Checkerboard Julia Sets

and the Families of Rational Maps

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

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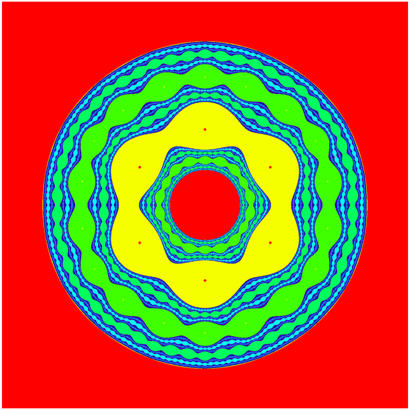
McMullen's Examples: A Cantor Set of Quasi-Circles

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$



$|\lambda|$ small but nonzero in $\mathbb{C}$, and

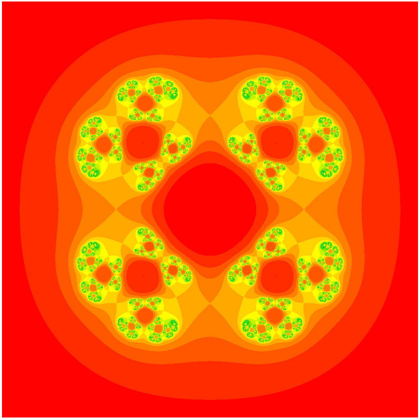
$$\frac{1}{n} + \frac{1}{d} < 1$$

For the Julia set on the left,

$$n = d = 3 \text{ and } \lambda = 0.01.$$

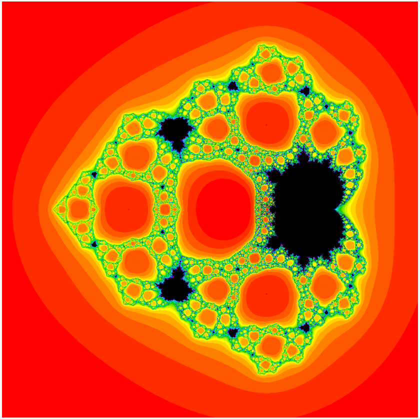
The $n = d = 2$ Case: A Cantor Set

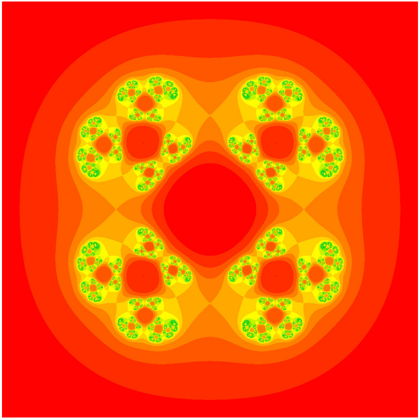
$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}, \quad \lambda = 0.2111$$



The $n = d = 2$ Case: The Connectedness Locus

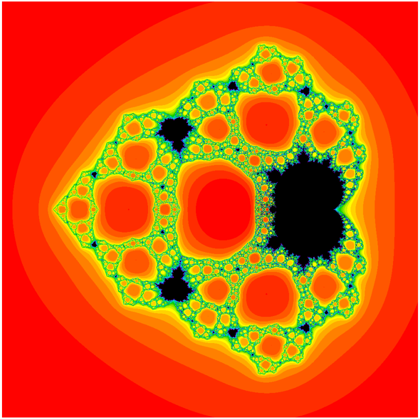
$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

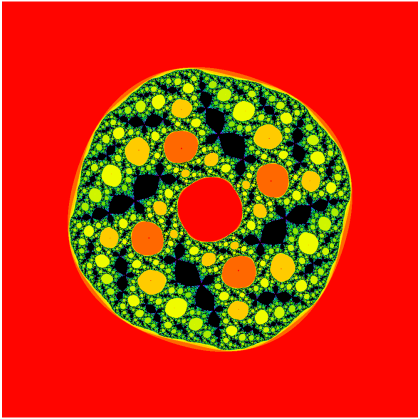




The $n = d = 2$ Case: Period-Three Example

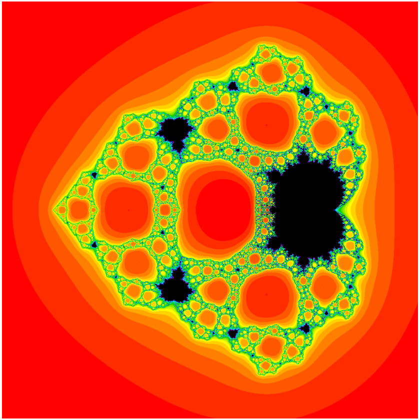
$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

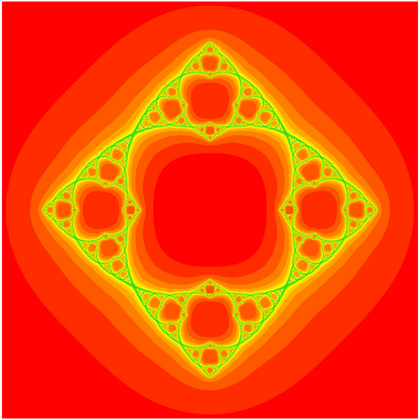




The $n = d = 2$ Case: A Misiurewicz Example

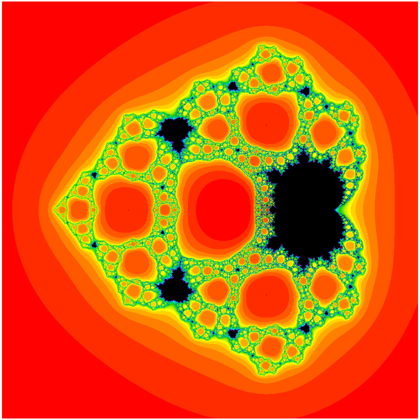
$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}, \quad \lambda = -\frac{3 + 2\sqrt{2}}{16}$$

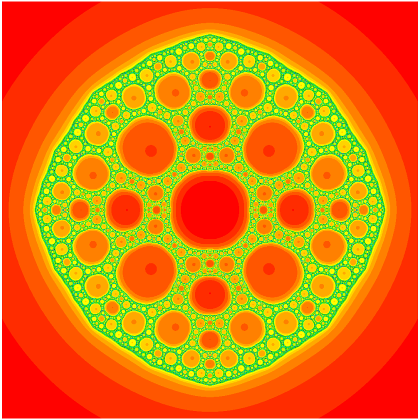




The $n = d = 2$ Case: A Sierpiński Carpet

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}, \quad \lambda = -\frac{1}{16}$$





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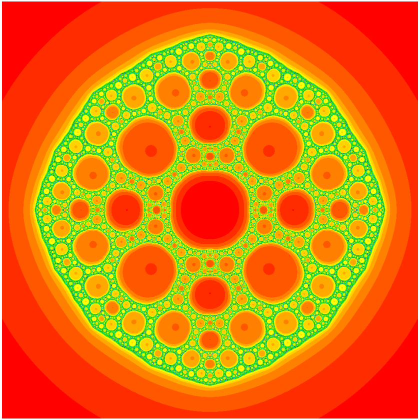
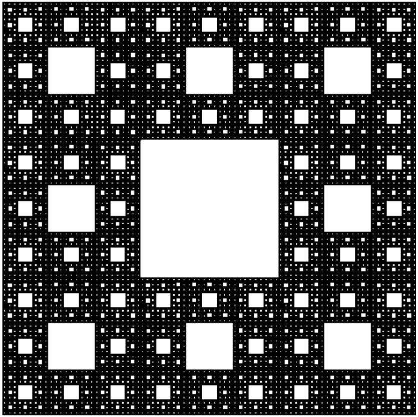




Sierpiński Carpets

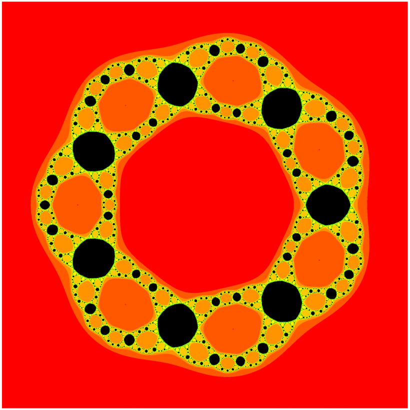
Sierpiński curve

A subset of the plane is called a *Sierpiński curve* if it is compact, connected, locally connected, nowhere dense, and the boundaries of its complementary domains are disjoint simple, closed curves.
In 1958, Whyburn proved the following two remarkable results about such sets.

1. Any two Sierpiński curves are homeomorphic.
2. The Sierpiński curve/carpet is universal in the sense that it contains a homeomorphic image of any compact, connected, one-dimensional, planar set.

A Checkerboard Julia Set

$$F_\lambda(z) = z^n + \frac{\lambda}{z^d}$$

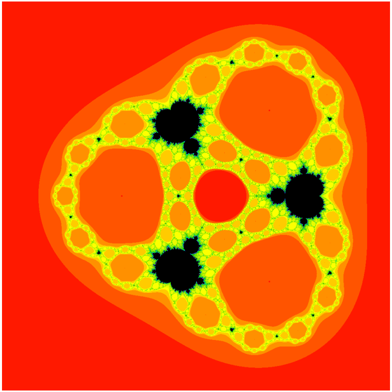


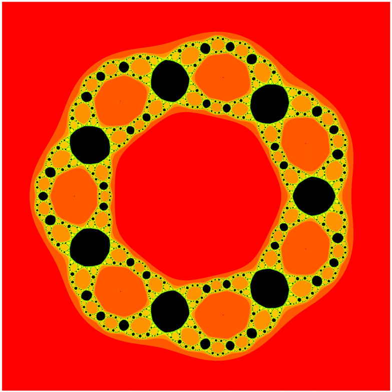
Here $n = 4$ and $d = 3$.

$$\lambda = 0.18$$

A Checkerboard Julia Set

$$F(z) = z^4 + \frac{0.18}{z^3}$$





A Checkerboard Julia Set

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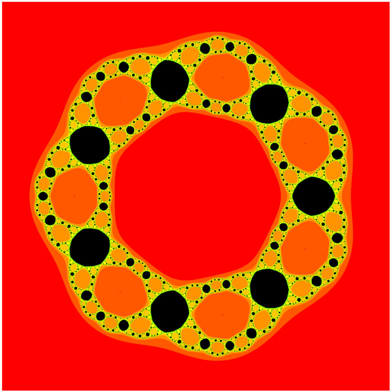
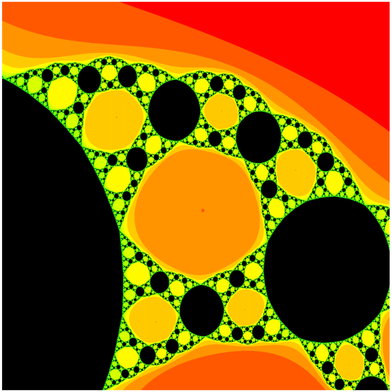




The Dynamical Trichotomy

The Escape Trichotomy (Devaney, Look, Uminsky)

Let $v_\lambda = F_\lambda(c_\lambda)$ be a critical value. Then:

1. if v_λ lies in B_λ, then $J(F_\lambda)$ is a Cantor set;
2. if v_λ lies in $T_\lambda \neq B_\lambda$, then $J(F_\lambda)$ is a Cantor set of disjoint simple closed curves surrounding the origin;
3. in all other cases, $J(F_\lambda)$ is a connected set. In addition, if $F_\lambda^j(v_\lambda) \in T_\lambda \neq B_\lambda$ for some $j \geq 1$, then $J(F_\lambda)$ is a Sierpiński curve.

Preliminaries for $F_\lambda(z) = z^n + \dfrac{\lambda}{z^d}$

Critical points:

- $n - 1$ critical points at ∞
- $d - 1$ critical points at 0
- $n + d$ "free" critical points that satisfy the equation

$$z^{n+d} = \left(\frac{d}{n}\right)\lambda.$$

Prepoles: The prepoles satisfy the equation

$$z^{n+d} = -\lambda.$$

Immediate basin of ∞ and the trap door

Immediate basin B_λ of ∞:

The component of the basin of ∞ that contains ∞.

Trap door T_λ:

The inverse image of B_λ that contains the pole at the origin. Note that it is possible that $B_\lambda = T_\lambda$.

Symmetries

F_λ has symmetries in both the dynamical plane and the parameter plane.

Symmetry Lemma I

F_λ is conjugate to $F_{\bar{\lambda}}$ by the conjugacy $z \mapsto \bar{z}$.

This symmetry implies that the parameter plane is symmetric under complex conjugation.

Symmetries—Lemma II

Symmetry Lemma II

If ω is a $(n+d)^{th}$ root of unity, then $F_\lambda(\omega z) = \omega^n F_\lambda(z)$.

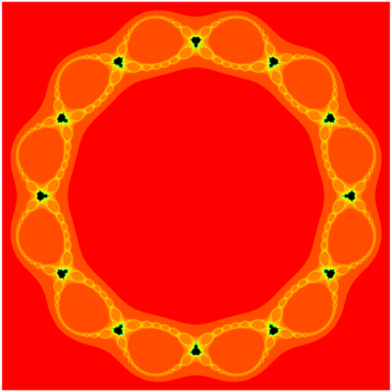
This second symmetry implies that the Julia set of F_λ is symmetric under the map $z \mapsto \omega z$. Similarly, B_λ and T_λ possess this $(n+d)$-fold symmetry.

Moreover, since the free critical points are arranged symmetrically with respect to $z \mapsto \omega z$, all of the free critical orbits behave symmetrically with respect to this rotation.

However, it is not necessarily true that all of these critical orbits behave in the same manner.

The most important consequence of Symmetry Lemma 2 is the fact that the orbits of all of the free critical points can be determined from the orbit of any one of them. So the one-dimensional λ-plane is a natural parameter plane for these maps.

Symmetries—Lemma II



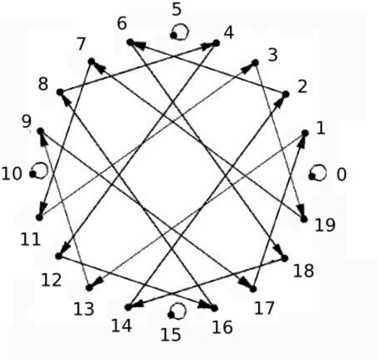
There is a Mandelbrot set centered on the positive real axis at

$$\lambda_0 = \left(\frac{d}{n} + 1\right)^{\frac{n+d}{1-n}} \left(\frac{d}{n}\right)^{\frac{d+1}{n-1}}$$

We denote it by $\mathcal{M}_0$.

Here, $n = 13$ and $d = 7$, and

$$\lambda_0 \approx 0.32.$$



The orbit diagram for the critical points of

$$F_{\lambda_0}(z) = z^{13} + \frac{\lambda_0}{z^7},$$

where λ_0 is the center of $\mathcal{M}_0$. The critical point on the positive real axis is a fixed point, and it is labeled with the number 0. The orbits of the remaining critical points are determined from the orbit of the fixed point using Symmetry Lemma 2.

Symmetries—Lemma III

Symmetry Lemma III

Suppose that η *is an* $(n+d)(n-1)$*st root of unity. Let* $\nu = \eta^{n+d}$ *and* $\omega = \eta^{n-1}$. *Then*

$$F^k_{\nu\lambda}(\eta z) = \eta^{n^k} \, F^k_\lambda(z)$$

for $k = 1, \ 2, \ 3, \ldots$.

Note that ν is an $(n-1)$st root of unity and ω is an $(n+d)$th root of unity.

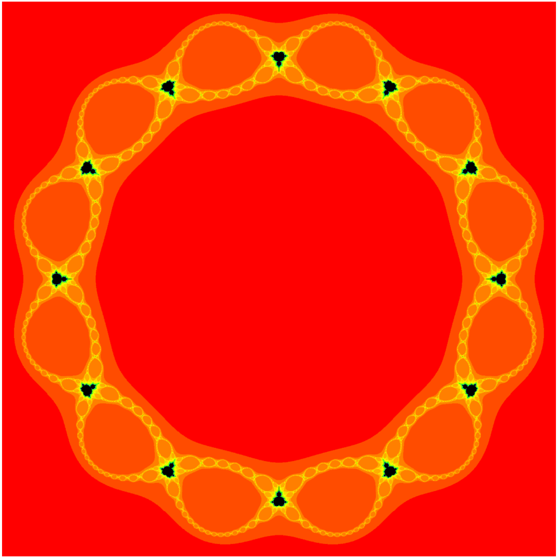
Symmetries—Lemma III

We can determine the orbit diagram of $F_{\nu\lambda}$ from the orbit diagram of λ. In particular, if c_λ is a critical point for F_λ, then ηc_λ is a critical point for $F_{\nu\lambda}$. We denote this critical point by $c_{\nu\lambda}$. From Symmetry Lemma 3, we have

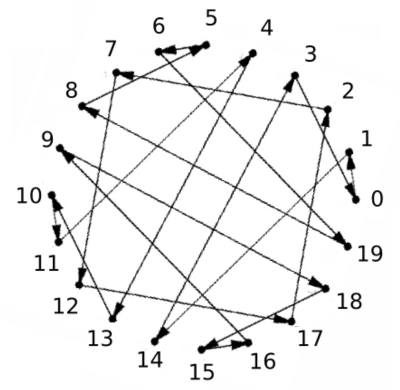
$$F_{\nu\lambda}^{k}\left(c_{\nu\lambda}\right) = F_{\nu\lambda}^{k}\left(\eta c_\lambda\right) = \eta^{n^k} F_{\lambda}^{k}\left(c_\lambda\right).$$

Therefore, the orbits of the critical points of F_λ and $F_{\nu\lambda}$ behave symmetrically with respect to rotation by some power of η. Consequently, the parameter plane is symmetric under the rotation $\lambda \mapsto \nu\lambda$.

Symmetries—Lemma III



Symmetries—Lemma III



The orbit diagram for the critical points of

$$F_{\lambda_1}(z) = z^{13} + \frac{\lambda_1}{z^7},$$

where $\lambda_1 = \nu\lambda_0$ is the center of the "next" principal Mandelbrot set $\mathcal{M}_1$, i.e., the image of $\mathcal{M}_0$ under the rotation $z \mapsto \nu z$.

Checkerboard Julia Sets

- All checkerboard Julia sets are homeomorphic.
- However, the maps F_λ restricted to their Julia sets are not always topologically conjugate.

Question: Is there an invariant that tells us when two parameter values drawn from the main cardioids of different principal Mandelbrot sets yield conjugate dynamics on their respective Julia sets?

Answer: Yes.

Some Notation

The principal Mandelbrot set intersecting the positive real axis is denoted $\mathcal{M}_0$.

The remaining $n - 2$ principal Mandelbrot sets is labeled $\mathcal{M}_1$ through $\mathcal{M}_{n-2}$ where the ordering is in the counterclockwise direction.

We write $\mathcal{M}_j \equiv \mathcal{M}_k$ if the parameters at the centers of $\mathcal{M}_j$ and $\mathcal{M}_k$ have conjugate dynamics on their Julia sets.

We denote the center of the main cardioid of $\mathcal{M}_j$ by λ_j. We only consider these λ-values.

Dynamical Invariant

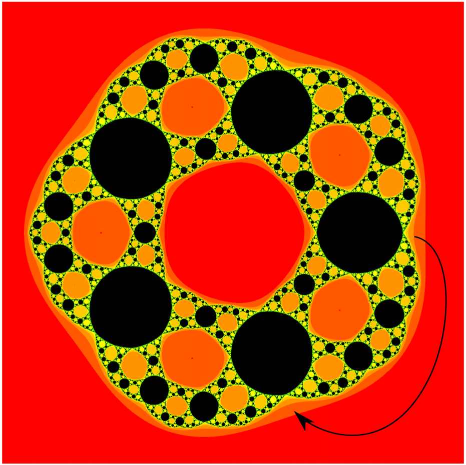
For each of the λ-values we are considering, we have $n + d$ connecting Fatou components, labeled C_λ^j for $j = 0, 1, \ldots, n + d - 1$ in the counterclockwise direction

Minimum Rotation Number

The rotation number for C_λ^j is the number of connecting components that C_λ^j is rotated through, in either direction, under the map F_λ.

The smallest rotation number over all the C_λ^j's for a given λ is the *minimum rotation number*. We denote it by $\rho(\lambda)$.

Example of Rotation Number



A C_λ^j with $\rho_j = -1$.

Minimum Rotation Number

Minimum Rotation Number and Conjugacies

For a given n and d, $\mathcal{M}_j \equiv \mathcal{M}_k$ if and only if $\rho(\lambda_j) = \rho(\lambda_k)$.

Conjugacies

Conjugacies

Let $\nu = e^{2\pi i/(n-1)}$. The map F_λ is conjugate to F_μ if and only if $\mu = \nu^{j(d+1)}\lambda$ or $\mu = \nu^{j(d+1)}\bar{\lambda}$ for some integer j.

So all centers whose parameters are of the form $\nu^k\lambda$ or $\nu^k\bar{\lambda}$ where $k \equiv j(d+1) \bmod (n-1)$ have conjugate dynamics.

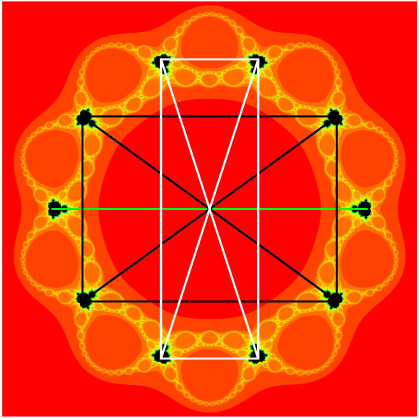
Conjugacies

gcd conjugacy

Let $g = gcd(n-1, d+1)$.

If $\lambda = \nu^g \mu$ or $\lambda = \nu^g \bar{\mu}$, then F_λ and F_μ have conjugate dynamics.
Therefore, $\mathcal{M}_i \equiv \mathcal{M}_{i+g}$ for all $i = 0, 1, \ldots, n-1$.

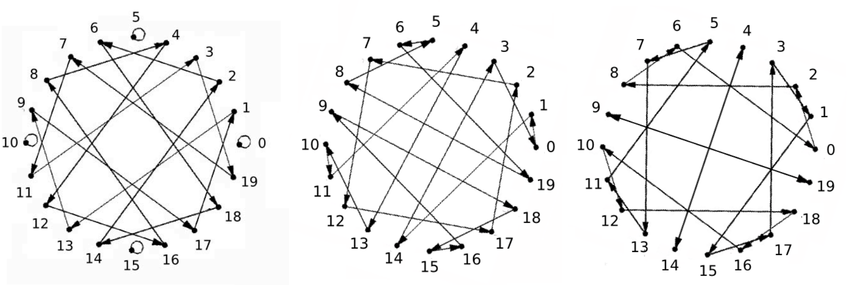
Example: $n = 11$ and $d = 4$



$n = 11,\ d = 4$

- Here, $n - 1 = 10$ and $d + 1 = 5$. Hence, $g = 5$.
- $\mathcal{M}_0 \equiv \mathcal{M}_5$,
  $\mathcal{M}_1 \equiv \mathcal{M}_6 \equiv \mathcal{M}_4 \equiv \mathcal{M}_9$,
  $\mathcal{M}_2 \equiv \mathcal{M}_7 \equiv \mathcal{M}_3 \equiv \mathcal{M}_8$
- As we will see, we have 3 conjugacies classes for this example.

Example: $n = 13$ and $d = 7$



If $n = 13$ and $d = 7$, then $g = 4$. There are three conjugacy classes. This figure contains one orbit diagram for each of the three classes.

Number of Conjugacy Classes

We can also obtain a count of the number of conjugacy classes for a given n and d.

Theorem

If g is even, there are $1 + g/2$ conjugacy classes for F_λ. If g is odd, there are $(g + 1)/2$ such conjugacy classes

Summary

- We have a classification of the dynamics for all of the principal Mandelbrot sets.
- What about the others? Various results are known, but so far there is no dynamical invariant that handles all cases.

Thank you for your attention.