MA 123
Calculus I
Midterm II Practice Exam
Answer Key

Note: Be aware that there may be *more* than one method to solving any one question. Keep in mind that the beauty in math is that you can often obtain the same answer from more than one approach. For this reason, the answers provided in this key are left in their most simplified form. Unless otherwise noted, the answers do not *need* to be simplified, yet simplification may be the key to dissecting a seemingly unworkable function.

1) Differentiate the following functions with respect to x

(a.) $f(x)=\dfrac{x^2-2\sqrt{x}}{x}$

→ $f(x)=\dfrac{x^2-2\sqrt{x}}{x}=x-2x^{-1/2}$

→ $f'(x)=1-(2)\left(\dfrac{-1}{2}\right)(x^{-3/2})=1+x^{-3/2}$ //

(b.) $f(x)=e^{3x}\sqrt{x^2+1}$

→ $f'(x)=(e^{3x})\left(\dfrac{1}{2}\right)(x^2+1)^{-1/2}(2x)+(x^2+1)^{1/2}(3e^{3z})$

→ $f'(x)=e^{3x}(x(x^2+1)^{-1/2}+3(x^2+1)^{1/2})$ //

(c.) $f(x)=e^x(\cos x+cx)$; Where c is some arbitrary constant

→ $f'(x)=e^x(-\sin x+c)+e^x(\cos x+cx)$

→ $f'(x)=e^x(\cos x-\sin x+c+cx)$ //

(d.) $f(x)=\sin(e^{\alpha x})$; Where α is some arbitrary constant

→ $f'(x)=\alpha e^{\alpha x}\cos(e^{\alpha x})$ //

2) Use Logarithmic Differentiation to find the derivative of the following function

$$y=e^x\sqrt{x}(x^2+1)^{10}$$

→ $\ln y=\ln(e^x\sqrt{x}\left(x^2+1\right)^{10})$; Separate using log rules

→ $\ln y=\ln e^x+\ln\sqrt{x}+\ln(x^2+1)^{10}$; Simplify using log rules

→ $\ln y=x+\dfrac{1}{2}\ln x+10\ln(x^2+1)$; Differentiate both sides

→ $\dfrac{1}{y}y'=1+\left(\dfrac{1}{2}\right)\left(\dfrac{1}{x}\right)+10\left(\dfrac{1}{x^2+1}\right)(2x)$; Simplify and solve for y '

→ $y'=y\left(1+\dfrac{1}{2x}+\dfrac{20x}{x^2+1}\right)=\left(e^x\sqrt{x}(x^2+1)^{10}\right)\left(1+\dfrac{1}{2x}+\dfrac{20x}{x^2+1}\right)$ //

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3) Use Implicit Differentiation to find the equation of the tangent line to the following curve at the given point.

$x^2 + 2xy - y^2 + x = 2$ at the point (1,2)

→ $2x + 2(xy' + y \cdot 1) - 2yy' + 1 = 0$; Differentiate both sides

→ $2x + 2xy' + 2y = 2yy' + 1 = 0$; Simplify

$2xy' - 2yy' = -1 - 2x - 2y$

$y'(2x - 2y) = -1 - 2x - 2y$

→ $y' = \dfrac{-1 - 2x - 2y}{2x - 2y}$; Determine y ‘ @ (1,2)

→ $y' = \dfrac{-1 - (2 \cdot 1) - (2 \cdot 2)}{(2 \cdot 1) - (2 \cdot 2)} = \dfrac{7}{2} = m$; Use Point Slope Formula

→ $y - 2 = \left(\dfrac{7}{2}\right)(x - 1)$; Solve for y

→ $y = \left(\dfrac{7}{2}x\right) - \left(\dfrac{3}{2}\right)$ //

4) Let $f(x) = x^4 - 2x^2$.

(a.) Find the first and second derivatives of $f(x)$.

→ $f'(x) = 4x^3 - 4x$

→ $f''(x) = 12x^2 - 4$ //

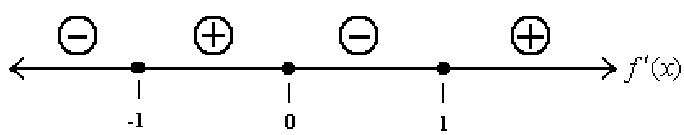
(b.) On which intervals is $f(x)$ increasing? decreasing ?

→ $f'(x) = 4x^3 - 4x = 0$; Find critical points

$4x(x^2 - 1) = 0$

$4x(x+1)(x-1) = 0$

→ $x = 0, \quad -1, \quad +1$; Plot the critical points on a number line



f(x) is increasing on the intervals (-1 , 0) and (+1 , ∞)

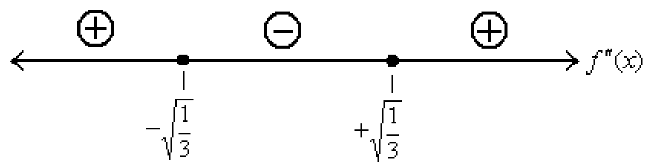
f(x) is decreasing on the intervals (-∞ , -1) and (0 , +1) //

(c.) On which intervals is $f(x)$ concave upward ? concave downward ?

→ $f''(x) = 12x^2 - 4 = 0$; Find inflection points

$$3x^2 - 1 = 0$$
$$3x^2 = 1$$

→ $x = \pm\sqrt{\frac{1}{3}}$; Plot the inflection points on a number line



f(x) is concave upward on the intervals $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and $\left(+\sqrt{\frac{1}{3}},\ +\infty\right)$

f(x) is concave downward on the interval $\left(-\sqrt{\frac{1}{3}},\ +\sqrt{\frac{1}{3}}\right)$ //

(d.) What are the inflection points of $f(x)$?

→ The inflection points are the x-values of which $f''(x) = 0$, namely

$x = \pm\sqrt{\frac{1}{3}}$ //

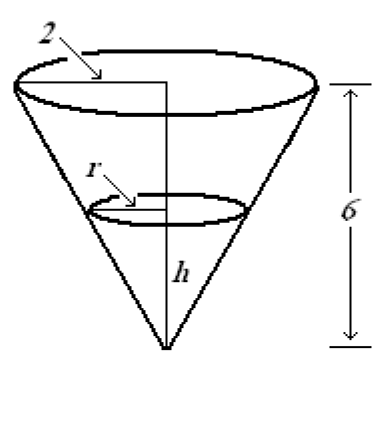
5) Water is leaking out of an inverted conical tank at a rate of 10,000 cm³/min at the same time that water is being pumped into the tank at a constant rate. The tank has a height of 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank. (Hint: See Figure 3 on Page 264 for a better visualization)

If C = the rate at which water is pumped in, then

$\frac{dV}{dt} = C - 10{,}000$, where $V = \frac{1}{3}\pi r^2 h$.

By similar triangles, $\frac{r}{2} = \frac{h}{6}$ ➔ $r = \frac{h}{3}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(h/3)^2 h = (\pi/27)h^3$



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$\frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$. When h = 200 cm

$\frac{dh}{dt}$ = 20 cm/min, so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20)$ ➔ $C = 10{,}000 + \frac{800{,}000}{9}\pi$ //

6) Let $f(x) = \frac{x^3}{x^2 - 4}$.

(a.) Find any/all local max and local min of $f(x)$.

→ $$f'(x) = \frac{(x^2-4)(3x^2)-(x^3)(2x)}{(x^2-4)^2} = \frac{x^4-12x^2}{(x^2-4)^2}$$

→ Critical Points: $f'(x) = \frac{x^4-12x^2}{(x^2-4)^2} = 0$

$$x^4 - 12x^2 = 0$$
$$x^2(x^2-12) = 0$$
$$x = 0, \quad -\sqrt{12}, \quad +\sqrt{12}$$

→ $$f''(x) = \frac{(x^2-4)^2(4x^3-24x)-(x^4-12x^2)(2)(x^2-4)(2x)}{(x^2-4)^4}$$

$$f''(x) = \frac{(x^2-4)[(x^2-4)(4x^3-24x)-(x^4-12x^2)(4x)]}{(x^2-4)^4}$$

$$f''(x) = \frac{8x^3+96x}{(x^2-4)^3}$$

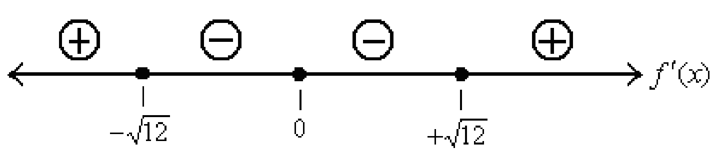
→ Since $f'(-\sqrt{12}) = 0$ and $f''(-\sqrt{12}) < 0$, then f has a local *maximum* at $x = -\sqrt{12}$.

Since $f'(+\sqrt{12}) = 0$ and $f''(+\sqrt{12}) > 0$, then f has a local *minimum* at $x = +\sqrt{12}$.

Since $f'(0) = 0$ and $f''(0) = 0$, then $f(0)$ is neither a local max or min. //

(b.) Determine when $f(x)$ is increasing.

→ Plot the Critical Points found in part (a.)



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f(x) is increasing on the intervals $\left(-\infty, \quad -\sqrt{12}\right)$ and $\left(+\sqrt{12}, \quad +\infty\right)$ //

(c.) Determine when $f(x)$ is decreasing.

→ See number line constructed in part (b.) above.

f(x) is decreasing on the intervals $\left(-\sqrt{12}, \quad 0\right)$ and $\left(0, \quad +\sqrt{12}\right)$ //

(d.) Find all asymptotes of $f(x)$.

→ Vertical asymptotes: When f(x) approaches ± ∞

$$x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = -2, \quad +2$$

Vertical asymptotes exist at x = ± 2.

→ Horizontal asymptotes:: What f(x) approaches as x → ± ∞

$$\lim_{x \to -\infty} \frac{x^3}{x^2 - 4} = \lim_{x \to -\infty} x = -\infty$$
$$\lim_{x \to -+\infty} \frac{x^3}{x^2 - 4} = \lim_{x \to +\infty} x = +\infty$$

Both limits do *not* exist due to the fact that they do not converge to a finite number, thus are no horizontal asymptotes. //

(e.) Determine when $f(x)$ is concave upward.

→ $f''(x) = \dfrac{8x^3 + 96x}{(x^2 - 4)^3} = 0$; Find inflection points

$$8x^3 + 96x = 0$$
$$8x(x^2 + 12) = 0$$

x = 0 ; Only *real* inflection point

Must also find when $f''(x)$ is undefined.

i.e when $(x^2 - 4)^3 = 0$

$$(x^2 - 4) = 0$$
$$(x+2)(x-2) = 0$$

$x = -2, \quad +2$; Note: These are the vertical asymptotes

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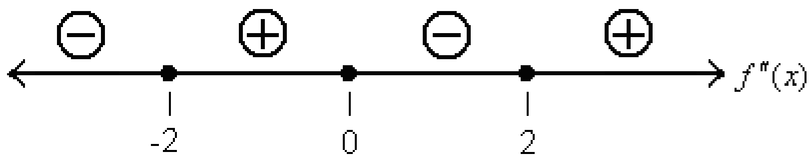
→ Plot the inflection points on a number line

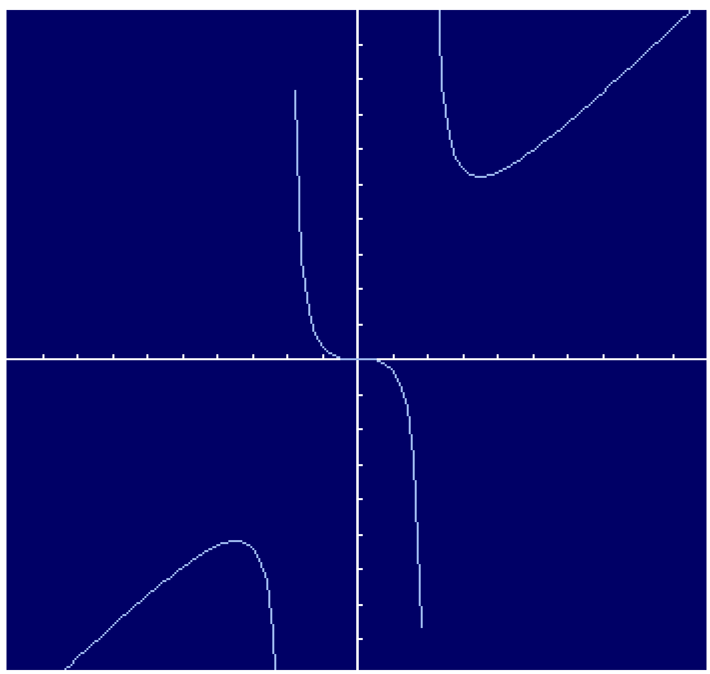


f(x) is concave upward on the interval $(-2,\ \ 0)$ and $(2,\ \ +\infty)$. //

(f.) Determine when $f(x)$ is concave downward.

f(x) is concave downward on the interval $(-\infty,\ \ -2)$ and $(0,\ \ 2)$. //

(g.) Use your results from parts A through F to sketch the graph of f(x) on the graph below.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Midterm II : Thurs, June 12th at 6 pm in MCS B33