LECTURE NOTES FOR THE KEIO/BOSTON UNIVERSITY DYNAMICAL SYSTEMS WORKSHOP: DYNAMICS OF COUPLED GYROSCOPES

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Abstract. Gyroscopes are mechanical device for measuring and maintaining orientation. Advances in manufacturing techniques in microelectromechanical systems (MEMS) allow for mass manufacturing of low-cost and miniaturized vibratory gyroscopes. Small disturbances, such as thermo interference, can increase phase drifts in the oscillatory signal and give inaccurate results. To remedy the aforementioned problem, researchers are considering networks of coupled MEMS gyroscopes. Experimental and numerical studies have shown that networked MEMS gyroscopes can increase the sensitivity while minimizing phase drift. In this note, we study a network of symmetrically coupled gyroscopes in a Hamiltonian setting. We first investigate the effects of coupling topology on the gyroscopic array. Normal form techniques are used to obtain the equations of motion of the reduced system. The techniques outlined here are applicable to generic symmetric Hamiltonian networks.

1. MEMs Gyroscopes

From the recent numerical work by Vu et al. [2011], we discovered that coupling multiple vibratory gyroscopes together as a navigation system can minimize the effects of noise, material imperfections, phase drift, and power consumption relative to a single device configuration. From experiments, we know that the damping and forcing coefficients have a relative small scale compared to other system parameters. The equations of the network model can be written as a Hamiltonian system and the dynamics can be studied as perturbations of the Hamiltonian structure. In this section, we develop a differential equation model of a network of gyroscopes in a Hamiltonian setting.

1.1. **A Single Gyrosocpe.** As seen from Figure 1, a vibratory gyroscope is represented as a spring-mass system. In this system, x and y represents the directions in the drive and

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sense modes, respectively and z represent the axis perpendicular to the xy-plane. Then, the governing equations can be written as

$$\begin{aligned} m\ddot{x} + c_x\dot{x} + \kappa_x x + \mu_x x^3 &= f_e(t) + 2m\Omega_z\dot{y} \\ m\ddot{y} + c_y\dot{y} + \kappa_y y + \mu_y y^3 &= - 2m\Omega_z\dot{x}. \end{aligned}$$

where m is the proof mass, Ω_z is the angular rate of rotation along the z-axis, c_x (c_y), κ_x (κ_y) and μ_x (μ_y) are the damping, spring and nonlinear constants along the x- (y-) directions, respectively. Typical forcing term has sinusoidal form $f_e(t) = A_d \cos w_d t$. Coriolis forces appear in the driving and sensing modes as $F_{cx} = 2m\Omega_z\dot{y}$ and $F_{cy} = -2m\Omega_z\dot{x}$.

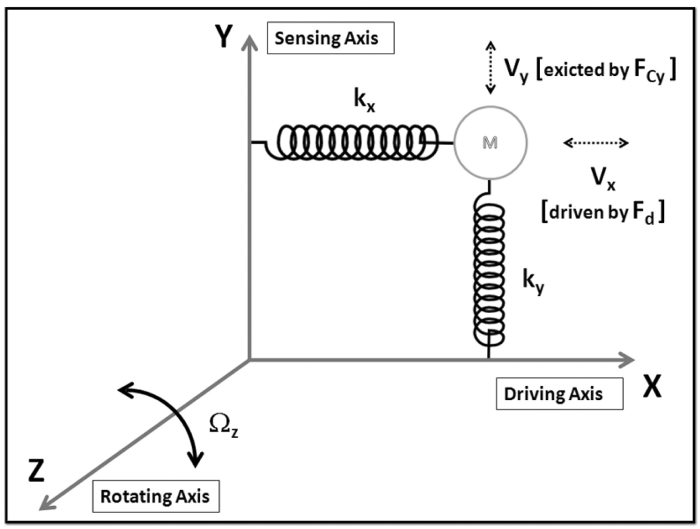


FIGURE 1. A diagram of a vibratory gyroscope system. A known driving force induces the spring-mass system to vibrate in the x-axis. An external rotating force, perpendicular to the xy-plane induces oscillations in the y-direction by transferring energy through the Coriolis force. Using the known driving force as a reference, the measured oscillations can be used to detect and quantify the rate of rotation

1.2. **A Network of Gyroscope.** To form a network of gyroscopes, we incorporate the coupling terms into the model as

$$(1) \quad \begin{aligned} m\ddot{x}_i + c_x\dot{x}_i + \kappa_x x + \mu_x x_i^3 &= f_e(t) + 2m_i\Omega_z\dot{y}_i + \sum_{i\sim j}\lambda_{ij}h(x_i, x_j) \\ m\ddot{y}_i + c_y\dot{y}_i + \kappa_y y + \mu_y y_i^3 &= - 2m_i\Omega_z\dot{x}_i, \end{aligned}$$

where $i \sim j$ denotes all the j^{th} gyroscopes that are coupled to the i^{th} gyroscope, λ_{ij} denotes the coupling strength constant, and $h(x_i, x_j)$ represents the coupling function.

1.3. **Hamiltonian Formulation.** As mentioned earlier, the damping and the forcing coefficients are relatively small in comparison to other parameters in the system. Thus, we may use the Hamiltonian approach to study the dynamics. First, we discard the damping and the forcing terms in (1) by assuming $c_x = c_y = 0$ and $f_e(t) = 0$. We further assume that the coefficients are identical in both directions and for each gyroscope and let $q_i = (q_{i1}, q_{i2})^T = (x_i, y_i)^T$ be the configuration components and $p_i = m\dot{q}_i + Gq_i$. We can write the Hamiltonian form of i-th gyroscope as

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} -\frac{G}{m} & \frac{1}{m} I_2 \\ -(K - \frac{1}{m}G^2 - \lambda\Gamma h_{x_i}(0,0,0)) & -\frac{G}{m} \end{pmatrix} \begin{pmatrix} q_i \\ p_i \end{pmatrix} + \begin{pmatrix} 0 \\ -f_i + \lambda\Gamma\left(h(x_{i-1}, x_i, x_{i+1}) - h_{x_i}(0,0,0)x_i\right) \end{pmatrix},$$

where $G = \begin{pmatrix} 0 & -m\Omega \\ m\Omega & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $K = \mathrm{diag}(\kappa, \kappa)$, and $f_i = \begin{pmatrix} \mu x_i^3 \\ \mu y_i^3 \end{pmatrix}$.

2. System Analysis

Given the complexity of the gyroscopic network, we will show the advantages of using the Hamiltonian approach in this section. First, topology of the coupled gyroscopic systems are considered. We discuss different topologies and their effects on the network. Then, taking advantage of the structure of the network, we use symmetry methods to simplify the system. Eigenvalues and critical bifurcation point of a symmetry-breaking bifurcation are obtained.

2.1. **Topological Considerations.** We begin our analysis by considering the system as unidirectionally and bidirectionally connected rings. These configurations are illustrated in Figure 2. Let $Y_i = (q_i, p_i)^T$, then we may write the system as

$$\dot{Y} = MY + F(Y),$$

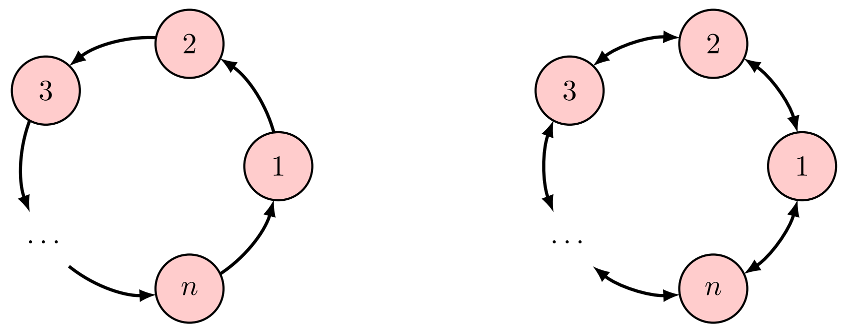
where $Y = (Y_1, \ldots, Y_n)$ and $F(Y) = (f_1, \ldots, f_n)$. Let 0_i and I_i denote the $i \times i$ zero and identity matrices. Then, we can define the skew-symmetric matrix

$$J_{2i} = \begin{bmatrix} 0_i & I_i \\ -I_i & 0_i \end{bmatrix}.$$

Recall that a Hamiltonian matrix must satisfy the condition

$$M^T J + JM = 0,$$

where $J = \text{diag}\,\overbrace{(J_4, \ldots, J_4)}^{n \text{ times}}$. We can show that the linear system of the unidirectional case does not satisfy the Hamiltonian condition, but the bidirectional case does. The proof is left to the reader as an exercise or see Buono et al. [2014].



(A) Unidirectionally coupled gyroscopes. Coupling function: $h(x_{i-1}, x_i, x_{i+1}) = x_{i+1} - x_i$.

(B) Bidirectionally coupled gyroscopes. Coupling function: $h(x_{i-1}, x_i, x_{i+1}) = (x_{i+1} - x_i) + (x_{i-1} - x_i)$.

FIGURE 2. The coupled gyroscopic systems represented as directed graphs.

2.2. **Isotypic Decomposition.** Given that the unidirectional ring is not Hamiltonian, we will not further consider this case. To take advantage the symmetric structure present in the bidirectional ring, we construct a transformation matrix P so that, under this coordinate transformation, the linear system of the network decomposes into its isotypic components. Details for constructing this matrix for the gyroscopic network can be found in Buono et al. [2014]. For general symmetric networks, please see Golubitsky et al. [1988] and the references therein.

Let P be a matrix with the aforementioned properties and $U = PY$, then the system can be written as

$$\dot{U} = \mathcal{M}U + \mathcal{F}(U).$$

When n is odd, the linear part of the system is

$$\mathcal{M} = \text{diag}\left(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_1, \ldots, \mathcal{M}_{\lfloor n/2 \rfloor}, \mathcal{M}_{\lfloor n/2 \rfloor}\right),$$

where

$$\mathcal{M}_j = \begin{pmatrix} -\frac{G}{m} & \frac{1}{m} I_2 \\ -(K - \frac{1}{m}G^2 + 2\lambda\Gamma(1 - \cos(2\pi j/N)) & -\frac{G}{m} \end{pmatrix}.$$

Similarly, when n is even, the linear part of the system is

$$\mathcal{M} = \text{diag}\left(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_1, \ldots, \mathcal{M}_{n/2-1}, \mathcal{M}_{n/2-1}, \mathcal{M}_{n/2}\right).$$

The block diagonalization of the linear system allows us to calculate the eigenvalues and they are

$$\rho_j^{\pm} = \frac{1}{\sqrt{m}}\sqrt{-\left(\kappa + 2m\Omega^2 + \lambda\left(1 - \cos\frac{2\pi j}{n}\right)\right) \pm \sqrt{s_j}} \text{ and } -\rho_j^{\pm},$$

where $s_j = 4m\Omega^2(\kappa + m\Omega^2 + \lambda(1 - \cos(2\pi j/n))) + \lambda^2(1 - \cos(2\pi j/n))^2$. More importantly, we also determined the critical point of a symmetry-breaking bifurcation and it is $\lambda^*_{\lfloor n/2 \rfloor} = \dfrac{-\kappa}{2(1 - \cos(2\pi j/n))}$.

3. Symmetric Normal Form

Due to the topological structure of the gyroscopic network, we were able to put the linear system in block diagonal form and obtain critical information on the bifurcation dynamics of the system. To better understand the bifurcating solutions, we will determine the corresponding normal form. The linear normal form can be obtained by finding a suitable symplectic linear transform. For the higher order terms, we apply invariant theory to simplify the calculations.

3.1. **Linear Hamiltonian Normal Form.**

To begin, we must find a transformation matrix Q that would put the linear part of the system into Hamiltonian linear normal form. Details of finding this symplectic transformation matrix can be found in Meyer et al. [2008] and Burgoyne and Cushman [1974]. Due to the length of the discussion, we again refer the reader to Buono et al. [2014] for the specific details of the symplectic transformation Q.

Assume that we have obtained Q and let $X = QU$, the system can be written in the following form

$$\dot{X} = \mathbf{M}X + \mathbf{F}(X).$$

More importantly, the Hamiltonian of the network can be written as

$$H(X) = \tilde{H}_0(X) + H_2(X).$$

where $\tilde{}$ denotes a function already in normal form, $\tilde{H}_0(X)$ and $H_2(X)$ represent polynomials of degree two and four, respectively. We note that H_1 terms are not present in the Hamiltonian function because there are no quadratic terms present in the system.

3.2. **Nonlinear Hamiltonian Normal Form with Symmetry.**

Using the linear terms, we may proceed to simplify the higher-order terms of the Hamiltonian. For Hamiltonian systems without symmetry, we need to use the method of Lie triangle [Meyer et al., 2008]. For discussion on the normal form of symmetric systems, please see Golubitsky et al. [1988]. In the bidirectional case, the system has dihedral (D_n) symmetry. We will take advantage of this topological structure and simplify our calculations.

Recall that a function is invariant under symmetry group Γ if $H(\gamma X) = H(X)$ for all $\gamma \in \Gamma$. Since the system is D_n-symmetric, the normal form of its Hamiltonian must be D_n-invariant. In order words, the terms in $H_2(X)$ that are also D_n invariant are the only terms that could be part of the symmetric normal form. Thus, we first calculate the invariants of the system. Comparing the list of invariants and the terms in $H_2(X)$, we may drop any terms that are not part of the invariant set and obtain the symmetric normal form for the higher order terms. Neusel [2007] and Paule and Sturmfels [2008] are references for invariant theory and they provide algorithms on calculating the invariants.

As a result, we may write a further simplified Hamiltonian function as

$$\tilde{H}(X) = \tilde{H}_0(X) + \tilde{H}_2(X) + \mathcal{O}(X^5).$$

For example, when $n = 3$, the Hamiltonian of the system in normal form is

$$\tilde{H}(X) = \frac{\rho_1}{2}(x_{1,1}^2 + x_{1,3}^2) + \frac{\rho_2}{2}(x_{1,2}^2 + x_{1,4}^2) + \frac{1}{2}(x_{2,2}^2 + x_{3,2}^2) + \frac{\sqrt{\kappa + 4\Omega^2}}{2}(x_{2,1}^2 + x_{3,1}^2) + \frac{\sqrt{\kappa + 4\Omega^2}}{2}(x_{2,3}^2 + x_{3,3}^2)\frac{1}{8}\,\frac{\mu\,\kappa^2}{(\kappa + 4\,\Omega^2)^2}\left(x_{2,4}^2 + x_{3,4}^2\right)^2.$$

4. Conclusion

In this note, we used MEMS gyroscopes as a motivating example to study coupled dynamical systems. We outlined the methods used in analyzing a bidirectionally coupled gyroscopic system. The steps outlined in this method can be used for other networks with symmetry [Matus-Vargas et al., 2014., Chan et al., 2014]. For future work on Hamiltonian networks, one can set the problem within the groupoid formalism by Golubitsky et al. [2005]. In this context, we may study networks that do not have symmetric spatial structure. Topological criteria and generic bifurcations for Hamiltonian coupled cell networks are currently underway in Chan [2014].

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