

# A Construction of Rigid Analytic Cohomology Classes for Congruence Subgroups of $SL_3(\mathbb{Z})$

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*Abstract.* We give a constructive proof, in the special case of  $GL_3$ , of a theorem of Ash and Stevens which compares overconvergent cohomology to classical cohomology. Namely, we show that every ordinary classical Hecke-eigenclass can be lifted uniquely to a rigid analytic eigenclass. Our basic method builds on the ideas of M. Greenberg; we first form an arbitrary lift of the classical eigenclass to a distribution-valued cochain. Then, by appropriately iterating the  $U_p$ -operator, we produce a cocycle whose image in cohomology is the desired eigenclass. The constructive nature of this proof makes it possible to perform computer computations to approximate these interesting overconvergent eigenclasses.

## 1 Introduction

Let  $k \geq 0$  be an integer and let  $\Gamma \subseteq SL_2(\mathbb{Z})$  be a congruence subgroup. By Eichler–Shimura theory, the cohomology group  $H^1(\Gamma, \text{Sym}^k(\mathbb{C}))$ , considered as a Hecke-module, contains the space of cuspforms of level  $\Gamma$  and weight  $k + 2$ . In [12], Stevens studied a much larger cohomology group, one with coefficients in a space of  $p$ -adic distributions  $\mathbf{D}_k$  equipped with a weight  $k$  action of the Iwahori subgroup  $\Gamma_0(p) \subseteq SL_2(\mathbb{Z})$ . The space  $\mathbf{D}_k$  admits a  $\Gamma_0(p)$ -equivariant map to  $\text{Sym}^k(\mathbb{Q}_p^2)$  and thus, if  $\Gamma \subseteq \Gamma_0(p)$ , we have a Hecke-equivariant map

$$H^1(\Gamma, \mathbf{D}_k) \xrightarrow{\rho_k} H^1(\Gamma, \text{Sym}^k(\mathbb{Q}_p^2))$$

on cohomology. While the target of  $\rho_k$  encodes information about classical modular forms, the source contains information about overconvergent modular forms of weight  $k + 2$ .<sup>1</sup> Moreover, in [12], Stevens proved that if one restricts  $\rho_k$  to the subspace where  $U_p$  acts with non-critical slope, the above map becomes an isomorphism. (This should be viewed as the analogue of Coleman’s theorem on small slope overconvergent forms being classical.)

Ash and Stevens in [5, 6] generalized the above comparison theorem to representations of  $GL_n(\mathbb{Q})$  (even of arbitrary  $\mathbb{Q}_p$ -split reductive groups). If  $\lambda$  is a character of a torus of  $GL_n$ , let  $V_\lambda$  denote the  $\mathbb{Q}_p$ -representation with highest weight  $\lambda$ . One then replaces  $\mathbf{D}_k$  with a space of  $p$ -adic distributions  $\mathbf{D}_\lambda$  endowed with a weight  $\lambda$  action.

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<sup>1</sup>G. Stevens, unpublished notes.

As before, this space maps equivariantly to  $V_\lambda$ , and Ash and Stevens proved that the corresponding map

$$H^r(\Gamma, \mathbf{D}_\lambda) \xrightarrow{\rho_\lambda} H^r(\Gamma, V_\lambda)$$

is an isomorphism if one restricts to a subspace where the slope of  $U_p$  is small enough (see [6]).

A more explicit study of the case of  $\mathrm{GL}_2(\mathbb{Q})$  was made in [10] via modular symbols. As a consequence of the above comparison theorem, any non-critical classical Hecke-eigensymbol lifts to a unique  $\mathbf{D}_k$ -valued Hecke-eigensymbol. In [10], the following constructive proof of this fact is given: form an arbitrary lift of the classical eigensymbol to a  $\mathbf{D}_k$ -valued modular symbol (but not necessarily to an eigensymbol). Explicit formulae for such lifts are given. Then iterate the  $U_p$ -operator to obtain a sequence that converges to the sought after  $\mathbf{D}_k$ -valued Hecke-eigensymbol.

One may hope then to use the methods of [10] to explicitly lift classical Hecke-eigenclasses for  $\mathrm{GL}_n(\mathbb{Q})$  with  $n > 2$ . One daunting part of such a task is generalizing the first step of lifting a  $V_\lambda$ -valued cohomology class to a  $\mathbf{D}_\lambda$ -valued cohomology class. For  $\mathrm{GL}_2(\mathbb{Q})$ , this was done by explicitly “solving the Manin relations” which required writing down an explicit fundamental domain for the action of a congruence subgroup on the upper-half plane. To repeat these arguments for  $\mathrm{GL}_n(\mathbb{Q})$  with  $n > 2$  would involve examining the geometry of certain higher dimensional symmetric spaces, which appears to be a difficult task.

However, in the case of  $\mathrm{GL}_2(\mathbb{Q})$ , M. Greenberg [9] simplified the arguments of [10] and managed to form liftings in a “geometry-free” manner. His basic idea is to lift modular symbols into a larger ambient space. This larger space is big enough that forming such lifts is trivial. He then uses the  $U_p$ -operator to force such lifts back into the space of interest. Iterating this process leads to a sequence of modular symbols that converges to the true Hecke-eigensymbol. These ideas were used by Trifković in [13] to compute lifts of eigenclasses corresponding to automorphic forms for  $\mathrm{GL}_2(K)$  with  $K/\mathbb{Q}$  an imaginary quadratic field; this again is a situation where lifting classes directly is made difficult by the complicated geometry that is present.

In this paper, we generalize these constructions to the cohomology of  $\mathrm{GL}_3(\mathbb{Q})$ . As a rich theory of  $p$ -adic automorphic forms for higher rank groups is beginning to emerge, we note that there are very few groups simple enough for which computations and numerical exploration are feasible. Along with  $\mathrm{Sp}_4(\mathbb{Q})$  and  $\mathrm{GL}_2(K)$  for  $K/\mathbb{Q}$  imaginary quadratic,  $\mathrm{GL}_3(\mathbb{Q})$  is a natural next step in complexity beyond  $\mathrm{GL}_2(\mathbb{Q})$ . It is a complicated enough group so that many of the new higher rank phenomena are observable in its theory, but well-enough understood that computational techniques exist for studying its  $V_\lambda$ -valued cohomology.

To carry out M. Greenberg’s lifting idea in the context of  $\mathrm{GL}_3$ -cohomology, we first axiomatize the situation as follows. Let  $R$  be a commutative ring and let  $\Gamma \subseteq G$  be abstract groups. Let  $\pi \in G$  be such that  $\Gamma$  and  $\pi^{-1}\Gamma\pi$  are commensurable. Set  $S$  equal to the semigroup generated by  $\Gamma$  and  $\pi$ . Then the double coset  $\Gamma\pi\Gamma$  induces an operator  $U$  on  $H^r(\Gamma, M)$  for any right  $R[S]$ -module  $M$ . Consider a surjective map  $D \rightarrow V$  of  $R[S]$ -modules and the induced map on cohomology

$$H^r(\Gamma, D) \rightarrow H^r(\Gamma, V).$$

Let  $\varphi$  be a  $U$ -eigenclass in  $H^r(\Gamma, V)$  whose  $U$ -eigenvalue is a unit in  $R/\text{Ann}_R(\varphi)$ . We prove the existence of a unique  $U$ -eigenclass  $\Phi$  in  $H^r(\Gamma, D)$  lifting  $\varphi$  under the assumption that  $D$  has a decreasing  $R[S]$ -stable filtration  $\text{Fil}^\bullet D$  such that:

- $D/\text{Fil}^0 D = V$ ,
- $\text{Fil}^n D \cdot \pi \subseteq \text{Fil}^{n+1} D$ ,
- the topology on  $D$  induced by  $\text{Fil}^\bullet D$  is separated and complete.

(This result corresponds to Theorem 3.1.)

For  $\text{GL}_2(\mathbb{Q})$ , a filtration on  $\mathbf{D}_k$  that satisfies the above conditions is given in [10] and it is precisely these properties that are used in [9] to produce explicit lifts of modular symbols. In this paper, we construct for every dominant weight  $\lambda$  of  $\text{GL}_3(\mathbb{Q})$  a filtration on  $\mathbf{D}_\lambda$  that satisfies the above axioms. In particular, we obtain another proof of the theorem of Ash and Stevens on lifting ordinary eigenclasses.

The proof of this general lifting theorem follows the methods of [9]. Indeed, the basic idea is to lift a  $V$ -valued cocycle representing  $\varphi$  to a  $D$ -valued cochain. There is no reason why such a lift should again be a cocycle. However, as  $V = D/\text{Fil}^0 D$ , this cochain will be a cocycle mod  $\text{Fil}^0 D$ . Applying the  $U$ -operator and dividing by the  $U$ -eigenvalue of  $\varphi$  forms a new cochain that still lifts  $\varphi$ , but is now a cocycle modulo  $\text{Fil}^1 D$ . This increase in accuracy is a consequence of the second assumption on the filtration. Iterating this process then leads to a sequence of cochains that converges to a cocycle whose image in cohomology is the desired Hecke-eigenlift.

For  $\Gamma \subseteq \text{SL}_3(\mathbb{Z})$ , the work of Ash and others [1, 4] gives a description of  $H^3(\Gamma, M)$  in terms of  $\text{GL}_3$ -modular symbols. These spaces are computable using the methods of [3, 4], and by iterating the  $U_p$ -operator one can actually compute approximations to  $\mathbf{D}_\lambda$ -valued lifts of Hecke-eigenclasses in  $H^3(\Gamma, V_\lambda)$ . We carried out such computations for some boundary classes of small level and trivial weight  $V_0$ , and indeed obtained sequences of improving approximations to  $\mathbf{D}_0$ -valued Hecke-eigenlifts of these classes. We intend in the near future to compute lifts of classes of trivial weight found in [4] that do not arise from  $\text{GL}_2$ .

Now that the beginnings of a computational theory exist for  $\text{GL}_3(\mathbb{Q})$ , many questions arise. For  $\text{GL}_2(\mathbb{Q})$ , a non-critical Hecke-eigensymbol in  $H^1(\Gamma, \mathbf{D}_k)$  encodes the  $p$ -adic  $L$ -function of the corresponding classical cuspform. Do these  $\mathbf{D}_\lambda$ -valued Hecke-eigenclasses encode some kind of  $p$ -adic  $L$ -function of the corresponding automorphic form? In [11],  $\mathbf{D}_k$ -valued Hecke-eigensymbols are used to attach a  $p$ -adic  $L$ -function to a critical slope modular form. Can critical slope  $\text{GL}_3(\mathbb{Q})$ -forms be studied using these methods? In [8], the algorithms of [10] were used to compute Stark–Heegner points on elliptic curves. Can one hope to use these Hecke-eigenlifts to (conjecturally) construct points on the (conjectural) motive attached to these automorphic forms à la Darmon [7]?

The format of the paper is as follows: in the following section we introduce the distribution spaces  $\mathbf{D}_\lambda$  for  $\text{GL}_3(\mathbb{Q})$ . In the third section, we prove our general lifting result. In the fourth section, we construct a filtration on  $\mathbf{D}_\lambda$  satisfying the properties mentioned above and obtain a lifting result for  $\text{GL}_3(\mathbb{Q})$ .

## 2 $p$ -Adic Distributions

In this section, we recall the notion of Ash and Stevens of  $p$ -adic distribution valued cohomology for  $GL_3(\mathbb{Q})$ . Our description will be fairly concrete, proving many of the basic facts by explicit computations. We refer the reader to [5] and the forthcoming [6] for the case of  $GL_n(\mathbb{Q})$  and for a more systematic treatment.

### 2.1 Notation

Let  $p$  be a positive prime. Let  $\mathbb{C}_p$  denote the completion of a fixed algebraic closure of  $\mathbb{Q}_p$ , and let  $\mathbb{O}_p$  denote its ring of integers.

Let  $G$  denote the algebraic group scheme  $GL_3$ . Let  $B$  (resp.  $B^{\text{opp}}$ ) denote the group of upper (resp. lower) triangular matrices in  $G$ . Let  $N$  (resp.  $N^{\text{opp}}$ ) denote the group of unipotent matrices in  $B$  (resp.  $B^{\text{opp}}$ ). Let  $T$  be the group of diagonal matrices so that  $B = TN$  and  $B^{\text{opp}} = N^{\text{opp}}T$ .

Let  $\mathcal{J}$  denote the Iwahori subgroup of  $G(\mathbb{O}_p)$ , that is, the collection of elements in  $G(\mathbb{O}_p)$  whose reduction modulo the maximal ideal of  $\mathbb{O}_p$  is upper triangular.

Let  $\Gamma_0(p) := \mathcal{J} \cap SL_3(\mathbb{Z})$  denote the  $p$ -Iwahori subgroup of  $SL_3(\mathbb{Z})$ , and fix a congruence subgroup  $\Gamma \subseteq SL_3(\mathbb{Z})$  contained in  $\Gamma_0(p)$ .

### 2.2 Spaces of Distributions

We set  $\mathfrak{X} := \text{im}(\mathcal{J} \rightarrow N^{\text{opp}}(\mathbb{C}_p) \backslash G(\mathbb{C}_p))$ . Since  $\mathcal{J} = (\mathcal{J} \cap N^{\text{opp}}(\mathbb{O}_p))B(\mathbb{O}_p)$ , we have that  $\mathfrak{X}$  is isomorphic to  $B(\mathbb{O}_p)$ .

Let  $\lambda$  denote some algebraic character of the torus  $T$ . Consider the collection of  $\mathbb{O}_p$ -valued functions

$$M_\lambda := \{f: \mathfrak{X} \rightarrow \mathbb{O}_p \mid f(tg) = \lambda(t)f(g) \text{ for } t \in T(\mathbb{O}_p) \text{ and } g \in \mathfrak{X}\}.$$

We wish to consider the subset of these functions that are  $\mathbb{Q}_p$ -rigid analytic. To make this precise, note that  $N(\mathbb{O}_p)$  maps injectively into  $\mathfrak{X}$ . We give  $N(\mathbb{O}_p)$  the structure of a  $\mathbb{Q}_p$ -rigid analytic space by identifying it with the unit polydisc in  $\mathbb{C}_p^3$  via

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathfrak{X} \longleftrightarrow (x, y, z) \in \mathbb{O}_p^3.$$

So explicitly, a function on  $N(\mathbb{O}_p)$  is  $\mathbb{Q}_p$ -rigid analytic if it is of the form

$$f\left(\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\right) = \sum_{i,j,k} c_{ijk} x^i y^j z^k$$

where  $c_{ijk} \rightarrow 0$  as  $i + j + k \rightarrow \infty$ .

We then define

$$A_\lambda := \left\{ f: \mathfrak{X} \rightarrow \mathbb{O}_p \mid \begin{array}{l} f \text{ restricted to } N(\mathbb{O}_p) \text{ is a } \mathbb{Q}_p\text{-rigid function,} \\ f(tg) = \lambda(t)f(g) \text{ for } t \in T(\mathbb{O}_p) \end{array} \right\}.$$

Note that any function in  $\mathbf{A}_\lambda$  is uniquely determined by its restriction to  $N(\mathbb{O}_p)$ .

Under our identification of  $N(\mathbb{O}_p)$  with  $\mathbb{O}_p^3$ , rigid functions on  $N(\mathbb{O}_p)$  correspond to elements of the Tate algebra  $\mathbb{Z}_p\langle X, Y, Z \rangle$ . Let  $f_{abc}$  denote the unique extension to  $\mathbf{A}_\lambda$  of the function that sends

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to} \quad x^a y^b z^c.$$

Under the above identification,  $f_{abc}$  corresponds to the element  $X^a Y^b Z^c$ . Since the  $\mathbb{Z}_p$ -span of these monomials is dense in  $\mathbb{Z}_p\langle X, Y, Z \rangle$ , the span of the  $f_{abc}$  forms a dense subset of  $\mathbf{A}_\lambda$ .

We then set  $\mathbf{D}_\lambda = \text{Hom}_{\text{cont}}(\mathbf{A}_\lambda, \mathbb{Z}_p)$ , the space of continuous  $\mathbb{Z}_p$ -linear functionals of  $\mathbf{A}_\lambda$  into  $\mathbb{Z}_p$ . By the above observations, an element  $\mu \in \mathbf{D}_\lambda$  is uniquely determined by its values on  $f_{abc}$  for all  $a, b, c \geq 0$ .

### 2.3 The Weight $\lambda$ Action

Let  $I = \mathcal{J} \cap \text{GL}_3(\mathbb{Z}_p)$  be the Iwahori subgroup of  $\text{GL}_3(\mathbb{Z}_p)$ . Let  $\pi$  be the diagonal matrix with diagonal entries 1,  $p$  and  $p^2$ , and let  $\Sigma$  be the semigroup generated by  $I$  and  $\pi$ .

Note that  $I$  acts on  $N^{\text{opp}}(\mathbb{O}_p) \backslash G(\mathbb{O}_p)$  by multiplication on the right. We extend this to an action of  $\Sigma$  by letting  $\pi$  act by  $N^{\text{opp}}g \cdot \pi := N^{\text{opp}}\pi^{-1}g\pi$ . This is well defined as  $\pi$  normalizes  $N^{\text{opp}}$ . We note that this action of  $\Sigma$  preserves  $\mathfrak{X} \subseteq N^{\text{opp}}(\mathbb{O}_p) \backslash G(\mathbb{O}_p)$ .

We then get a left action of  $\Sigma$  on  $M_\lambda$  by  $(\gamma f)(g) = f(g \cdot \gamma)$ . The following lemma describes this action explicitly on the functions  $f_{abc}$ . In particular, it will imply that this action induces a left action on  $\mathbf{A}_\lambda$  and thus a right action on  $\mathbf{D}_\lambda$  by  $(\mu | \gamma)(f) = \mu(\gamma f)$ .

Let  $\lambda(k_1, k_2, k_3)$  be the character of the torus that sends  $\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$  to  $d_1^{k_1} d_2^{k_2} d_3^{k_3}$ .

Also, for  $f \in M_\lambda$ , we write  $f\left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) = f(x, y, z)$ .

**Lemma 2.1** *Let  $\lambda = \lambda(k_1, k_2, k_3)$ . For  $\gamma \in I$ , the weight  $\lambda$  action of  $\gamma$  on  $f \in M_\lambda$  is given by:*

$$\begin{aligned} (\gamma f)(x, y, z) &= \det(\gamma)^{k_3} (a_{11} + a_{21}x + a_{31}y)^{k_1 - k_2} (m_{33} - m_{13}y - m_{23}z + m_{13}xz)^{k_2 - k_3} \\ &\quad \times f\left(\frac{a_{12} + a_{22}x + a_{32}y}{a_{11} + a_{21}x + a_{31}y}, \frac{a_{13} + a_{23}x + a_{33}y}{a_{11} + a_{21}x + a_{31}y}, \frac{-m_{32} + m_{12}y + m_{22}z - m_{12}xz}{m_{33} - m_{13}y - m_{23}z + m_{13}xz}\right) \end{aligned}$$

Here  $m_{ij}$  is the  $ij$ -th minor of  $\gamma$ . Also  $(\pi f)(x, y, z) = f(px, p^2y, pz)$ .

**Remark 2.2** In the case of  $\text{GL}_2(\mathbb{Q})$ , consider  $\lambda = \lambda(k_1, k_2)$ , the highest weight of  $\text{Sym}^{k_1 - k_2}(\mathbb{Q}_p^2) \otimes \det^{k_2}$ . Then the corresponding weight  $\lambda$  action is given by

$$(\gamma f)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = (\gamma f)(x) = (\det \gamma)^{k_2} (a + cx)^{k_1 - k_2} f\left(\frac{b + dx}{a + cx}\right).$$

**Proof of Lemma 2.1.** For  $\gamma \in I$ , we have

$$\begin{aligned} & \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + a_{21}x + a_{31}y & a_{12} + a_{22}x + a_{32}y & a_{13} + a_{23}x + a_{33}y \\ a_{21} + a_{31}z & a_{22} + a_{32}z & a_{23} + a_{33}z \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \equiv \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & \frac{b_{22}b_{11} - b_{21}b_{12}}{b_{11}} & \frac{b_{23}b_{11} - b_{21}b_{13}}{b_{11}} \\ 0 & \frac{b_{32}b_{11} - b_{31}b_{12}}{b_{11}} & \frac{b_{33}b_{11} - b_{31}b_{13}}{b_{11}} \end{pmatrix} \\ &\equiv \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & \frac{b_{22}b_{11} - b_{21}b_{12}}{b_{11}} & \frac{b_{23}b_{11} - b_{21}b_{13}}{b_{11}} \\ 0 & 0 & \frac{\det(\gamma)}{b_{22}b_{11} - b_{21}b_{12}} \end{pmatrix} \\ &\equiv \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & \frac{b_{22}b_{11} - b_{21}b_{12}}{b_{11}} & 0 \\ 0 & 0 & \frac{\det(\gamma)}{b_{22}b_{11} - b_{21}b_{12}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{b_{12}}{b_{11}} & \frac{b_{13}}{b_{11}} \\ 0 & 1 & \frac{b_{23}b_{11} - b_{21}b_{13}}{b_{22}b_{11} - b_{21}b_{12}} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The congruences above are taking place in  $\mathfrak{X}$ , i.e., modulo  $N^{\text{opp}}(\mathbb{C}_p)$  (on the left).

Thus,

$$(\gamma f) \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \lambda \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & \frac{b_{22}b_{11} - b_{21}b_{12}}{b_{11}} & 0 \\ 0 & 0 & \frac{\det(\gamma)}{b_{22}b_{11} - b_{21}b_{12}} \end{pmatrix} f \begin{pmatrix} 1 & \frac{b_{12}}{b_{11}} & \frac{b_{13}}{b_{11}} \\ 0 & 1 & \frac{b_{23}b_{11} - b_{21}b_{13}}{b_{22}b_{11} - b_{21}b_{12}} \\ 0 & 0 & 1 \end{pmatrix}$$

A direct computation finds that

$$\begin{aligned} b_{23}b_{11} - b_{21}b_{31} &= -m_{32} + m_{12}y + m_{22}z - m_{12}xz, \text{ and} \\ b_{22}b_{11} - b_{21}b_{12} &= m_{33} - m_{13}y - m_{23}z + m_{13}xz \end{aligned}$$

where  $m_{ij}$  are the minors of  $\gamma$ . Plugging back in establishes the first formula of the lemma.

For  $\pi$ , we have

$$(\pi f) \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = f \left( \pi^{-1} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \pi \right) = f \begin{pmatrix} 1 & px & p^2y \\ 0 & 1 & pz \\ 0 & 0 & 1 \end{pmatrix}$$

as claimed. ■

**Corollary 2.3** *The action of  $\Sigma$  on  $M_\lambda$  preserves  $\mathbf{A}_\lambda$ .*

**Proof** In the formulae of Lemma 2.1, the only possibly troublesome terms are  $(a_{11} + a_{21}x + a_{31}y)^{-1}$  and  $(m_{33} - m_{13}y - m_{23}z + m_{13}xz)^{-1}$ . For  $\gamma \in I$ , one checks that  $a_{11}$  and  $m_{33}$  are units while all other coefficients present are divisible by  $p$ . In particular, the power series expansion of these two functions is again rigid analytic in  $x, y$ , and  $z$ . ■

**2.4 Specialization to Weight  $\lambda$**

This section follows closely [5, Section 4], where the general case of  $GL_n$  is treated. We note that we will be considering *right* representations of  $G = GL_3$  exclusively.

Let  $\lambda$  be an algebraic character of the torus  $T$  which is dominant with respect to the Borel  $B^{\text{opp}}$ , and let  $V_\lambda$  be the finite dimensional right representation of  $G$  with highest weight  $\lambda$  (with respect to  $B^{\text{opp}}$ ). Fix  $v_\lambda \in V_\lambda(\mathbb{Q}_p)$  a highest weight vector; that is,

$$v_\lambda \cdot t = \lambda(t)v_\lambda \text{ for } t \in T(\mathbb{Q}_p), \quad \text{and} \quad v_\lambda \cdot n = v_\lambda \text{ for } n \in N^{\text{opp}}(\mathbb{Q}_p).$$

Thus, the function

$$f_\lambda: G(\mathbb{O}_p) \rightarrow V_\lambda(\mathbb{C}_p) \text{ given by } f_\lambda(g) = v_\lambda \cdot g$$

descends to  $N^{\text{opp}}(\mathbb{O}_p) \backslash G(\mathbb{O}_p)$ , and by restriction gives a function on  $\mathfrak{X}$ .

**Remark 2.4**

1. We now describe the analogous map in the case of  $GL_2$ . Let  $V_k = \text{Sym}^k(\mathbb{Q}_p^2)$  be the space of homogeneous polynomials in  $X_1$  and  $X_2$  of degree  $k$ , and let  $GL_2$  act on  $V_k$  on the right by

$$(P|\gamma)(X_1, X_2) = P(aX_1 + bX_2, cX_1 + dX_2).$$

Then  $V_k$  has highest weight  $\lambda = \lambda(k, 0)$  and highest weight vector  $X_1^k$ . The map  $f_\lambda$  restricted to  $\mathfrak{X}$  is given by  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto X_1^k \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = (X_1 + xX_2)^k$ .

2. For  $G = GL_3$ , let  $\lambda = \lambda(k, 0, 0)$  so that  $V_\lambda = \text{Sym}^k(\mathbb{Q}_p^3)$ . We view  $V_\lambda$  as the space of homogeneous polynomials in  $X_1, X_2$ , and  $X_3$  of degree  $k$  with the analogous right action of  $G$ . Then  $V_\lambda$  has  $X_1^k$  as a highest weight vector and the map  $f_\lambda$  is given by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto X_1^k \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = (X_1 + xX_2 + yX_3)^k.$$

3. More generally, if  $\lambda = \lambda(a, b, 0)$ , then  $V_\lambda$  arises as a subrepresentation of  $\text{Sym}^a(\mathbb{Q}_p^3) \otimes \text{Sym}^b(\mathbb{Q}_p^3)$  and has highest weight vector

$$v_\lambda = \sum_{i=0}^b (-1)^i \binom{b}{i} X_1^{a-i} X_2^i \otimes X_1^i X_2^{b-i}.$$

From this explicit formula for  $v_\lambda$ , one could write down the map  $f_\lambda$  as above.

**Lemma 2.5** *The restriction of  $f_\lambda$  to  $\mathfrak{X}$  is in  $\mathbf{A}_\lambda \otimes V_\lambda(\mathbb{Q}_p)$ .*

**Proof** For  $t \in T(\mathbb{O}_p)$  and  $g \in G(\mathbb{O}_p)$ ,

$$f_\lambda(tg) = v_\lambda \cdot tg = \lambda(t)v_\lambda \cdot g = \lambda(t)f_\lambda(g)$$

since  $v_\lambda$  is a weight vector of weight  $\lambda$ . From this, it follows that  $f_\lambda$  is in  $M_\lambda \otimes V_\lambda(\mathbb{Q}_p)$ .

Further, let  $v_1, v_2, \dots, v_d$  be a basis of  $V_\lambda(\mathbb{Q}_p)$ . Then since  $V_\lambda$  is an algebraic representation of  $\mathrm{GL}_3$ ,

$$v_\lambda \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \sum_i P_i(x, y, z)v_i,$$

where  $P_i(x, y, z) \in \mathbb{Q}_p[x, y, z]$ . Since these coefficients are polynomials, they are in particular  $\mathbb{Q}_p$ -rigid analytic. Hence,  $f_\lambda$  is in  $\mathbf{A}_\lambda \otimes V_\lambda$ . ■

We evaluate a distribution  $\mu \in \mathbf{D}_\lambda$  on functions in  $\mathbf{A}_\lambda \otimes V_\lambda(\mathbb{Q}_p)$  by setting  $\mu(\sum f_i \otimes v_i)$  to be  $\sum \mu(f_i)v_i$  where  $\{v_i\}$  is a basis of  $V$ . It is clear that this definition is independent of the choice of basis.

Evaluation at  $f_\lambda$  then gives a map  $\mathbf{D}_\lambda \rightarrow V_\lambda(\mathbb{Q}_p)$  which we will see below is  $I$ -equivariant. However, this map is not  $\pi$ -equivariant, and for this reason we introduce the  $\star$ -action on  $V_\lambda$  as in [2].

For  $v \in V_\lambda$ , we define

$$v \star \gamma = v \cdot \gamma \text{ for } \gamma \in I \quad \text{and} \quad v \star \pi = \lambda(\pi)^{-1}v \cdot \pi.$$

This action extends uniquely to an action of  $\Sigma$ , and we write  $V_\lambda^\star$  when we view  $V_\lambda$  as a  $\Sigma$ -module under this action.

We now have the following analogue of [5, Lemma 4.1].

**Lemma 2.6** *Evaluating at  $f_\lambda$  gives a  $\Sigma$ -equivariant map  $\rho_\lambda: \mathbf{D}_\lambda \rightarrow V_\lambda^\star(\mathbb{Q}_p)$ .*

**Proof** Note that for any  $\gamma \in I$ ,

$$f_\lambda(x\gamma) = v_\lambda \cdot (x\gamma) = (v_\lambda \cdot x) \cdot \gamma = f_\lambda(x) \cdot \gamma.$$

Recall that  $\pi$  acts on  $\mathfrak{X}$  by  $x \cdot \pi = \pi^{-1}x\pi$ . Thus,

$$f_\lambda(x \cdot \pi) = f_\lambda(\pi^{-1}x\pi) = v_\lambda \cdot (\pi^{-1}x\pi) = \lambda(\pi^{-1})f_\lambda(x) \cdot \pi = f_\lambda(x) \star \pi.$$

We have thus proven that for all  $\gamma \in \Sigma$ ,  $\gamma f_\lambda = e_\gamma \circ f_\lambda$ , where  $e_\gamma: V_\lambda \rightarrow V_\lambda$  is the linear map  $e_\gamma(v) = v \star \gamma$ .

Now, if  $L: V_\lambda(\mathbb{Q}_p) \rightarrow V_\lambda(\mathbb{Q}_p)$  is a linear map, one sees immediately that  $\mu(L \circ f) = L(\mu(f))$  for  $f$  in  $\mathbf{A}_\lambda \otimes V_\lambda(\mathbb{Q}_p)$ . Thus,

$$(\mu|_\gamma)(f_\lambda) = \mu(\gamma f_\lambda) = \mu(e_\gamma \circ f_\lambda) = e_\gamma(\mu(f_\lambda)) = \mu(f_\lambda) \star \gamma$$

which proves the  $\Sigma$ -equivariance of  $\rho_\lambda$ . ■

**Remark 2.7** We continue with the second example of Remark 2.4, namely  $\lambda = \lambda(k, 0, 0)$ . The map  $\rho_\lambda$  is then given by

$$\begin{aligned} \mu \mapsto \mu((X_1 + xX_2 + yX_3)^k) &= \mu\left(\sum_{r+s+t=k} \binom{k}{r, s, t} x^s y^t X_1^r X_2^s X_3^t\right) \\ &= \sum_{r+s+t=k} \binom{k}{r, s, t} \mu(f_{st0}) X_1^r X_2^s X_3^t. \end{aligned}$$



Set  $L_\lambda$  equal to the image of  $\mathbf{D}_\lambda$  under  $\rho_\lambda$ ; then  $L_\lambda$  is a  $\Sigma$ -stable  $\mathbb{Z}_p$ -lattice of  $V_\lambda^*(\mathbb{Q}_p)$ . Recall that  $\Gamma \subseteq \Gamma_0(p) = \mathcal{J} \cap \mathrm{SL}_3(\mathbb{Z})$  is a congruence subgroup. The map  $\rho_\lambda$  then induces a map on cohomology  $\rho_\lambda^r: H^r(\Gamma, \mathbf{D}_\lambda) \rightarrow H^r(\Gamma, L_\lambda)$ , which we refer to as specialization to weight  $\lambda$ .

These cohomology groups carry a natural action of Hecke operators  $U_p$  and  $T(\ell, k)$  for  $\ell \neq p$  prime and  $k = 1, 2, 3$  (see, for instance, [2]). We will primarily be interested in the operator  $U_p$ , which is associated to the diagonal matrix  $\pi$  whose diagonal entries are 1,  $p$ , and  $p^2$ . This operator will correspond to the operator  $U$  carefully defined in the following section. We point out that since  $\rho_\lambda$  is  $\Sigma$ -equivariant,  $\rho_\lambda^r$  is automatically a Hecke-equivariant map.

The following theorem of Ash and Stevens analyzes the specialization map restricted to the  $U_p$ -ordinary subspace. (See [6], which also treats the non-ordinary case; see [5] and [2] for analogous results.)

**Theorem 2.8** *The natural map  $\rho_\lambda^r: H^r(\Gamma, \mathbf{D}_\lambda)^{\mathrm{ord}} \xrightarrow{\sim} H^r(\Gamma, L_\lambda)^{\mathrm{ord}}$  is an isomorphism. Here  $M^{\mathrm{ord}}$  denotes the direct sum of all generalized  $U_p$ -eigenspaces whose  $U_p$ -eigenvalue is a  $p$ -adic unit.*

As a consequence of this theorem, any  $U_p$ -ordinary Hecke-eigensymbol in  $H^r(\Gamma, L_\lambda)$  lifts uniquely to a Hecke-eigensymbol in  $H^r(\Gamma, \mathbf{D}_\lambda)$ . In the following sections, we will prove this fact in a constructive manner analogous to Greenberg’s work in [9].

### 3 Lifting Cohomology Classes

In this section, we present a general lemma on lifting cohomology classes. The notation of this section is meant to mirror that of the previous section with the aim of making transparent the intended application to the case of interest.

Let  $\Gamma \subseteq G$  be two groups, let  $\pi$  be some element in  $G$ , and let  $S$  be the semi-group generated by  $\Gamma$  and  $\pi$ . Let  $R$  be a commutative ring and let  $D$  be any right  $R[S]$ -module. For  $\gamma \in S$  and  $\mu \in D$ , we write the action of  $S$  on  $D$  by  $\mu \cdot \gamma$ .

If we assume that  $\Gamma$  and  $\pi^{-1}\Gamma\pi$  are commensurable, there is an operator  $U = U(\pi)$  on  $H^i(\Gamma, D)$  defined as follows. Let  $D^\pi$  denote the  $\pi^{-1}\Gamma\pi$ -module whose underlying set is just  $D$  and whose action by  $s \in \pi^{-1}\Gamma\pi$  is given by  $\mu \cdot_\pi s = \mu \cdot \pi s \pi^{-1}$ . Acting by  $\pi$  gives a map  $H^i(\Gamma, D) \xrightarrow{\pi} H^i(\pi^{-1}\Gamma\pi, D^\pi)$  and restriction and transfer yield maps

$$H^i(\pi^{-1}\Gamma\pi, D^\pi) \xrightarrow{res} H^i(\Delta, D^\pi) \xrightarrow{tr} H^i(\Gamma, D)$$

where  $\Delta = \Gamma \cap \pi^{-1}\Gamma\pi$ . We define  $U$  as the composition of these three maps.

For  $x$  in an  $R$ -module  $M$ , we set  $\mathrm{Ann}_R(x)$  equal to the ideal of elements of  $R$  that annihilate  $x$ . If  $M$  is a right  $R$ -module and  $T: M \rightarrow M$  is a linear map, we call a non-zero element  $x \in M$  an eigenvector for  $T$  with eigenvalue  $\alpha \in R$ , if  $x|T = \alpha x$ . Note that  $\alpha$  is only determined modulo  $\mathrm{Ann}_R(x)$ . We say that  $x$  is ordinary for  $T$  if the image of  $\alpha$  is a unit in  $R/\mathrm{Ann}_R(x)$ .

The following theorem is the main result of the section.

**Theorem 3.1** *Let  $D$  be a right  $R[S]$ -module with a decreasing  $R[S]$ -filtration  $\mathrm{Fil}^\bullet D$  such that*

- (1)  $\text{Fil}^n D \cdot \pi \subseteq \text{Fil}^{n+1} D$  for each  $n \geq 0$ ,
- (2) the natural map  $D \rightarrow \varprojlim D/\text{Fil}^n D$  is an isomorphism.

Let  $\varphi_0$  in  $H^r(\Gamma, D/\text{Fil}^0 D)$  be an ordinary eigenvector for  $U$  with eigenvalue  $\alpha$ . Then there exists  $\Phi \in H^r(\Gamma, D)$  such that

- (1) the image of  $\Phi$  in  $H^r(\Gamma, D/\text{Fil}^0 D)$  equals  $\varphi_0$ ,
- (2)  $\Phi$  is an eigenvector for  $U$  with eigenvalue  $\alpha$ ,
- (3)  $\text{Ann}_R(\Phi) = \text{Ann}_R(\varphi_0)$ .

Moreover, if  $\Phi'$  is any ordinary  $U$ -eigenlift of  $\varphi_0$ , then  $\Phi' = \Phi$ .

The proof of this theorem will occupy the remainder of the section. We will make use of a non-canonical lift of  $U$  to the level of cochains. To this end, let

$$\cdots \rightarrow F_r \xrightarrow{d_r} F_{r-1} \rightarrow \cdots \rightarrow F_0 \rightarrow R \rightarrow 0$$

be a free resolution of  $R$  by right  $R[\Gamma]$ -modules. Applying  $\text{Hom}_\Gamma(-, D)$  yields

$$0 \rightarrow \text{Hom}_\Gamma(F_0, D) \rightarrow \cdots \rightarrow \text{Hom}_\Gamma(F_{r-1}, D) \xrightarrow{d_r} \text{Hom}_\Gamma(F_r, D) \rightarrow \cdots$$

Set

$$C^r(\Gamma, D) = \text{Hom}_\Gamma(F_r, D), \quad Z^r(\Gamma, D) = \ker(d_{r+1}), \quad \text{and } B^r(\Gamma, D) = \text{im}(d_r).$$

So, by definition,  $H^r(\Gamma, D) \cong Z^r(\Gamma, D)/B^r(\Gamma, D)$ .

Note that  $F_\bullet^\pi \rightarrow R \rightarrow 0$  is a free resolution of  $R[\pi^{-1}\Gamma\pi]$ -modules. Also, both  $F_\bullet \rightarrow R \rightarrow 0$  and  $F_\bullet^\pi \rightarrow R \rightarrow 0$  are free  $R[\Delta]$ -resolutions of  $R$ . In particular, there exists an  $R[\Delta]$ -chain complex map  $\tau$ :

$$\begin{array}{ccccc} F_\bullet & \longrightarrow & R & \longrightarrow & 0 \\ \downarrow \tau_\bullet & & \downarrow = & & \\ F_\bullet^\pi & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

lifting the identity map on  $R$ .

Unraveling the definition of restriction and transfer gives the following description of  $U$  on the level of cocycles, as in [5, Formulae 4.3]. Let  $\varphi \in H^r(\Gamma, D)$  and let  $\tilde{\varphi} \in Z^r(\Gamma, D)$  be a cocycle representing  $\varphi$ . Decompose the double coset  $\Gamma\pi\Gamma$  as a union of right cosets  $\cup_i \Gamma\pi\gamma_i$  for  $\gamma_i \in \Gamma$ . Then

$$(\tilde{\varphi}|U)(f_r) \equiv \sum_i \tilde{\varphi}(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi\gamma_i \pmod{B^r(\Gamma, D)}.$$

To lift  $U$  to the level of cochains, we simply define an operator  $U: \text{Hom}(F_r, D) \rightarrow \text{Hom}(F_r, D)$  by

$$(\varphi|U)(f_r) := \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi\gamma_i$$

for  $\varphi \in \text{Hom}(F_r, D)$ .

Note that this operator depends on the choice of  $\tau$  and on the choice of coset representatives for  $\Gamma\pi\Gamma$ . We will see in the proof of the following lemma, however, that its restriction to  $\text{Hom}_\Gamma(F_r, D)$  is independent of the choice of coset representatives.

**Lemma 3.2** *The operator  $U: \text{Hom}(F_r, D) \rightarrow \text{Hom}(F_r, D)$  induces a map of chain complexes  $U: C^r(\Gamma, D) \rightarrow C^r(\Gamma, D)$  and hence a map of cohomology groups  $H^r(\Gamma, D) \rightarrow H^r(\Gamma, D)$ .*

**Proof** We first check that the action of  $U$  on an element of  $C^r(\Gamma, D)$  does not depend on the choice of coset representatives for  $\Gamma\pi\Gamma$ . So assume for each  $i$  that we have  $\Gamma\pi\gamma_i = \Gamma\pi\hat{\gamma}_i$  and write  $\eta_i\pi\gamma_i = \pi\hat{\gamma}_i$  with  $\eta_i \in \Gamma$ . Note that  $\pi^{-1}\eta_i^{-1}\pi = \hat{\gamma}_i\gamma_i^{-1}$  and thus is in  $\Delta$ . We then have for  $\varphi \in C^r(\Gamma, D) = \text{Hom}_\Gamma(F_r, D)$ ,

$$\begin{aligned} \sum_i \varphi(\tau_r(f_r \cdot \hat{\gamma}_i^{-1})) \cdot \pi\hat{\gamma}_i &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1} \pi^{-1} \eta_i^{-1} \pi)) \cdot \eta_i \pi \gamma_i \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1}) \cdot \pi \pi^{-1} \eta_i^{-1} \pi) \cdot \eta_i \pi \gamma_i \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1}) \cdot \eta_i^{-1}) \cdot \eta_i \pi \gamma_i \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi \gamma_i, \end{aligned}$$

which establishes the independence.

We now verify that  $U$  induces a map from  $C^r(\Gamma, D)$  to  $C^r(\Gamma, D)$ ; that is, we must verify that if  $\varphi$  is invariant under  $\Gamma$ , then so is  $\varphi|U$ . We have

$$\begin{aligned} ((\varphi|U) \cdot \gamma)(f_r) &= ((\varphi|U)(f_r \cdot \gamma^{-1})) \cdot \gamma \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma^{-1} \gamma_i^{-1})) \cdot \pi \gamma_i \gamma \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi \gamma_i = (\varphi|U)(f_r). \end{aligned}$$

Here the second to last equality follows from the independence of coset representatives established above as  $\Gamma\pi\Gamma = \cup_i \Gamma\pi\gamma_i = \cup_i \Gamma\pi\gamma_i\gamma$ .

Lastly, the fact that  $U$  commutes with  $d$  is immediate from its definition. ■

The following simple lemma forms the basis of our argument.

**Lemma 3.3** *Assume that  $D$  has a decreasing  $R[S]$ -filtration  $\text{Fil}^\bullet D$  satisfying hypothesis (1) of Theorem 3.1. Then  $\varphi \in C^r(\Gamma, \text{Fil}^n D)$  implies  $\varphi|U \in C^r(\Gamma, \text{Fil}^{n+1} D)$ .*

**Proof** We have

$$(\varphi|U)(f) = \sum_i \varphi(\tau(f \cdot \gamma_i^{-1})) \cdot \pi \gamma_i,$$

which is in  $\text{Fil}^{n+1} D$  as  $\varphi$  takes values in  $\text{Fil}^n D$  and, by (1),  $\text{Fil}^n D \cdot \pi \subseteq \text{Fil}^{n+1} D$ . ■

**Lemma 3.4** Assume that  $D$  has a decreasing  $R[S]$ -filtration  $\text{Fil}^\bullet D$  satisfying hypotheses (1) and (2) of Theorem 3.1.

If  $\Psi$  is in the kernel of  $H^r(\Gamma, D) \rightarrow H^r(\Gamma, D/\text{Fil}^0 D)$  and  $\Psi|U = \alpha\Psi$  with  $\alpha \in (R/\text{Ann}_R(\Psi))^\times$ , then  $\Psi = 0$ . That is, there are no ordinary  $U$ -eigenclasses in the kernel of  $H^r(\Gamma, D) \rightarrow H^r(\Gamma, D/\text{Fil}^0 D)$ .

**Proof** Since  $\alpha$  is a unit modulo  $\text{Ann}_R(\Psi)$ , there is an element  $\beta \in R$  such that  $\alpha\beta \equiv 1 \pmod{\text{Ann}_R(\Psi)}$ . We then have  $\Psi = \beta^n \Psi|U^n$  for all  $n \geq 0$ . If  $\hat{\Psi}$  is a cocycle in  $Z^r(\Gamma, \text{Fil}^0 D)$  representing  $\Psi$ , by Lemma 3.3,  $\hat{\Psi} \in Z^r(\Gamma, \text{Fil}^n D) + B^r(\Gamma, D)$  for all  $n \geq 0$ . Thus, the image of  $\Psi$  in  $H^r(\Gamma, D/\text{Fil}^n D)$  vanishes for all  $n \geq 0$ . By hypothesis (2) on  $\text{Fil}^\bullet D$ , we have

$$H^r(\Gamma, D) = \varprojlim_n H^r(\Gamma, D/\text{Fil}^n D)$$

and thus  $\Psi = 0$ . ■

We are now prepared to prove our main theorem.

**Proof of Theorem 3.1** Let  $\hat{\varphi}_0 \in Z^r(\Gamma, D/\text{Fil}^0 D)$  denote a cocycle representing  $\varphi_0 \in H^r(\Gamma, D/\text{Fil}^0 D)$  and let  $\tilde{\varphi}_0 \in C^r(\Gamma, D)$  denote an arbitrary lift of  $\hat{\varphi}_0$ . Note that  $d\tilde{\varphi}_0$  takes values in  $\text{Fil}^0 D$  as  $\hat{\varphi}_0$  is a cocycle.

Since  $\varphi_0$  is an ordinary  $U$ -eigenclass, there is some  $\beta \in R$  such that  $\alpha\beta \equiv 1 \pmod{\text{Ann}_R(\varphi_0)}$ . Define  $\tilde{\varphi}_n := \beta^n \tilde{\varphi}_0|U^n \in C^r(\Gamma, D)$ . We claim that the image of  $\tilde{\varphi}_n$  in  $C^r(\Gamma, D/\text{Fil}^n D)$  is a cocycle. Indeed,

$$d\tilde{\varphi}_n = \beta^n d(\tilde{\varphi}_0|U^n) = \beta^n (d\tilde{\varphi}_0)|U^n,$$

which by Lemma 3.3 takes values in  $\text{Fil}^n D$  as  $d\tilde{\varphi}_0$  takes values in  $\text{Fil}^0 D$ .

Let  $\varphi_n \in Z^r(\Gamma, D/\text{Fil}^n D)$  denote the reduction of  $\tilde{\varphi}_n$  modulo  $\text{Fil}^n D$  and let  $\varphi_n$  denote the image of  $\tilde{\varphi}_n$  in  $H^r(\Gamma, D/\text{Fil}^n D)$ . We will show that for each  $n > 0$ ,  $\varphi_n$  is  $U$ -eigenvector with eigenvalues  $\alpha$  and that  $\varphi_n$  is a lift of  $\varphi_{n-1}$ .

For the first claim, since  $\varphi_0|U = \alpha\varphi_0$ , we have  $\tilde{\varphi}_0|U - \alpha\tilde{\varphi}_0$  is in  $C^r(\Gamma, \text{Fil}^0 D) + B^r(\Gamma, D)$ . Thus

$$\tilde{\varphi}_n|U - \alpha\tilde{\varphi}_n = \beta^n \tilde{\varphi}_0|U^{n+1} - \alpha\beta^n \tilde{\varphi}_0|U^n = \beta^n (\tilde{\varphi}_0|U - \alpha\tilde{\varphi}_0)|U^n,$$

which, by Lemma 3.3, is in  $C^r(\Gamma, \text{Fil}^n D) + B^r(\Gamma, D)$ . Therefore,  $\varphi_n|U = \alpha\varphi_n$ .

For the second claim, we have

$$\tilde{\varphi}_n - \tilde{\varphi}_{n-1} = \beta^n \tilde{\varphi}_0|U^n - \beta^{n-1} \tilde{\varphi}_0|U^{n-1} = \beta^{n-1} (\beta\tilde{\varphi}_0|U - \tilde{\varphi}_0)|U^{n-1},$$

which, by Lemma 3.3, is in  $C^r(\Gamma, \text{Fil}^n D) + B^r(\Gamma, D)$  as  $\beta\tilde{\varphi}_0|U = \varphi_0$ .

Thus,

$$\{\varphi_n\} \in \varprojlim_n H^r(\Gamma, D/\text{Fil}^n D) \cong H^r(\Gamma, \varprojlim_n D/\text{Fil}^n D) \cong H^r(\Gamma, D),$$

where the last equality follows from hypothesis (2) on  $\text{Fil}^\bullet D$ . The collection of classes  $\{\varphi_n\}$  therefore corresponds to a single class  $\Phi \in H^r(\Gamma, D)$ . It is immediate that  $\Phi$  lifts  $\varphi_0$  and that  $\Phi|U = \alpha\Phi$ .

The equality of the  $R$ -annihilators of  $\Phi$  and  $\varphi_0$ , and the uniqueness of  $\Phi$  now follow from Lemma 3.4. We check these two facts simultaneously. To this end, let  $\Phi'$  in  $H^r(\Gamma, D)$  be any ordinary  $U$ -eigenlift of  $\varphi_0$  with  $\Phi'|U = \alpha'\Phi'$ . Then  $\text{Ann}_R(\Phi') \subseteq \text{Ann}_R(\varphi_0)$  as  $\Phi'$  maps onto  $\varphi_0$ . To see the reverse inclusion, let  $x \in \text{Ann}_R(\varphi_0)$ . Then, by Lemma 3.4,  $x\Phi' = 0$  since otherwise  $x\Phi'$  would be a  $U$ -ordinary eigenclass which maps to 0 in  $H^r(\Gamma, D/\text{Fil}^0 D)$ . Thus,  $\text{Ann}_R(\varphi_0) = \text{Ann}_R(\Phi')$ .

Further, since  $U$  scales  $\Phi'$  by  $\alpha'$  and scales  $\varphi_0$  by  $\alpha$ , it follows that  $\alpha' - \alpha$  is in  $\text{Ann}_R(\varphi_0) = \text{Ann}_R(\Phi')$ . In particular,  $\Phi'$  also satisfies  $\Phi'|U = \alpha\Phi'$ . Again, by Lemma 3.4, the difference  $\Phi - \Phi'$  must vanish as otherwise  $\Phi - \Phi'$  would be a  $U$ -ordinary eigenclass which maps to 0 in  $H^r(\Gamma, D/\text{Fil}^0 D)$ . Hence,  $\Phi = \Phi'$  and  $\text{Ann}_R(\Phi) = \text{Ann}_R(\varphi_0)$ . ■

**Remark 3.5** If  $R$  is assumed to be a local ring, we can further deduce that there is a unique  $U$ -eigenclass lifting  $\varphi_0$  (as opposed to a unique ordinary  $U$ -eigenclass). Indeed,  $\alpha \in R$  is a unit if and only if  $\alpha$  is a unit modulo any proper ideal. From this it follows easily that any eigenlift of an ordinary eigenclass is automatically ordinary. This strengthening will be used in the proof of the main theorem of the following section.

### 4 Filtrations and Liftings

We return now to the setting of Section 2. In order to invoke the results of the previous section, we introduce a filtration on  $\mathbf{D}_\lambda$  which satisfies the hypotheses of Theorem 3.1.

#### 4.1 A $\Sigma$ -Stable Filtration on $\mathbf{D}_\lambda$

We define a filtration on  $\mathbf{D}_\lambda$  as follows. For  $N \in \mathbb{Z}^{\geq 0}$ , set

$$\text{Fil}^N \mathbf{D}_\lambda := \left\{ \mu \in \mathbf{D}_\lambda : \text{ord}_p(\mu(f_{rst})) \geq \left\lceil \frac{N - (r + s + t)}{2} \right\rceil \text{ for } r, s, t \geq 0 \right\},$$

where  $\lceil x \rceil$  is the smallest integer greater than or equal to  $x$ . Recall that the functions  $f_{rst}$  are defined at the end of Subsection 2.2.

We note that this filtration is too large to satisfy the hypotheses of Theorem 3.1. In the following subsection, we will replace it by a slightly smaller filtration.

**Proposition 4.1** *The filtration  $\text{Fil}^\bullet \mathbf{D}_\lambda$  is stable under the action of  $\Sigma$ .*

**Proof** By definition,  $(\mu|_\gamma)(f_{rst}) = \mu(\gamma f_{rst})$ . If  $\lambda = \lambda(k_1, k_2, k_3)$ , by Lemma 2.1, for

$\gamma \in I$  we have

$$\begin{aligned} (\gamma f_{rst}) \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} &= \det(\gamma)^{k_3} (a_{12} + a_{22}x + a_{32}y)^r (a_{13} + a_{23}x + a_{33}y)^s \\ &\quad \times (-m_{32} + m_{12}y + m_{22}z - m_{12}xz)^t (a_{11} + a_{21}x + a_{31}y)^{k_1 - k_2 - r - s} \\ &\quad \times (m_{33} - m_{13}y - m_{23}z + m_{13}xz)^{k_2 - k_3 - t}. \end{aligned}$$

We expand out the last terms with the binomial theorem (keeping in mind that the exponents may be negative). We have

$$\begin{aligned} (\gamma f_{rst}) \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} &= \det(\gamma)^{k_3} (a_{12} + a_{22}x + a_{32}y)^r \\ &\quad \times (a_{13} + a_{23}x + a_{33}y)^s (-m_{32} + m_{12}y + m_{22}z - m_{12}xz)^t \\ &\quad \times (a_{11})^{k_1 - k_2 - r - s} \left( \sum_{j=0}^{\infty} \binom{k_1 - k_2 - r - s}{j} \left( \frac{a_{21}}{a_{11}}x + \frac{a_{31}}{a_{11}}y \right)^j \right) \\ &\quad \times (m_{33})^{k_2 - k_3 - t} \left( \sum_{j=0}^{\infty} \binom{k_2 - k_3 - t}{j} \left( -\frac{m_{13}}{m_{33}}y - \frac{m_{23}}{m_{33}}z + \frac{m_{13}}{m_{33}}xz \right)^j \right) \\ &= \sum_{a,b,c} \alpha_{abc} x^a y^b z^c. \end{aligned}$$

Binomial coefficients  $\binom{a}{b}$  with  $a \geq 0$  and  $b > a$  should be interpreted as 0. Note that  $a_{11}, a_{22}, a_{33}, m_{22}$ , and  $m_{33}$  are all units while  $a_{21}, a_{31}, a_{32}, m_{12}, m_{13}$ , and  $m_{23}$  are all multiples of  $p$ .

We now consider the possible valuations of  $\alpha_{abc}$  for a fixed triple  $(a, b, c)$ . Note that the coefficient of a monomial of degree  $d$  appearing in the expansion of the first three nonconstant factors of this product has valuation at least  $d - (r + s + t)$ . (This follows as the only non-linear term that appears in these factors has a coefficient divisible by  $p$ .) Also note that the coefficient of a monomial of degree  $m$  in the expansion of the last two nonconstant factors will have valuation at least equal to  $\lceil \frac{m}{2} \rceil$ . (This follows as all coefficients that appear in these expressions are divisible by  $p$  including the coefficient of the  $xz$  term.) Thus, the valuation of  $\alpha_{abc}$  is at least  $\lceil \frac{(a+b+c) - (r+s+t)}{2} \rceil$ .

Note that  $\mu(\gamma f_{rst}) = \sum_{a,b,c} \alpha_{a,b,c} \mu(f_{abc})$ ; we wish to show that this sum has valuation at least  $\lceil \frac{N - (r+s+t)}{2} \rceil$ . We do this term-by-term. Fix a triple  $(a, b, c)$ . If  $a + b + c \geq N$ , then by the above computation, we have that the valuation of  $\alpha_{abc}$  is large enough. If  $a + b + c < N$ , then, as  $\mu$  is in  $\text{Fil}^N \mathbf{D}_\lambda$ , we have that  $\mu(f_{abc})$  has valuation at least  $\lceil \frac{N - (a+b+c)}{2} \rceil$ . Since

$$\left\lceil \frac{N - (a + b + c)}{2} \right\rceil + \left\lceil \frac{(a + b + c) - (r + s + t)}{2} \right\rceil \geq \left\lceil \frac{N - (r + s + t)}{2} \right\rceil,$$

we have that  $\mu|_\gamma$  is in  $\text{Fil}^N \mathbf{D}_\lambda$ .

Lastly, we consider the action by the element  $\pi$ . In this case,

$$(\mu|_\pi)(f_{rst}) = \mu(\pi f_{rst}) = p^{r+2s+t} \mu(f_{rst}).$$

Thus, if  $\mu$  is in  $\underline{\text{Fil}}^N \mathbf{D}_\lambda$  so is  $\mu|_\gamma$ . ■

**Lemma 4.2** *If  $\mu \in \underline{\text{Fil}}^N \mathbf{D}_\lambda$  and  $\mu(f_{000}) = 0$ , then  $\mu|_\pi \in \underline{\text{Fil}}^{N+1} \mathbf{D}_\lambda$ .*

**Proof** Since  $(\mu|_\pi)(f_{rst}) = p^{r+2s+t} \mu(f_{rst})$ , if  $(r, s, t) \neq (0, 0, 0)$ , then

$$\text{ord}_p((\mu|_\pi)(f_{rst})) \geq \text{ord}_p(\mu(f_{rst})) + 1 \geq \left\lceil \frac{(N+1) - (r+s+t)}{2} \right\rceil.$$

Moreover,

$$(\mu|_\pi)(f_{000}) = \mu(\pi \cdot f_{000}) = \mu(f_{000}) = 0$$

and thus  $\mu|_\pi \in \underline{\text{Fil}}^{N+1} \mathbf{D}_\lambda$ . ■

### 4.2 The Main Filtration

Recall the map  $\rho_\lambda: \mathbf{D}_\lambda \rightarrow L_\lambda$  defined in Subsection 2.4. Set  $K_\lambda := \ker(\rho_\lambda)$ , which is a  $\Sigma$ -module, as  $\rho_\lambda$  is  $\Sigma$ -equivariant. We then define our main filtration on  $\mathbf{D}_\lambda$  by

$$\text{Fil}^N \mathbf{D}_\lambda := \underline{\text{Fil}}^N \mathbf{D}_\lambda \cap K_\lambda.$$

Before checking that this filtration satisfies the hypotheses of Theorem 3.1, we introduce one lemma.

**Lemma 4.3** *If  $\mu \in K_\lambda$ , then  $\mu(f_{000}) = 0$ .*

**Proof** Let  $v_\lambda, v_2, \dots, v_d$  be a basis of  $V_\lambda(\mathbb{Q}_p)$ . Since  $v_\lambda$  is a highest weight vector for  $N^{\text{opp}}$ , we have

$$v_\lambda \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = v_\lambda + \sum_{i=2}^d P_i(x, y, z) v_i.$$

Thus,

$$\rho_\lambda(\mu) = \mu(f_{000}) v_\lambda + \sum_{i=2}^d \mu(P'_i) v_i$$

where  $P'_i$  is the unique extension of  $P_i$  to  $\mathfrak{X}$ . Therefore,  $\mu(f_{000}) = 0$  as  $\mu$  is in the kernel of  $\rho_\lambda$ . ■

**Proposition 4.4** *We have that  $\text{Fil}^\bullet \mathbf{D}_\lambda$  is a decreasing  $\mathbb{Z}_p[\Sigma]$ -stable filtration such that*

- (1)  $\text{Fil}^N \mathbf{D}_\lambda|_\pi \subseteq \text{Fil}^{N+1} \mathbf{D}_\lambda$  for all  $N \geq 0$ ,
- (2) the natural map  $\mathbf{D}_\lambda \rightarrow \varprojlim \mathbf{D}_\lambda / \text{Fil}^N \mathbf{D}_\lambda$  is an isomorphism.

**Proof** The first part of the proposition follows from Lemmas 4.2 and 4.3. For the second part, it is clear from the definitions that  $\cap \text{Fil}^N \mathbf{D}_\lambda = 0$ . Conversely, if  $\{\mu_N\} \in \varprojlim \mathbf{D}_\lambda / \text{Fil}^N \mathbf{D}_\lambda$ , for a fixed triple  $(a, b, c)$  the sequence  $\{\mu_N(f_{abc})\}$  is Cauchy converging to say  $\alpha_{abc} \in \mathbb{Z}_p$ . Let  $\mu$  be the unique distribution such that  $\mu(f_{abc}) = \alpha_{abc}$  for all  $a, b, c$ . Then  $\mu$  projects to  $\mu_N$  for each  $N \geq 0$ . ■

### 4.3 Lifting Hecke-Eigensymbols

Let  $\mathcal{H}$  denote the abstract Hecke-algebra generated over  $\mathbb{Z}_p$  by  $U_p$  and by  $T(\ell, k)$  for all primes  $\ell \neq p$  and  $k = 1, 2, 3$ . Recall that the specialization map

$$\rho_k^r: H^r(\Gamma, \mathbf{D}_\lambda) \rightarrow H^r(\Gamma, L_\lambda)$$

is  $\mathcal{H}$ -equivariant. We offer the following theorem (which is implied by Theorem 2.8).

**Theorem 4.5** *If  $\varphi \in H^r(\Gamma, L_\lambda)$  is an  $\mathcal{H}$ -eigenvector whose  $U_p$ -eigenvalue is a unit, then there exists a unique  $\mathcal{H}$ -eigenvector  $\Phi \in H^r(\Gamma, \mathbf{D}_\lambda)$  that specializes to  $\varphi$ .*

**Proof** We apply Theorem 3.1 to the distribution module  $\mathbf{D}_\lambda$  and its filtration  $\text{Fil}^\bullet \mathbf{D}_\lambda$ . By Proposition 4.4, this filtration satisfies the hypotheses of Theorem 3.1 with respect to  $\pi$ , the diagonal matrix with entries 1,  $p$  and  $p^2$ . Note that  $\mathbf{D}_\lambda / \text{Fil}^0 \mathbf{D}_\lambda \cong L_\lambda$  and thus, by Theorem 3.1, there exists a unique  $U_p$ -eigenvector  $\Phi$  lifting  $\varphi$  (see Remark 3.5).

We moreover claim that  $\Phi$  is automatically an  $\mathcal{H}$ -eigenvector. Indeed, let  $T \in \mathcal{H}$  and let  $\varphi|T = a\varphi$ . Then, since  $T$  and  $U_p$  commute, both  $\Phi|T$  and  $a\Phi$  are  $U_p$ -eigenlifts of  $a\varphi$ . By uniqueness, we conclude that  $\Phi|T = a\Phi$ . ■

**Remark 4.6** The above argument works equally well if one has a  $\mathcal{H}$ -eigenvector  $v \in H^r(\Gamma, L_\lambda) \otimes \mathcal{O}$  for some finite extension  $\mathcal{O}$  of  $\mathbb{Z}_p$ .

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