Variational Methods and
Open Problems in
Celestial Mechanics

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Strict Instructions:
Design talk for non-specialists, particularly graduate students.

Useful tip-
If you are asked for a title and abstract way in advance of a talk...
Either

1. Be vague...
2. Change the title when you give the talk...

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A Variational Technique and Saari’s Conjecture for the N-Body Problem

Even change the author: G.D. Hall

Work of Boston U.

Eric Wahl with Aaron Hoffman
Ph.D. Boston U Post doc B.U.
Lincoln Labs
M.I.T.

Reference Eric Wahl’s Thesis - BU - 2003

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Recall

The Newtonian N-body Problem

Let \( q_1(t), q_2(t), \ldots, q_n(t) \in \mathbb{R}^n \) (or \( \mathbb{R}^3 \))

"positions"

and

\( m_1, m_2, \ldots, m_n > 0 \) (or \( \geq 0 \), see below)

"masses"

Newtonian N-body problem

\[
m_i \ddot{q}_i(t) = \sum_{j \neq i} m_i m_j \frac{(q_i - q_j)}{||q_i - q_j||^3} \frac{1}{||q_i - q_j||^2} \\
\]

\( i = 1, \ldots, n \)
More concise form
\[ m_i \ddot{q}_i = \sum_{j \neq i} m_i m_j \frac{(q_j - q_i)}{\|q_j - q_i\|^3} \]

Let \[ U = \sum_{i,j} \frac{m_i m_j}{\|q_i - q_j\|} \]
then \[ \frac{dU}{dt} = \sum_{i,j} \frac{m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3} \frac{1}{\|q_j - q_i\|^2} \]

So rewrite \[ m_i \ddot{q}_i = \frac{dU}{dt} \quad i = 1, \ldots, n \]

Or \[ \ddot{q} = (\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_n) \]

\[ M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_n \end{bmatrix} \]

Build inner product \[ M \ddot{q} = \nabla U \quad \text{in space} \]

"Solving" in impossible
Happy with special cases:
1. Limiting problems where look at system when some of the masses \( \to 0 \)

"Restricted 3-body Problem"

\[ m_1, m_2 \text{ attract each other and } m_3, \]
\[ m_3 \text{ has no affect on } m_1, m_2 \]

"2+1" Body problem
Study "N+M" body problems.
\[ N+M+5 \]
2. Limit dimension
(all N bodies on a line
or N bodies and the velocities
have some symmetry).

5. Look for solutions with
some specific property

All of these yield interesting
and difficult questions with
many open problems...

Focus on one "recent" problem...
(In my life time...)

Define \[ I(q) = I(q_1, q_2, ..., q_N) = \sum_{i=1}^{N} m_i \| q_i \|^2 \]

Moment of inertia
measures the "size" of a
configuration \( q_1, q_2, ..., q_N \).
Remark: $I(\tilde{q})$ just one number
\[ I \geq 1 \]
so there are many different configurations of the same size.

Special Solution:
A solution $q_1(t), \ldots, q_n(t)$ is called a relative equilibrium solution if $q_1(t), q_2(t), \ldots, q_n(t)$ in a fixed point (equilibrium solution) in rotating coordinates.

That is, each $q_i(t)$ follows a circle with center at $0$ and all bodies have the same period.

For example,

Many open questions remain concerning relative equilibria (!!!)
Saari observed that if $q_1(t), \ldots, q_n(t)$ in a relative equilibrium solutions then $I(q_1(t), \ldots, q_n(t)) = \text{constant}$. 
Saari's Conjecture (1970):
The only solutions of the $n$-body problem with constant moment of inertia are relative equilibria.

**Known**
McCord: True - 3-Body
Equal Mass

Moeckel: True - 3-Body
Arbitrary Mass
Arbitrary Dimension
(Alg. Geo techniques)

**Subtle Problem:**

Roberts: False: If you allow negative "masses"
False: For inverse square potential (inverse cube force)
Make life easier...

Set \( m_1, m_2, \ldots, m_n = 1 \). (Equal mass).

Then \( U(g) = U(g_1, \ldots, g_n) \)
\[ = \sum_{i=1}^{n} \frac{1}{\| g_i - g_j \|} \]

Note \( U(g^0) = \sum_{i=1}^{n} \frac{1}{\| g_i - g_i \|} \)
\[ = \frac{1}{2} U(g) \]

U in homogeneous...-

So \( \nabla U_\tilde{g} : \tilde{g} \): directional derivative in direction \( \tilde{g} \)
\[ = \lim_{h \to 0} \frac{U(g + h\tilde{g}) - U(g)}{h} \]
\[ = \lim_{h \to 0} \frac{1}{h} \left[ U(g) - U(g) \right] \]
\[ = -U(g). \]

So if \( \bar{g}(t) \) is a solution
\( (\bar{g} = \nabla U_{\bar{g}}) \)
with \( I(g(t)) = \sum_{i=1}^{n} \| g_i \|^2 = \bar{g} \cdot \bar{g} \) constant.

Then \( \frac{d}{dt} I(g(t)) = 2\bar{g} \cdot \dot{g} = 0 \)

\( \frac{d^2}{dt} I(g(t)) = 2\ddot{g} \cdot \bar{g} + 2\dot{g} \cdot \dot{g} = 0 \)
i.e. \( 2 \nabla U_{\bar{g}} \cdot \bar{g} + 2 \| \dot{g} \|^2 = 0 \)
i.e. \( U(\bar{g}) = -\| \dot{g} \|^2. \)
But \( \| \dot{x} \|^2 = 2 \cdot \text{Kinetic Energy} - U(x) = \text{ Potential Energy} \)

And Total Energy

\[ H(x) = \| \dot{x} \|^2 - U(x) \]

is conserved (constant) along solutions.

So... both \( \| x \|^2 \) and \( U(x) \) are constant along solutions...

I.E. A solution with constant moment of inertia also has constant potential energy. \( U(x(t)) \).

But wait

If \( \dot{x}(t) \) is in a level set \( \{ x : U(x) = c \} \) (so \( U(x(t)) = c \) for all \( t \) ) then \( \dot{x} \) is tangent to the level set and \( \dot{x} \) has components tangent and \( L \) to surface (parallel to \( \Delta U \)).
Recall: a curve $\tilde{y}(t)$ is a geodesic on a level set $U = \text{constant}$ if $U(\tilde{y}(t)) = \text{constant}$ and 
$\tilde{y}(t) = \lambda(t) \Delta U|_{\tilde{y}(t)}$
for scalar function $\lambda(t)$...

So, to find counter examples to Saari's conjecture:
1. Find geodesics on $U = \text{constant}$ surface $\tilde{y} = \lambda(t) \Delta U|_{\tilde{y}(t)}$
2. Check if $\lambda(t) \equiv 1$

... $\Rightarrow I(\tilde{y}(t))$ constant and $\tilde{y} = \Delta U|_{\tilde{y}(t)}$ is a solution. ($\lambda(t) = \text{constant}$ O.K.)

Only such geodesics
I know about are relative equilibria...

But - gives us another idea...
Suppose you find a geodesic on $U = \text{constant}$, say $\tilde{y}(t)$

So $\tilde{y}(t) = \lambda(t) \Delta U|_{\tilde{y}(t)}$

and $|\lambda(t) - 1|$ is small?
Use that geodesic to find a near by solution
\[ |\tilde{\gamma}(t) - \gamma(t)| \text{ small} \]
\[ \tilde{\gamma}(t) = \nabla U |_{\tilde{\gamma}} \]
(So far \( U(\tilde{\gamma}(t)) \) not constant)

**Idea Newton's Method:**
Suppose \( \tilde{\gamma}(t) \) has \( U(\tilde{\gamma}(t)) \in \mathcal{C} \)
and \( \tilde{\gamma}(t) = \lambda(t) \nabla U |_{\tilde{\gamma}} \)
(i.e. \( \tilde{\gamma} \) is a geodesic in \( \{ U = c \} \))
Also suppose \( |1 - \lambda(t)| \text{ small} \)

Look for \( \tilde{\varphi}(t) = \gamma(t) + \tilde{\eta}(t) \)
such that \( \tilde{\varphi}(t) = \nabla U |_{\tilde{\varphi}} \)
(a solution!)
(Not requiring \( U(\tilde{\varphi}(t)) \) constant
Hope \( \tilde{\varphi} \approx \gamma \) or \( \tilde{\eta} \) small)

Want \( \tilde{\varphi} = \nabla U |_{\tilde{\varphi}} \)
or \( \nabla U |_{\tilde{\varphi}} = \tilde{\varphi} = \gamma + \tilde{\eta} \)
\[ = \lambda(t) \nabla U |_{\tilde{\gamma}} + \tilde{\eta} \]
So we need
\[ \tilde{\eta} = \nabla U |_{\tilde{\varphi}} - \lambda(t) \nabla U |_{\tilde{\gamma}} \]
\[ \ddot{y} = \nabla \nabla \bar{y} - \lambda(t) \nabla \nabla \bar{y} \]

So,
\[ \ddot{y} = \nabla \nabla \bar{y} + \ddot{y} \nabla \nabla \bar{y} + O(\|\eta\|^2) \]
\[ - \lambda(t) \nabla \nabla \bar{y} \]

So, \[ \ddot{y} = (1 - \lambda(t)) \nabla \nabla \bar{y} + \ddot{\eta} \nabla \nabla \bar{y} + O(\|\eta\|^2) \]

(Hope) Process converges to a solution \( \bar{y}(t) \) with \( \|\bar{y} - \bar{y}\| \) bounded with \( \|1 - \lambda(t)\| \).

In fact: Works - Converges very quickly

Choose \( \bar{y}(t) \) so that
\[ \ddot{\eta} = \ddot{\eta} \nabla \nabla \bar{y} + (1 - \lambda(t)) \nabla \nabla \bar{y} \]

Let \( \bar{y} = \bar{y} + \bar{\eta} \)

Show \( \bar{y} \) closer to being a solution than \( y \). Repeat...
What has been swept under the rug?

1. Better to work with a 1st order system -

\[
\begin{align*}
\ddot{q}_i &= \frac{\partial H}{\partial \dot{q}_i} \\
\dot{p}_i &= \frac{\partial H}{\partial q_i}
\end{align*}
\]

\[
H(q, p) = \sum \frac{m_i \dot{q}_i^2}{2} - U(q)
\]

Given \((\tilde{y}(t), \tilde{\dot{y}}(t))\) with \(U(\tilde{y}(t))\)

constant solution for

\[
H(y, \dot{y}) = \sum \frac{m_i \dot{y}_i^2}{2} - \lambda(t) U(y)
\]

2. Need initial or boundary condition to solve D.E. of Newton's Method step -- Start with periodic geodesic - look for periodic solution OR look at geodesics and solutions with specific boundary conditions

3. Troublesome direction - Equations degenerate in rotation direction because of symmetry... choose good coordinates

4. Avoid singularities (collisions!)
#3 above choosing good coordinates to deal with rotation symmetry...

Planar 3-body Problem (back to \( m_1 = m_2 = m_3 = 1 \))

\[ U : (\mathbb{R}^3)^3 \rightarrow \mathbb{R} \]

Fix center of mass = \( \overline{0} \)

\( (\overline{0}, \overline{0}, \overline{0}) \)

Nine linear condition

\[ \tilde{U} : \mathbb{R}^9 \rightarrow \mathbb{R} \]

\( U : \mathbb{R}^n \rightarrow \mathbb{R} \) invariant under rotation

Form "shape space"

\( \mathbb{R}^y / S^1 : 3 \)-dimensional

\( \bigtriangleup \cong \bigtriangleup \neq \bigtriangleup \)

(shape shape and size of configuration - not orientation (up to rotation)

\( \tilde{U} \): shape space \( \rightarrow \) \( \mathbb{R} \).

Rk. Lost information --

E.g. Relative equilibrium solutions don't change shape = pts in shape space
What does \( \{ \hat{U} = c \} \) look like in shape space?

Fix \( c \) -- bring \( g \) close to \( g_c \).

\( U \) goes up so must push \( g_3 \) away
to make \( U \) go back down to \( c \).
This geodesic has \( \lambda(t) \approx 1 \)
-- at least close enough for
method to converge!

Converges to

\[ \text{Figure 8} \]

\[ \text{(Chenciner + Montgomery)} \]

Then Eric graduated...

But not before realizing these
pictures give a different
way to think about Saari's
conjecture (and relative
equilibria)

\( U = \text{constant in shape space} \)

\( I = \text{constant in shape space} \)
So counter examples of Saari's conjecture live on the intersection of $U^\perp$ constant and a sphere.

Also relative equilibria appear when ever intersection is a point.

Open Questions

General Case Saari's Conjecture.

Generalized Saari's Conjectures...

Eq. Seek solutions with $U(g) \cdot (I(g))$ constant.

Which geodesics on $U^\perp$ constant correspond to solutions?...

(what does "correspond" mean?)

(www.uci.edu/~dsaari/conjecture-revisited.pdf)