasons of degree, an r-spin structure $\mathcal{L}$ will exist if and only if $2g-2-\sum_{i=1}^n m_i$ is divisible by r. When this condition is fulfilled, there are r^{2g} choices. Let $\mathcal{M}^{1/r}_{g,n}(\mathbf{m})$ denote the moduli space of smooth r-spin structures of type $\mathbf{m}$. It will be empty unless the degree condition is satisfied.

Since the $r = 2$ case may be thought of as a kind of holomorphic spin structure, we sometimes call $(\mathcal{L} \to C; p_1, \dots, p_n; b)$ a *higher spin structure or a higher spin curve.*

Jarvis [Ja] introduced a smooth compactification $\overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{m})$ of $\mathcal{M}^{1/r}_{g,n}(\mathbf{m})$. When a smooth curve C degenerates to acquire a node then the r-spin structure $\mathcal{L} \to C$ can either stay locally free at the node (called a *Ramond node*) or it can fail to be locally free at the node (called a *Neveu-Schwarz node*).[§] The morphism b fails to be an isomorphism at the node in the latter case, however, it will be an isomorphism wherever $\mathcal{L}$ is locally free. $\overline{\mathcal{M}}^{1/r}_{g,n}$ is a smooth, compact, complex orbifold of complex dimension $3g-3+n$. The forgetful morphism $\mathrm{st} : \overline{\mathcal{M}}^{1/r}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ which forgets the r-spin structure is a branched cover.

When r is not prime then one must include not only an r-th root $\mathcal{L}$ but also a d-th root for all d which divides r together with a collection of isomorphisms generalizing b. These extra roots are necessary for smoothness of the moduli spaces but will otherwise not play much of a role in our discussion.

Let $\mathcal{H}'$ be a vector space with a basis $\{e_0, \dots, e_{r-1}\}$. Let η' be a metric on $\mathcal{H}'$ such that $\eta'(e_a, e_b) = 1$ if $a + b \equiv (r-2) \mod r$. An important subspace will be $\mathcal{H}$ which is spanned by $\{e_0, \dots, e_{r-2}\}$ and let η be the restriction of η' to $\mathcal{H}$.

In [JKV], axioms are stated for a cohomology class (called the *virtual class*) $c^{1/r}$ in $H^{2D}(\overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{m}))$ where

$$D = \frac{1}{r}((r-2)(g-1) + \sum_{i=1}^n m_i) \tag{22}$$

[§]The physical terminology associated to the nodes is due to Witten.[Wi] such that it gives rise to a CohFT. The number D appears in the following manner. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}^{1/r}_{g,n}$ be the universal curve and let $\mathcal{L} \to \mathcal{C}$ be the universal r-spin structure. Let $R^\bullet\pi_*(\mathcal{L}) \to \overline{\mathcal{M}}^{1/r}_{g,n}$ be its K-theoretic pushforward. Recall that for $i = 0, 1$, the fiber of $R^i\pi_*(\mathcal{L})$ over a point in $\overline{\mathcal{M}}^{1/r}_{g,n}$ consisting of a curve C is $H^i(C;\mathcal{L})$. The Euler characteristic $\dim_{\mathbb{C}} R^0\pi_*(\mathcal{L}) - \dim_{\mathbb{C}} R^1\pi_*(\mathcal{L}) =: -D$. When $g = 0$, $R^0\pi_*(\mathcal{L})$ vanishes for degree reasons. Therefore, $R^1\pi_*(\mathcal{L})^* \to C$ is a vector bundle of rank D. The class $c^{1/r}$ in $H^{2D}(\overline{\mathcal{M}}^{1/r}_{0,n}(\mathbf{m}))$ is then its top Chern class.

Construction of the virtual class in higher genera has been carried out in [PoVa] but it is rather involved since it is not the top Chern class of the index bundle.

Theorem 5.6. [JKV] *Let $(\mathcal{H}', \eta')$ and $(\mathcal{H}, \eta)$ be defined as above. Let $c^{1/r}(\mathbf{m})$ denote the virtual class on $\overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{m})$, let $\Lambda'_{g,n}$ in $H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes (\mathcal{H}')^{*\otimes n}$ be defined via*

$$\Lambda'_{g,n}(e_{m_1}, \dots, e_{m_n}) := r^{1-g}\mathrm{st}_* c^{1/r}(\mathbf{m})$$

for all $m_1, \dots, m_n = 0, \dots, r-1$, and $\Lambda' := \{\Lambda'_{g,n}\}$. Let $\Lambda_{g,n}$ in $H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{H}^{*\otimes n}$ be given by the restriction of $\Lambda'_{g,n}$ to $\mathcal{H}$ and $\Lambda := \{\Lambda_{g,n}\}$ then $(\mathcal{H}, \eta, \Lambda)$ and $(\mathcal{H}', \eta', \Lambda')$ are CohFTs. Furthermore,

$$\Lambda'_{g,n}(e_{m_1}, \dots, e_{m_n}) = 0 \tag{23}$$

if $m_i = r - 1$ for some i.

In particular, the CohFT associated to $r = 2$ is isomorphic to the the CohFT associated to the Gromov-Witten invariants of a point.

Remark 5.7. The reason that we make a distinction between the two CohFTs above is that e_0 plays the role of an identity element, which we have been suppressing, of the Frobenius algebra associated to the CohFT $(\mathcal{H}, \eta, \Lambda)$. This is analogous to the role in Gromov-Witten theory played by the usual identity element in $H^0(V)$. However, because of Equation (23), e_0 cannot be the identity element in larger space $(\mathcal{H}', \eta')$. In fact, the one dimensional vector space spanned by e_{r-1} decouples from the rest of the theory. There does not appear to be a counterpart to this phenomenon in Gromov-Witten theory.

Following Witten's argument, the $g = 0$ potential function can be calculated [JKV] using WDVV and an identity between divisor classes on $\overline{\mathcal{M}}^{1/r}_{0,n}$.

In the previous, we have always restricted the m_i's to lie in the range $0, \dots, r-1$. It is reasonable to ask what happens if we allow arbitrary positive values instead. Let δ_i be an n-tuple with a 1 in the i-th slot and 0's everywhere else. It is easy to see that $\overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{m})$ is canonically isomorphic to $\overline{\mathcal{M}}^{1/r}_{g,n}(\mathbf{m} + r\delta_i)$ for all $\mathbf{m}$ with nonnegative components by taking the tensor product of the higher spin structure with $\mathcal{O}(-p_i)$. However, the class $c^{1/r}(\mathbf{m} + r\delta_i)$ has a different dimension from $c^{1/r}(\mathbf{m})$. It turns out that these classes satisfy [JKV2] the equation

$$c^{1/r}(\mathbf{m} + r\delta_i) = -\frac{m_i + 1}{r}\,\psi_i\, c^{1/r}(\mathbf{m})$$

for all $i = 1, \dots, n$ where ψ_i in $H^2(\overline{\mathcal{M}}^{1/r}_{g,n})$ is the first Chern class of the tautological line bundle associated to the i-th marked point. This complex line bundle over $\overline{\mathcal{M}}_{g,n}$ has a fiber over $(C; p_1, \dots, p_n)$ which is the complex tangent bundle $T^*_{x_i}C$. The classes ψ_i are called *gravitational descendants*.

If one computes the potential function but where we now allow all nonnegative values of m's then one obtains integrals of products of these ψ classes with the virtual class $c^{1/r}$. The resulting potential function Ψ is called the *large phase space potential*. Witten had a very interesting conjecture about this coming from physics.

Conjecture 5.8 (The Generalized Witten Conjecture). *The large phase space potential function associated to the integer $r \geq 2$, Ψ, is the unique solution of the r-th KdV integrable hierarchy satisfying one more equation called the string equation.*

When $r = 2$, this conjecture can be shown to reduce to the original Witten conjecture which was proven by Kontsevich [Ko] using transcendental techniques and, more recently, by Okounkov-Pandharipande [OkPa] using algebraic techniques.

Theorem 5.9. [JKV] *The generalized Witten conjecture holds in genus 0.*

The above follows from a calculation of the genus 0 potential function. While some progress has been made towards this conjecture in low genera, it is still open.

5.3. **Some generalizations.** Here, we mention a few generalizations to the above. The construction of Gromov-Witten invariants has been extended to the case where the target V is a smooth orbifold by Chen-Ruan [CR1, CR2]. An algebraic version of this theory appears in [AGV]. In this theory, the moduli space of stable maps must be enlarged to include curves with an orbifold structure at its marked points and nodes. These curves are responsible for what physicists call the *twisted sectors of the theory*. Tantalizingly enough, the notion of r-spin curves admits a clean description in terms of bundles over curves with orbifold structure [AbJa]. There are thus many fascinating connections between the two theories.

The notion of CohFT itself has recently been generalized to a G-equivariant CohFT (or G-CohFT) when G is a finite group [JKK]. This has the property that when G is the trivial group then one recovers the usual notion of a CohFT. Furthermore, a G-CohFT is related to a G-equivariant version of a Frobenius algebra in precisely the same way that a CohFT is related to a Frobenius algebra. A G-equivariant Frobenius algebra, on the other hand, has been given an equivariant topological field theory interpretation by [Tu]. The relevant moduli space necessary to define a G-CohFT is called the moduli spaces of pointed, admissible G-covers over a genus g curve with n marked points $\overline{\mathcal{M}}^G_{g,n}$. A pointed admissible G-cover over a stable curve is a generalization of a principal G-bundle over a curve but where the group action on the total space may fail to be free over the marked points and nodes. The associated moduli space $\overline{\mathcal{M}}^G_{g,n}$ forms a special kind of partial operad called a colored operad. A G-equivariant CohFT is defined in terms of $\overline{\mathcal{M}}^G_{g,n}$ in a way analogous to the way that a usual CohFT is defined in terms of $\overline{\mathcal{M}}_{g,n}$ except that G-equivariance is maintained throughout the construction. Furthermore, we prove that after taking an appropriate "quotient" by G, one obtains a (usual) CohFT but with an additional grading by conjugacy classes of G corresponding to the twisted sectors. An example of a G-equivariant CohFT can be constructed from a smooth, projective variety X with a finite group action by introducing the appropriate G-equivariant version of stable maps into X. Restricting to only degree zero stable maps, we recover a G-equivariant theory due to [FG]. Taking the appropriate "quotient" by G, we recover the Chen-Ruan orbifold cohomology of the quotient $[X/G]$.

References

AbJa. *Moduli of twisted spin curves*, Proc. Amer. Math. Soc. 131 (2003), no. 3, 685–699

Ar. Arnold, V.I., *The cohomology ring of the colored braid group.* Mat. Zametki **5**, 227–231 (1969) (English translation: Math. Notes **5**, 138-140 (1969))

AGV. D. Abramovich, T. Graber, and A. Vistoli, *Algebraic orbifold quantum products.* In A. Adem, J. Morava, and Y. Ruan (eds.), Orbifolds in Mathematics and Physics, *Contemp. Math.*, Amer. Math. Soc., Providence, RI. **310**, (2002), 1–25. `math.AG`/0112004.

BaKi. B. Bakalov and A. Kirillov, *Lectures on tensor categories and modular functors.* Univ. Lecture Series **21**, Amer. Math. Soc., Providence, RI, 2001.

Co. F. R. Cohen, *Artin's braid groups, classical homotopy theory and sundry other curiosities.* Contemp. Math. **78**, 167–206 (1988)

CR1. W. Chen and Y. Ruan, *A new cohomology theory for orbifold.* `math.AG/0004129`.

CR2. ———, *Orbifold Gromov-Witten theory.* In A. Adem, J. Morava, and Y. Ruan (eds.), Orbifolds in Mathematics and Physics, *Contemp. Math.*, Amer. Math. Soc., Providence, RI. **310**, (2002), 25–85. `math.AG/0103156`.

Du. B. Dubrovin, *Geometry of 2D topological field theories,* in *Integrable systems and Quantum Groups,* Lecture Notes in Math. 1620, Springer, Berlin, 1996, pp. 120–348.

FG. B. Fantechi and L. Göttsche, *Orbifold cohomology for global quotients.* `math.AG/0104207`.

Ge. E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories, Commun. Math. Phys.* **159** (1994) 265–285.

Ge2. *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. **10** (1997), no. 4, 973–998.

GeKa. E. Getzler, M. Kapranov, *Cyclic operads and cyclic homology,* in *Geometry, topology, & physics,* Conf. Proc. Lecture Notes Geom. Topology, VI, Internat. Press, Cambridge, MA, 1995, pp. 167–201.

GeKa2. E. Getzler, M. Kapranov, *Modular operads, Compositio Math.* 110 (1998) 65–126.

Hu. Y. Z. Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, 148. Birkhaüser Boston, Inc., Boston, MA, 1997. xiv+280 pp. ISBN: 0-8176-3829-6

Ja. T. Jarvis, *Geometry of the moduli of higher spin curves*, Internat. J. Math. **11** (2000), no. 5, 637–663, `math.AG`/9809138.

Ja2. T. Jarvis, *Torsion-free sheaves and moduli of generalized spin curves,* Compositio Math. **110** (1998) no. 3, 291-333.

JKK. T. Jarvis, R. Kaufmann, and T. Kimura, *Pointed Admissible G-Covers and G-equivariant Cohomological Field Theories*, `math.AG`/0302316

JKV. T. Jarvis, T. Kimura, and A. Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies*, Compositio Math. **126** (2001), no. 2, 157–212. `math.AG`/9905034.

JKV2. T. Jarvis, T. Kimura, and A. Vaintrob, *Gravitational descendants and the moduli space of higher spin curves*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), 167–177, Contemp. Math., 276, Amer. Math. Soc., Providence, RI, 2001.

KSV. T. Kimura, J. Stasheff, and A. A. Voronov, *On operad strucures on moduli spaces and string theory, Comm. Math. Phys.*171 (1995), 1–25.

Ko. Kontsevich, M., *Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys.* 147 (1992) 1–23.

KoMa. M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry. Commun. Math. Phys.* **164** (1994), 525–562.

LZ. B. Lian, and G. J. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory, Comm. Math. Phys.* **154** (1993), 613 – 646.

Ma. Yu. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces.* Colloquium Publ., **47**, Amer. Math. Soc., Providence, RI, 1999.

MSS. M. Markl, S. Shnider, J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002. x+349 pp. ISBN: 0-8218-2134-2

OkPa. A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and matrix models, I*, `math.AG`/0101147

PoVa. A. Polishchuk and A. Vaintrob, *Algebraic construction of Witten's top Chern class.* In E. Previato (ed.), Contemp. Math. **276**, Amer. Math. Soc., (2001), 229–249. `math.AG`/0011032.

Se. G. Segal, *The definition of a conformal field theory*, Preprint. Oxford.

Se2. G. Segal, *Two dimensional conformal field theories and modular functors,* IXth Int. Congr. on Math. Phys. (Bristol; Philadephia) (B. Simon, A. Gruman, and I. M. Davies, eds.), IOP Publishing Ltd, 1989, 22–37.

Tu. V. Turaev, *Homotopy field theory in dimension 2 and group-algebras.* `math.QA`/99100110.

Wi. E. Witten, *Algebraic geometry associated with matrix models of two dimensional gravity*, Topological models in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX 1993, 235-269.

Zw. B. Zweibach, *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation.* Nucl. Phys. B **390** (1993), 33–152

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