Patterns and Inhomogeneities in Spatially Extended Systems

Gabriela Jaramillo
University of Minnesota

1 Introduction

We study patterns and inhomogeneities in spatially extended systems. Examples of such phenomena are roll patterns in Rayleigh-Bénard convection, Turing patterns, which help explain spots and stripes in animals, spirals as models of cardiac fibrillation, patterns in large network of neurons, or target patterns in the Belousov-Zhabotinsky reaction. In this notes we will focus in two examples, striped patterns in Rayleigh-Bénard convection and target patterns in the Belousov-Zhabotinsky reaction.

In the following sections we will briefly describe the model equations we will be using. We will take the point of view that the inhomogneities, or defects, can be modeled as a perturbation of these model equations and explain the difficulty we face when trying to use methods from perturbation theory to study their effects on pattern formation. In particular, we will see that the main difficulty comes from zero being in the continuous spectrum of the linearization. To resolve this problem, we recover Fredholm properties of the linearization by considering it in the context of weighted Sobolev spaces. Finally, we will show some results for striped patterns and target patterns which were obtained using this approach.

2 Modulation Equation

In this section we describe how one derives the complex Ginzburg-Landau equation as a modulation equation for the Belousov-Zhabotinksy reaction using the multiple scales formalism. We start from a general reaction-diffusion equation

$$U_t = \Delta U + F(U,\mu), \quad U \in \mathbb{R}^m, \quad \mu \in \mathbb{R}, \tag{2.1}$$

and assume that for negative values of the parameter μ there exist a steady solution, $U_*(\mu)$. Furthermore, we suppose that for $\mu = 0$ the Jacobian of F evaluated at $U_*(0)$ has two purely imaginary eigenvalues, $\lambda_{1,2} = \pm\omega_0$, which undergo a supercritical Hopf bifurcation, and that the rest of the eigenvalues have negative real part. Since $F(U,\mu)$ is assumed to be smooth, we know there exists a center manifold that can be parametrized using the center space E^c, that is by the eigenvectors corresponding to $\lambda = \pm\omega_0$. Moreover, our assumptions imply that all solutions which are not on this center manifold will converge exponentially to it, and we can therefore distinguish between two different time scales. If we now add back a slow spatial dependence, it is reasonable to expect that the coordinates of the center manifold are of the form

$$U_*(x,t;\varepsilon) = \varepsilon A(\varepsilon x, \varepsilon^2 t)\mathrm{e}^{-\mathrm{i}\omega_0 t} + c.c., x \in \mathbb{R}^n$$

where $\varepsilon = \sqrt{\mu}$ and A is a complex number.

Inserting this Ansatz back into (2.1) we obtain a hierarchy of equations at each order of ε, and find that at order $\mathrm{O}(\varepsilon^3)$ the equations are solvable provided that A satisfies the complex Ginzburg-Landau equation,

$$A_t = (1 + \mathrm{i}\alpha)\Delta A + A - (1 + i\gamma)A|A|^2. \tag{2.2}$$

Here α and γ are parameters that are specific to the original reaction diffusion system. For more information regarding the derivation of the complex Ginzburg-Landau equation using the multiple scales formalisms see Kuramoto's book [4], and for a more detailed description of how one parametrizes the center manifold see [1].

In the case of striped patterns in Rayleigh-Bénard convection we turn to a phenomenological model, the anisotropic Swift-Hohenberg equation,

$$u_t = -(\Delta + 1)^2 u + \partial_{yy} u + u - u^3. \tag{2.3}$$

A similar method as described above shows that this equation has as a modulation equation the real Ginzburg-Landau equation

$$A_t = \Delta A + A - A|A|^2, \tag{2.4}$$

which we will use to study the effects of inhomogeneities in striped patterns.

3 Defects and inhomogeneities as perturbations

We first look at the complex Ginzburg-Landau equation in dimension 3 and ask whether defects produce target patterns in this system. In the two dimensional case, experiments show that these type of patterns only form if the chemical reaction is pocked, that is if there is an inhomogeneity in the medium. If we consider a reaction-diffusion system as a continuum of locally coupled oscillators, then the inhomogeneity represents a change in frequency of a small patch, which can be modeled as a perturbation,

$$A_T = (1 + \mathrm{i}\alpha)\Delta A + (1 + \mathrm{i}\gamma)A - (1 + \mathrm{i}\gamma)A|A|^2 + \mathrm{i}\varepsilon g(\mathbf{x})A, \quad \mathbf{x} \in \mathbb{R}^3. \tag{3.1}$$

Here ε is a small parameter and $g(\mathbf{x})$ is an algebraically localized function. Notice that we have set up the above equation in a co-rotating frame so that the spatially homogenous solution is $A_* = 1$.

On the other hand, when looking at the real Ginzburg-Landau equation as a model for convection rolls, we ask whether inhomogeneities shift the pattern's frequency. More precisely we want to know whether the stripes move to accommodate for the defect. In dimension 2, these patterns can be described as steady solutions to the real Ginzburg-Landau equation that are periodic in x and which have the form $A_*(x) = \sqrt{1 - k^2}\mathrm{e}^{\mathrm{i}kx}$. To answer this question we model the defect as a perturbation of the form

$$A_T = \Delta A + A - A|A|^2 + \varepsilon g(x, y), \tag{3.2}$$

where again $g(x, y)$ is an algebraically localized function.

What we would like to do now is to use the Implicit Function Theorem to find steady solutions to the perturbed equations. The general idea is that we want to find zeros of an operator $F(u, \varepsilon) : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$, given by,

$$F(u, \varepsilon) = Lu + N(u) + \varepsilon g.$$

where L represents the linear part, $N(u)$ represents the nonlinear terms and $\varepsilon g(x)$ is the perturbation. After expanding solutions in powers of ε,

$$u = \varepsilon u_1 + \varepsilon^2 u^2 + \varepsilon^3 u_3 + \mathrm{O}(\varepsilon^4)$$

we find that at order $\mathrm{O}(\varepsilon)$ we need to solve

$$Lu_1 = g(x).$$

This is possible provided our operator $L : \mathcal{X} \subset \mathcal{Y} \to \mathcal{Y}$ is invertible. Unfortunately, we will see that the linearizations of (3.1) and (3.2) do not have closed range.

The next best thing is for the linear operator to be a Fredholm operator. Fredholm operators have closed range, and finite dimensional kernel and cokernel. As a result we can define a projection, $P : \mathcal{Y} \to \mathrm{Ran}(\mathcal{Y})$, onto the range of L and split into two parts, an invertible operator (3.3) , which will give us solutions that depend on ε and elements of the kernel, and also a finite dimensional operator (3.4).

$$PF(u_1 + u_2; \varepsilon) = 0 \quad \Longrightarrow \quad u_2 = u_2(u_1; \varepsilon) \tag{3.3}$$

$$(I - P)F(u_1 + u_2; \varepsilon) = 0 \tag{3.4}$$

Depending on the Fredholm index of L, i = dim Ker - dim cKer, this last operator is underdetermined (if $i < 0$), or over determined (if $i > 0$). So that in the case of $i < 0$, we may need to add variables to our Ansatz in order to find full solutions, or if $i > 0$ we need to add more equations.

4 The Linearized operator

The goal now is to find steady solutions to (3.1) and (3.2) near A_*. In the case of the complex Ginzburg-Landau equation, linearizing the system with $A(\mathbf{x}) = (1 + s(\mathbf{x}))\mathrm{e}^{\mathrm{i}\phi(\mathbf{x})}$ and after a change of coordinates we obtain $L_1 : \mathcal{X} \to \mathcal{Y}$,

$$L_1 \begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} \Delta - 2 & -\alpha\Delta \\ \alpha\Delta - 2\gamma & \Delta \end{bmatrix} \begin{bmatrix} s \\ \phi \end{bmatrix} \tag{4.1}$$

(See [3] for more details on the change of coordinate used and its motivation). Notice that in usual Sobolev spaces what prevents this operator from being invertible is the Laplacian.

On the other hand, for the real Ginzburg-Landau, using $A(x, y) = (\sqrt{1 - k^2} + s(x, y))\mathrm{e}^{\mathrm{i}\phi(x,y)}$, we obtain $L_2 : \mathcal{X} \subset \mathcal{Y} \to \mathcal{Y}$.

$$L_2 \begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} \Delta - 2\tau^2 & -2k\tau\partial_x \\ \frac{2k}{\tau}\partial_x & \Delta \end{bmatrix} \begin{bmatrix} s \\ \phi \end{bmatrix} \tag{4.2}$$

where $\tau = \sqrt{1 - k^2}$. Again the Laplacian, and additionally the term ∂_x, make this operator not invertible in any translation invariant space. It is possible, by taking derivatives of each component of L, to construct an operator which replaces ∂_x with Δ, so that again the main difficulty we face is inverting the Laplacian. Here is were Kondratiev spaces come in.

5 Kondratiev Spaces

Kondratiev spaces where developed in the context of fluid problems in exterior domains and domains with corners. We denote them by $M^{s,p}_\gamma$, where $s \in \mathbb{Z}$, $p \in (1, \infty)$, and $\gamma \in \mathbb{R}$. They can be defined as the completion of $C^\infty_0(\mathbb{R}^n)$ under the norm

$$\|u\|_{M^{2,p}_\gamma} = \sum_{|\alpha| \leqslant s} \|\langle \mathbf{x} \rangle^{\gamma + |\alpha|} \partial^\alpha u\|_{L^p}.$$

where $\langle \mathbf{x} \rangle = (1 + |\mathbf{x}|^2)^{1/2}$. The main result we use comes from McOwen [5], where he shows that the Laplace operator, $\Delta : M^{2,p}_\gamma \to L^p_{\gamma+2}$, is a Fredholm operator.

Theorem 1 *Let $1 < p = \frac{q}{q-1} < \infty$, $n \geqslant 2$, and $\delta \neq -2 + n/q + m$ or $\delta \neq -n/p - m$, for some $m \in \mathbb{N}$. Then*

$$\Delta : M^{2,p}_\delta \to L^p_{\delta+2},$$

is a Fredholm operator and

(i) for $-n/p < \delta < -2 + n/q$ the map is an isomorphism;

(ii) for $-2 + n/q + m < \delta < -2 + n/q + m + 1$, $m \in \mathbb{N}$, the map is injective with closed range equal to

$$R_m = \left\{ f \in L^p_{\delta+2} : \int f(y)H(y) = 0 \, \textit{for all} \;\; H \in \bigcup_{j=0}^{m} \mathcal{H}_j \right\};$$

(iii) for $-n/p - m - 1 < \delta < -n/p - m$, $m \in \mathbb{N}$, the map is surjective with kernel equal to

$$N_m = \bigcup_{j=0}^{m} \mathcal{H}_j.$$

Here, $\mathcal{H}_j$ denote the harmonic homogeneous polynomials of degree j.

On the other hand, if $\delta = -n/p - m$ or $\delta = -2 + n/q + m$ for some $m \in \mathbb{N}$, then Δ does not have closed range.

To understand why these weighted spaces work we first look at the Laplacian in regular Sobolev spaces. Recall that $\lambda = 0$ is in the continuous spectrum of the operator $\Delta : W^{2,2} \to L^2$. In particular, this means that the range of this operator is not closed in L^2, and therefore its inverse $\Delta^{-1} : Ran(\Delta)L^2 \to L^2$ is not bounded. One way to see this is by constructing what are called Weyl sequences. For example, let $u_n = \dfrac{\chi(|x|/n)h(\mathbf{x})}{\|h(\mathbf{x})\chi(|x|/n)\|_{L^2}}$ in L^2, where $h(\mathbf{x})$ is any harmonic polynomial, and $\chi \in C_0^\infty(\mathbb{R})$ is given by

$$\chi(x) = \begin{cases} 1 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| > 2. \end{cases}$$

A short calculation shows that the sequence $y_n = \Delta u_n \to 0$ in L^2, yet $\|\Delta^{-1} y_n\|_{L^2} = \|u_n\|_{L^2} = 1$, proving that Δ^{-1} is an unbounded operator.

Roughly speaking, by defining the domain of the Laplacian as a Kondratiev space with a negative weight we remove most harmonic polynomials and prevent the construction of Weyl sequences. This makes the Laplacian's kernel a finite dimensional space and its range closed. Using the adjoint we see that increasing the weight γ, and as a result prescribing more localization, makes the Laplacian's cokernel finite dimensional and its range closed.

6 Results

Going back to our two examples, where we considered inhomogeneities with an algebraic localization, we set up the equations in Kondratiev and weighted Sobolev spaces with positive weights. Then using the results from the previous section one can show that the linearized operators are now Fredholm with a negative index. Because our system is underdetermined, we need to add variables in order to span the cokernel. These extra variables correspond to far field corrections in the phase, and in the case of the real Ginzburg-Landau equation they tell us that the striped pattern shifts is phase in order to accommodate for the inhomogeneities. On the other hand, in the case of the complex Ginzburg-Landau equation in 3 dimensions the far field corrections tell us that target patterns do not form.

Here we also give a precise statement of the two Theorems we proved using this method. In the case of striped patterns (see [3]) we show that

Theorem 2 *For $|k| < 1/\sqrt{3}$, and $g \in W^{2,2}_{\beta}(\mathbb{R}^2)$, $\beta > 2$, there exists an ε_0 small and a family of solutions of the form,*

$$A(x, y; \varepsilon, \varphi) = S(x, y; \varepsilon, \varphi)\mathrm{e}^{-\mathrm{i}\Phi(x,y;\varepsilon,\varphi)}, \quad |\varepsilon| < \varepsilon_0.$$

- *The amplitude S decays to $\sqrt{1-k^2}$ as $|\mathbf{x}| \to \infty$.*
- *The phase in the far field satisfies:*

$$\Phi(x, y; \varepsilon, \varepsilon) \sim kx + \boxed{\frac{c(\varepsilon, \varphi)}{2k\sqrt{1-k^2}} \log(|\mathbf{x}|) + \varphi} + \Phi_\infty(\varepsilon)$$

and $c(\varepsilon, \varphi) = \varepsilon c_1(\varphi) + \mathrm{O}(\varepsilon^2)$,

$$c_1(\varphi) = \frac{\sqrt{1-3k^2}}{\pi(1-k^2)} \int Im[g\ \mathrm{e}^{-\mathrm{i}(kx+\varphi)}]\ \mathrm{d}x.$$

The terms that have been highlighted represent the far field corrections. Notice that if $\varphi \neq 0$ then the phase grows as $|\mathbf{x}| \to \infty$, which is not physically possible since in this case the system would have an increase in energy. This implies that the system selects a phase shift φ_* such that $c_1(\varphi_*) = 0$.

On the other hand, in the case of the complex Ginzburg-Landau equation (see [2]) we show that

Theorem 3 *For $1 + \alpha\gamma > 0$, $\delta \in (-1/2, 1/2)$, and $g \in L^2_{\delta+2}(\mathbb{R}^3)$, there exists an ε_0 small and a family o solutions of the form,*

$$A(\mathbf{x}, t; \varepsilon) = S(\mathbf{x}; \varepsilon)\mathrm{e}^{\Phi(\mathbf{x},t;\varepsilon)}, \quad \mathbf{x} \in \mathbb{R}^3, \quad |\varepsilon| < \varepsilon_0.$$

- *The amplitude S decays to 1 as $|\mathbf{x}| \to \infty$.*
- *The phase in the far field satisfies:*

$$\Phi(\mathbf{x}, t; \varepsilon) \sim -\mathrm{i}\gamma t + \boxed{\mathrm{i}\frac{c(\varepsilon)}{|\mathbf{x}|}},$$

and $c(\varepsilon) = c_1\varepsilon + \mathrm{O}(\varepsilon^2)$,

$$c_1 = \frac{1}{4\pi(1+\alpha\gamma)} \int g\ \mathrm{d}\mathbf{x}.$$

In this case, target patterns are associated with solutions that at infinity can be described as waves propagating away from the inhomogeneity. In particular, this means that for large $|\mathbf{x}|$, target patterns satisfy $\phi(x, t; \varepsilon) \sim \gamma t - kx$, where γ is the frequency and k is the wavenumber. It is then clear from our asymptotic expansion of ϕ that as $|\mathbf{x}| \to \infty$ the wave number $k \sim \nabla\phi \to 0$. Therefore, in 3 dimensions target patterns do not form.

6.1 Extensions

Here we briefly mentioned that these methods can be extended to the case of nonlocal operators described by convolution kernels whose Fourier symbol is equivalent to $\dfrac{k^2}{1+k^2}$ modulo an isomorphism.

In particular, one can look at a large 1 dimensional array of oscillators with nonlocal coupling whose phase dynamics is represented by the equation

$$\phi_t = -\phi + J(x) * \phi + (\partial_x K(x) * \phi)^2$$

and ask what happens if the system is perturbed by an inhomogeneity. In this case, we can show that target patterns do form (see [6]).

References

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