Variational Methods and Open Problems in Celestial Mechanics

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Strict Instructions:

Design talk for non-specialists, particularly graduate students.

Useful tip -

If you are asked for a title and abstract way in advance of a talk...

Either

1. Be vague...

2. Change the title when you give the talk...

A Variational Techniques and Saari's Conjecture for the N-Body Problem

Even change the author ~~G.R. Hall~~

Work of Boston U.

Eric Wahl with Aaron Hoffman

Ph.D. Boston U — Post doc B.U.

Lincoln Labs — Olin College

Reference Eric Wahl's Thesis - BU - 2008

Recall

The Newtonian N-body Problem

Let $q_1(t), q_2(t), \ldots, q_N(t) \in \mathbb{R}^2$ (or $\mathbb{R}^3$)

"positions"

and

$m_1, m_2, \ldots m_N > 0$ (or ≥ 0, see below)

"masses"

Newtonian N-body problem

$$m_i \ddot{q}_i(t) = \sum_{j \neq i} m_i m_j \frac{(q_j - q_i)}{\|q_j - q_i\|} \frac{1}{\|q_j - q_i\|^2}$$

$i = 1, \ldots, n$

More concise form

$$m_i \ddot{q}_i = \sum_{j \neq i} m_i m_j \frac{(q_j - q_i)}{\|q_j - q_i\|} \frac{1}{\|q_j - q_i\|^2}$$

Let $U = \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}$

then $\frac{\partial U}{\partial q_i} = \sum_{j \neq i} m_i m_j \frac{(q_j - q_i)}{\|q_j - q_i\|^2} \frac{1}{\|q_j - q_i\|^2}$

So rewrite

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} \qquad i = 1, \ldots, n$$

Or $\bar{q} = (q_1, q_2, \ldots, q_n)$

$$M = \begin{bmatrix} m_1 m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix}$$

$$M \ddot{\bar{q}} = \nabla U|_{\bar{q}}$$

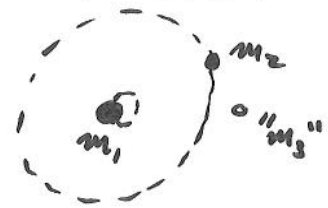
Build M into inner product $\bar{q}$ space

"Solving" is impossible

Happy with special cases:

1. Limiting problems where look at system when some of the masses $\to 0$

"Restricted 3-body Problem"



m_1 and m_2 attract each other and m_3, m_3 has no effect on m_1, m_2

"2+1" Body problem

Study "N+M" body problems.
N+M+J

2. Limit dimension

(all N bodies on a line

or N bodies and the velocities

have some symmetry).

3. Look for solutions with

some specific property

All of these yield interesting

and difficult questions with

many open problems...

Focus on one "recent" problem...

(In my life time...)

Define $I(\bar{q}) = I(q_1, q_2, \dots q_N)$

$$= \sum_{i=1}^{N} m_i \|q_i\|^2$$

<u>Moment</u> <u>of</u> <u>inertia</u>

measures the "size" of a

configuration $q_1, q_2, \dots q_N$.

Remark: $I(\bar{q})$ just one number

$$\overset{\text{same}}{\underset{I}{=}}$$

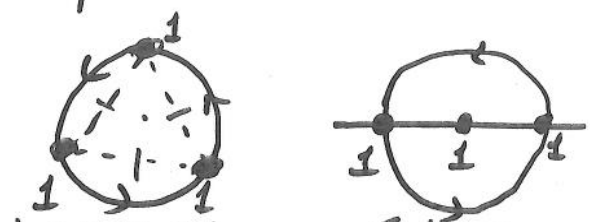
so there are many different configurations of the same size.

Special Solution:

A solution $q_1(t), \dots, q_N(t)$ is called a <u>relative equilibrium</u> solution if $q_1(t), q_2(t), \dots, q_N(t)$ is a fixed point (equilibrium solution) <u>in rotating coordinates</u>.

That is, each $q_i(t)$ follows a circle with center at $\bar{0}$ and all bodies have the same period.

<u>For Example</u>



Lagrange Euler

<u>Many</u> open questions remain concerning relative equilibria (!!)

Saari observed that if $q_1(t), \dots, q_N(t)$ is a relative equilibrium solution, then $I(q_1(t), \dots q_N(t)) = \text{constant}$.

Saari's Conjecture (1970):

The only solutions of the n-body problem with constant moment of inertia are relative equilibria.

Known

McCord (04): True - 3-Body
Equal Mass

Moeckel (05): True - 3-Body
Arbitrary Mass
Arbitrary Dimension
(Alg. Geom techniques)

Subtle Problem:

Roberts (05): False: If you allow negative "masses"

False: For inverse square potential (inverse cube force)

Make life easier...

Set $m_1, m_2, \ldots m_n = 1$. (Equal mass).

Then $U(\bar{q}) = U(q_1, \ldots, q_n)$

$$= \sum_{i<j} \frac{1}{\|q_i - q_j\|}$$

Note $U(\alpha \bar{q}) = \sum_{i<j} \frac{1}{\|\alpha q_i - \alpha q_j\|}$

$$= \frac{1}{\alpha} U(\bar{q})$$

U is homogenious. --

So $\nabla U|_{\bar{q}} \cdot \bar{q}$ = directional derivative in direction $\bar{q}$

$$= \lim_{h \to 0} \frac{U(\bar{q} + h\bar{q}) - U(\bar{q})}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{1+h} U(\bar{q}) - U(\bar{q})}{h}$$

$$= -U(\bar{q}).$$

So if $\bar{q}(t)$ is a solution

$(\ddot{\bar{q}} = \nabla U|_{\bar{q}})$

with $I(\bar{q}(t)) = \sum_{i=1}^{n} \|q_i\|^2 = \bar{q} \cdot \bar{q}$

constant.

Then $\frac{d I(\bar{q}(t))}{dt} = 2\dot{\bar{q}} \cdot \bar{q} = 0$

$\frac{d^2 I(\bar{q}(t))}{dt} = 2\ddot{\bar{q}} \cdot \bar{q} + 2\dot{\bar{q}} \cdot \dot{\bar{q}} = 0$

i.e. $2 \nabla U|_{\bar{q}} \cdot \bar{q} + 2\|\dot{\bar{q}}\|^2 = 0$

i.e. $U(\bar{q}) = -\|\dot{\bar{q}}\|^2$.

But $\|\dot{\bar{q}}\|^2 = 2 \cdot$ Kinetic Energy

$-U(\bar{q}) =$ potential Energy

and Total Energy

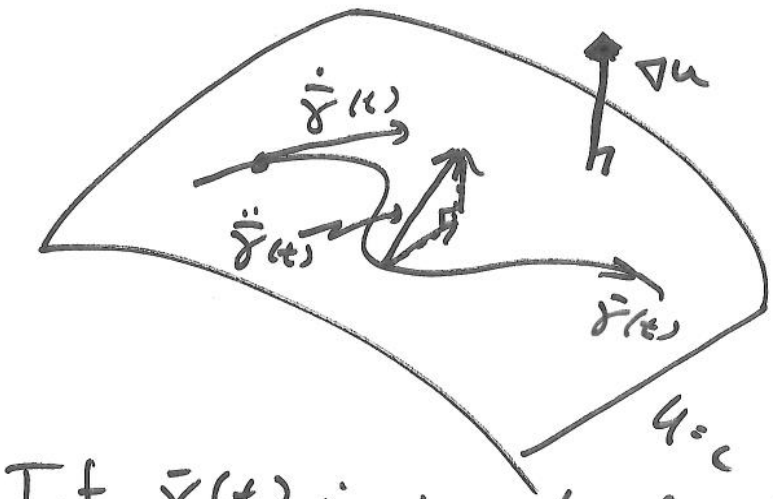
$$H(\bar{q}) = \frac{\|\dot{\bar{q}}\|^2}{2} - U(\bar{q})$$

is conserved (constant) along solutions.

So... both $\|\dot{\bar{q}}\|^2$ and $U(\bar{q})$ are constant along solutions...

I.E. a solution with constant moment of inertia also has constant potential energy. $(U(\bar{q}(t)))$.

But wait



If $\bar{\gamma}(t)$ is in a level set $\{\bar{q} : U(\bar{q}) = c\}$ (so $U(\bar{\gamma}(t)) = c$ for all t) then $\dot{\bar{\gamma}}$ is tangent to the level set and $\ddot{\bar{\gamma}}$ has components tangent and $\perp$ to surface (parrallel to ∇U)

Recall: a curve $\bar{\gamma}(t)$ is a geodesic on a level set U = constant if $U(\bar{\gamma}(t))$ = constant and

$$\ddot{\bar{\gamma}}(t) = \lambda(t)\, \nabla U\big|_{\bar{\gamma}(t)}$$

for scalar function $\lambda(t)$...

So, to find counter examples to Saari's conjecture

1. Find geodesics on U = constant surface $\ddot{\bar{\gamma}} = \lambda(t)\, \nabla U\big|_{\bar{\gamma}(t)}$
2. Check if $\lambda(t) \equiv 1$.

... $\Rightarrow I(\bar{\gamma}(t))$ constant and $\ddot{\bar{\gamma}} = \nabla h\big|_{\gamma}$

$\bar{\gamma}$ solution. ($\lambda(t)$ = constant O.K.)

Only such geodesics I know about are relative equilibria...

But - gives us another idea...

Suppose you find a geodesic on U = constant, say $\bar{\gamma}(t)$

So $\ddot{\bar{\gamma}}(t) = \lambda(t)\, \nabla U\big|_{\bar{\gamma}(t)}$

and $|\lambda(t) - 1|$ is small?

Use that geodesic to find a nearby solution

$|\bar{q}(t) - \bar{\gamma}(t)|$ small

$\ddot{\bar{q}}(t) = \nabla U|_{\bar{q}}$

(So far $U(\bar{q}(t))$ not constant).

<u>Idea</u> Newton's Method:

Suppose $\bar{\gamma}(t)$ has $U(\bar{\gamma}(t)) \equiv c$

and $\ddot{\bar{\gamma}}(t) = \lambda(t) \nabla U|_{\bar{\gamma}(t)}$

i.e. $\bar{\gamma}$ is a geodesic in $\{U = c\}$

Also suppose $|1 - \lambda(t)|$ small

Look for $\bar{q}(t) = \bar{\gamma}(t) + \bar{\eta}(t)$

such that $\ddot{\bar{q}}(t) = \nabla U|_{\bar{q}(t)}$

(a solution!)

(<u>Not</u> requiring $U(\bar{q}(t))$ constant

Hope $\bar{q} \approx \bar{\gamma}$ or $\bar{\eta}$ small)

Want $\ddot{\bar{q}} = \nabla U|_{\bar{q}}$

or $\nabla U|_{\bar{q}} = \ddot{\bar{q}} = \ddot{\bar{\gamma}} + \ddot{\bar{\eta}}$

$= \lambda(t) \nabla U|_{\bar{\gamma}} + \ddot{\eta}$

So we need

$\ddot{\bar{\eta}} = \nabla U|_{\bar{q}} - \lambda(t) \nabla U|_{\bar{\gamma}}$

$$\ddot{\bar{\eta}} = \nabla U|_{\bar{q}} - \lambda(t)\nabla U|_{\bar{\gamma}}$$

So

$$\ddot{\bar{\eta}} = \nabla U|_{\bar{\gamma}+\bar{\eta}} - \lambda(t)\nabla U|_{\bar{\gamma}}$$

$$\ddot{\bar{\eta}} = \nabla U|_{\bar{\gamma}} + \bar{\eta} D^2 U|_{\bar{\gamma}} + \mathcal{O}(\|\eta\|^2) - \lambda(t)\nabla U|_{\bar{\gamma}}$$

So $$\ddot{\bar{\eta}} = (1-\lambda(t))\nabla U|_{\bar{\gamma}} + \bar{\eta} D^2 U|_{\bar{\gamma}} + \mathcal{O}(\|\bar{\eta}\|^2)$$

...

Choose $\bar{\eta}_1(t)$ so that

$$\ddot{\bar{\eta}}_1 = \bar{\eta}_1 D^2 U|_{\bar{\gamma}} + \underbrace{(1-\lambda(t))\nabla U|_{\bar{\gamma}}}_{\text{Know } \bar{\gamma}(t)}$$

Let $\bar{q}_1 = \bar{\gamma} + \bar{\eta}_1$

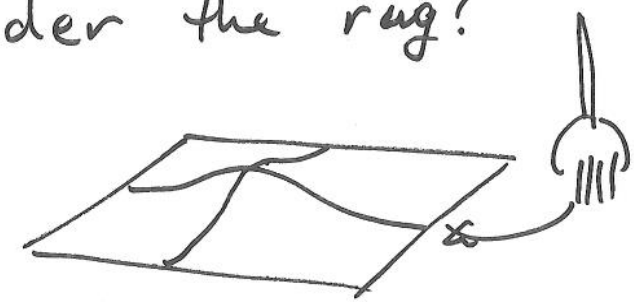
Show $\bar{q}_1$ closer to being a solution than $\bar{\gamma}$. Repeat...

(Hope) Process converges to a solution $\bar{q}(t)$ with $\|\bar{q}-\bar{\gamma}\|$ bounded with $\|1-\lambda(t)\|$.

In fact: Works.

Converges very quickly

What has been swept under the rug?



1. Better to work with a 1st order system -

$$\begin{cases} \dot{\bar{q}}_i = p_i/m_i \\ \dot{p}_i = \frac{\partial U}{\partial q_i} \end{cases} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix}$$

$$H(\bar{q}, \bar{p}) = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i} - U(\bar{q})$$

Given $(\bar{\gamma}(t), \bar{\xi}(t))$ with $U(\bar{\gamma}(t))$ constant solution for

$$H_1(\gamma, \xi) = \sum \frac{\|\xi_i\|^2}{m_i} - \lambda(t) U(\bar{\gamma})$$

2. Need initial or boundary condition to solve D.E. of Newton's Method step --

Start with periodic geodesic - Look for periodic solution

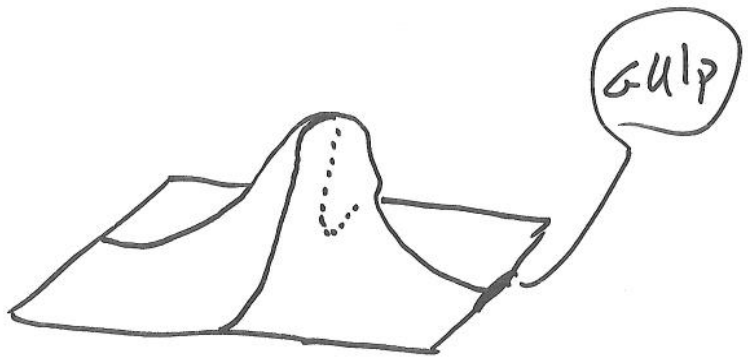
<u>OR</u>

Look at geodesics and solutions with specific boundary conditions

3. Troublesome direction - Equations degenerate in rotation direction because of symmetry...

choose good coordinates

4. Avoid singularities (collisions!)



#3 above choosing good coordinates to deal with rotation symmetry...

<u>Planar</u> 3-body Problem (back to $m_1 = m_2 = m_3 = 1$) in $\mathbb{R}^2$

$$U : (\mathbb{R}^2)^3 = \mathbb{R}^6 \to \mathbb{R}$$

Fix center of mass $= \bar{0}$

$(q_1 + q_2 + q_3 = \bar{0})$

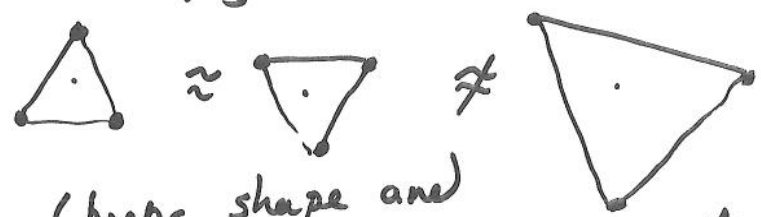
Nice linear condition

$$\tilde{U} : \mathbb{R}^4 \to \mathbb{R}$$

$\tilde{U} : \mathbb{R}^4 \to \mathbb{R}$ invariant under rotation

Form "shape space"

$\mathbb{R}^4 / S^1$: 3dimensional



(keeps shape and size of configuration - not orientation (up to rotation)

$$\hat{\tilde{U}} : \underbrace{\text{shape Space}}_{\text{3-dim}} \to \mathbb{R}.$$

Rk. Lost information --

E.g. Relative equilibrium solutions don't change shape = pts in shape space

What does

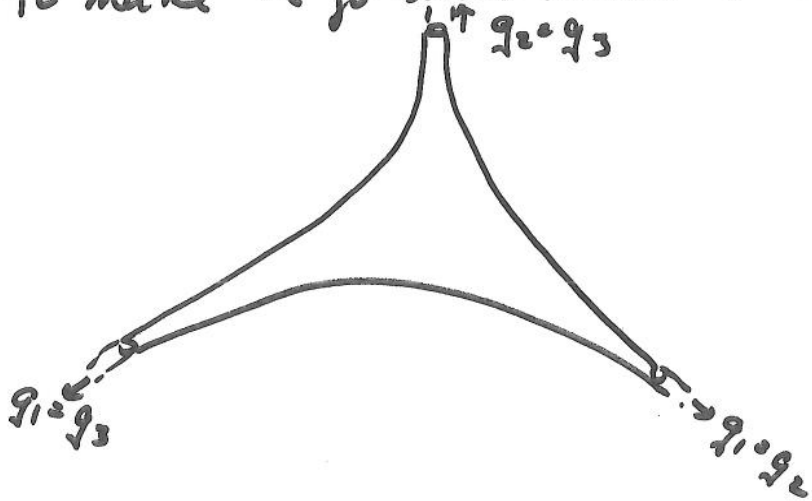
$$\{\hat{\hat{U}} = c\}$$

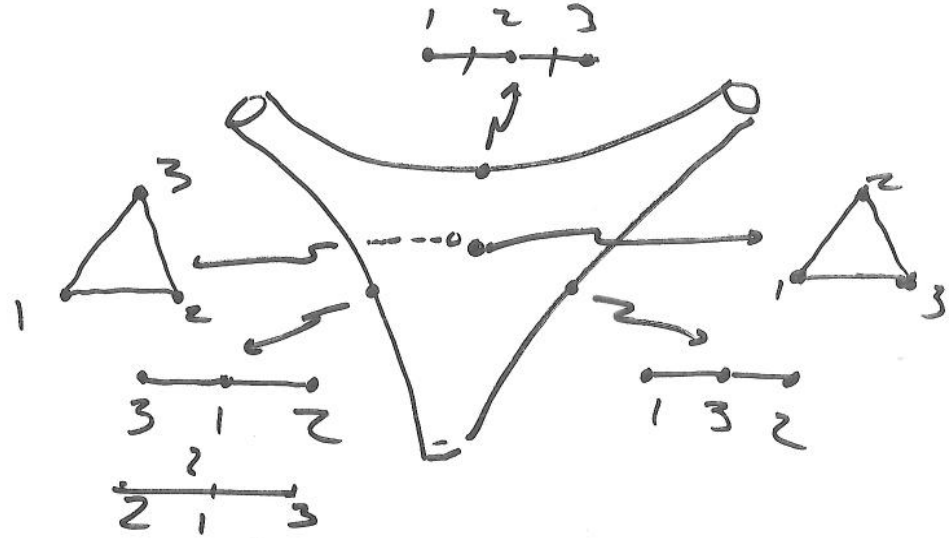
Look like in shape space?

Fix c -- bring q_1 close to q_2

U goes up so must push q_3 away

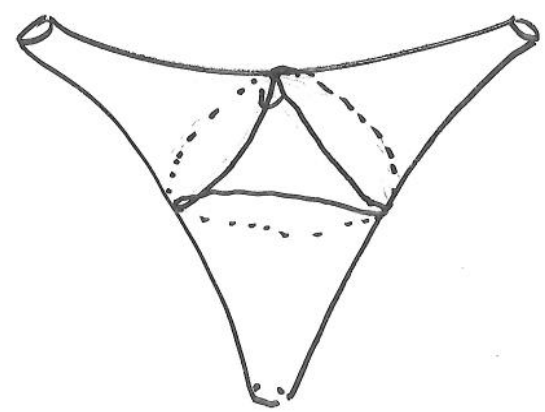
to make U go back down to c





Geodesics

(with $\lambda(t) \approx 1$)?



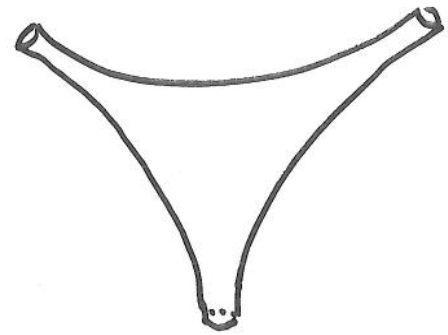
This geodesic has $\lambda(t) \approx 1$
-- at least close enough for method to converge!

Converges to

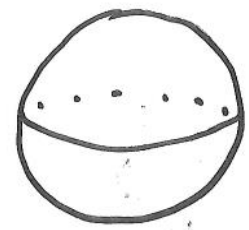
Figure 8
(Chenciner + Montgomery)

Then Eric graduated...
But not before realizing these pictures give a different way to think about Saari's conjecture (and relative equilibria)

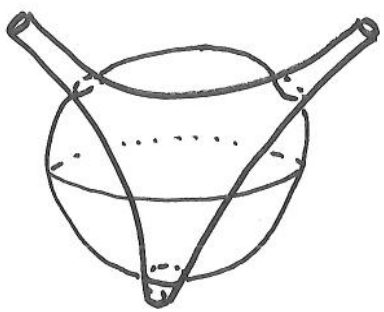
U = constant in shape space



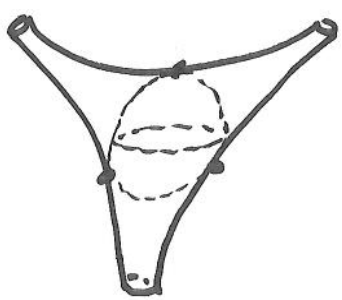
I = constant in shape space



So counter examples of Saari's conjecture live on the intersection of $U \cdot$ constant and a sphere



Also relative equilibria appear when ever intersection is a point



Open Questions

General Case Saari's Conjecture.

Generalized Saari's Conjectures...

Eg. Seek solutions with

$$U(\bar{q}) \cdot (I(\bar{q}))^{1/2}$$

constant.

(www.uci.edu/~dsaari/conjecture-revisited.pdf)

Which geodesics on $U \cdot$ constant correspond to solutions?..

(what does "correspond" mean?)