# Finite and Infinite Sequences in Tilings

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## 1 Introduction

A tiling is a cover of $\mathbb{R}^d$ by tiles such as polygons that intersect only on their borders. Many interesting tilings lack translational symmetry; that is, if $\mathcal{T}$ is one of such tilings and $\mathcal{T}+x=\mathcal{T}$ then $x=0$. However, they often have "almost translational symmetry"; for example, they are repetitive, that is, for any finite subset $\mathcal{P}\subset\mathcal{T}$ there exists $R>0$ such that for any $x\in\mathbb{R}^d$ there is $y\in\mathbb{R}^d$ with $\mathcal{P}+y\subset\mathcal{T}\cap B(x,R)$. Here $B(x,R)$ is the open ball with its center $x$ and radius $R$ and $\mathcal{T}\cap S=\{T\in\mathcal{T}\mid T\subset S\}$ for any $S\subset\mathbb{R}^d$. This repetitively means that, although the translate $\mathcal{T}-x$ is not equal to $\mathcal{T}$, they may be almost equal around the origin.

Limit-peiorid tilings (Definition 3.4) seem to have high degree of translational symmetry; they are limits of unions of symmetric patches. In this note we prove, on the other hand, if a tiling is from substitution the expansion factor of which is *irrational Pisot*, the situation is converse: there are no symmetric sub-patch in it.

## 2 Preliminaries

This section is for preliminaries. For details see [5].

### 2.1 Definition of Tiling

Throughout the article we write $\mathbb{Z}_{>0}=\{1,2,\ldots\}$ and $\mathbb{Z}_{\geq 0}=\{0,1,2,\ldots\}$. If $S$ is a set, $\operatorname{card} S$ denotes its cardinality. If $(X,\rho)$ is a metric space, $x\in X$ and $r>0$, we write the open ball of radius $r$ with its center $x$ by $B(x,r)$. For a metric space $(X,\rho)$ and $S\subset X$, its diameter is by definition $\operatorname{diam} S=\sup\{\rho(x,y)\mid x,y\in S\}$. For a topological space $X$ and its subset $A$, its closure, interior, and boundary are denoted by $\overline{A}, A^\circ, \partial A$, respectively.

**Definition 2.1.** Take $d\in\mathbb{Z}_{>0}$ and fix it. A tile of $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ which is nonempty, open and bounded.

A patch of $\mathbb{R}^d$ is a set $\mathcal{P}$ of tiles of $\mathbb{R}^d$ such that, if $S,T\in\mathcal{P}$ and $S\neq T$, then $S\cap T=\emptyset$. For a patch $\mathcal{P}$, its support is the subset $\overline{\bigcup_{T\in\mathcal{P}}T}$ of $\mathbb{R}^d$ and denoted by $\operatorname{supp}\mathcal{P}$.

A patch $\mathcal{P}$ is called a tiling if $\operatorname{supp} \mathcal{P} = \mathbb{R}^d$.

**Remark 2.2.** In the literature, a tile is defined in various ways. For example, it is defined as (1) a subset of $\mathbb{R}^d$ which is homeomorphic to a closed unit ball of $\mathbb{R}^d$ ([1]), (2) a closed polygonal subset of $\mathbb{R}^d$ ([9]), or (3) a subset of $\mathbb{R}^d$ which is compact and equal to the closure of its interior ([3]).

In all these definitions tiles are defined as compact sets. However we can do the same argument regarding tilings by considering the interiors of tiles. This way has one virtue. Often in tiling theory one has to consider labels for tiles, so that one can distinguish tiles of the same shape. If we deal with interiors, instead of compact sets, we can remove different points from two copies of the same tile, so that we can distinguish these two tiles by their "punctures" and do not need to consider labels separately.

**Definition 2.3.** For a patch $\mathcal{P}$ and a vector $x \in \mathbb{R}^d$, define a translate of $\mathcal{P}$ by $x$ via $\mathcal{P} + x = \{T + x \mid T \in \mathcal{P}\}$. We set $\mathcal{P}_1 \sim_t \mathcal{P}_2$ if there is $x \in \mathbb{R}^d$ such that $\mathcal{P}_1 + x = \mathcal{P}_2$.

**Definition 2.4.** A tiling $\mathcal{T}$ is said to be sub-periodic if there is $x \in \mathbb{R}^d \setminus \{0\}$ such that its translate by $x$ coincide with itself, that is, $\mathcal{T} + x = \mathcal{T}$. Otherwise a tiling is said to be non-periodic. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is said to be periodic if there is a basis $\{b_1, b_2, \ldots, b_d\}$ of $\mathbb{R}^d$ such that $\mathcal{T} + b_i = \mathcal{T}$ for all $i$.

**Example 2.5** (Square tiling)**.** For any dimension $d \in \mathbb{Z}_{>0}$, a tiling $\mathcal{T}_s = \{(0,1)^d + v \mid v \in \mathbb{Z}^d\}$ is called Square tiling. This is an example of periodic tiling.

**Definition 2.6.** Given a tiling $\mathcal{T}$, a patch $\mathcal{P}$ is $\mathcal{T}$-legal if there is $x \in \mathbb{R}^d$ such that $\mathcal{P} + x \subset \mathcal{T}$.

Next we introduce an important concept in tiling theory, which is called FLC (Definition 2.8).

**Definition 2.7.** For a patch $\mathcal{P}$ and a subset $S \subset \mathbb{R}^d$, set $\mathcal{P} \cap S = \{T \in \mathcal{P} \mid T \subset S\}$.

**Definition 2.8.** A tiling $\mathcal{T}$ has finite local complexity (FLC) if for any $R > 0$ the set $\{\mathcal{T} \cap B(x,R) \mid x \in \mathbb{R}^d\}/\sim_t$ is finite.

### 2.2 Tiling Space and Tiling Dynamical Systems

To study the nature of tilings, researching corresponding continuous hulls and tiling dynamical systems is useful. Let $\|\cdot\|$ be the standard norm of $\mathbb{R}^d$.

**Definition 2.9.** The set of all patches of $\mathbb{R}^d$ is denoted by $\Omega(\mathbb{R}^d)$.

First we define a metric on $\Omega(\mathbb{R}^d)$, which is based on a well-known idea: to regard two patches close if, after small translation, they precisely agree on a large ball about the origin.

Recall that for a patch $\mathcal{P}$ and $S \subset \mathbb{R}^d$, we set $\mathcal{P} \cap S = \{T \in \mathcal{P} \mid T \subset S\}$. For two patches $\mathcal{P}_1, \mathcal{P}_2$ of $\mathbb{R}^d$, set

$$\Delta(\mathcal{P}_1, \mathcal{P}_2) = \Big\{0 < r < \frac{1}{\sqrt{2}} \,|\text{there exist } x, y \in B(0, r) \text{ such that} \\ (\mathcal{P}_1 + x) \cap B(0, \frac{1}{r}) = (\mathcal{P}_2 + y) \cap B(0, \frac{1}{r})\Big\}.$$

Then define

$$\rho(\mathcal{P}_1, \mathcal{P}_2) = \inf(\Delta(\mathcal{P}_1, \mathcal{P}_2) \cup \{\frac{1}{\sqrt{2}}\}). \tag{1}$$

**Proposition 2.10.** *The metric space $(\Omega(\mathbb{R}^d), \rho)$ is complete.*

*Proof.* Take a Cauchy sequence $(\mathcal{P}_n)_{n \in \mathbb{Z}_{>0}}$. We may assume

$$\rho(\mathcal{P}_n, \mathcal{P}_{n+1}) < \frac{1}{2^n}$$

holds for each $n \in \mathbb{Z}_{>0}$. For any $n \in \mathbb{Z}_{>0}$ there are $x_n, y_n \in B(0, \frac{1}{2^n})$ such that

$$(\mathcal{P}_n + x_n) \cap B(0, 2^n) = (\mathcal{P}_{n+1} + y_n) \cap B(0, 2^n).$$

Set $z_n = \sum_{k=n}^{\infty}(x_k - y_k)$. Then $\|z_n\| < \frac{1}{2^{n-2}}$. Set

$$\mathcal{Q}_n = (\mathcal{P}_n \cap B(0, 2^n - 1)) + z_n$$

for each $n \in \mathbb{Z}_{>0}$. For any $n \in \mathbb{Z}_{>0}$ we have $Q_n \subset Q_{n+1}$ and $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{Q}_n$ is a patch. Also, one can show that if $n < m$, $\mathcal{Q}_{m+1} \cap B(0, 2^n) = \mathcal{Q}_m \cap B(0, 2^n)$. From this we can show

$$\mathcal{P} \cap B(0, 2^{n-1}) = (\mathcal{P}_n + z_n) \cap B(0, 2^{n-1}).$$

and $\mathcal{P} = \lim \mathcal{P}_n$. □

**Definition 2.11.** For a tiling $\mathcal{T}$, its continuous hull is $\Omega_{\mathcal{T}} = \overline{\{\mathcal{T} + x \mid x \in \mathbb{R}^d\}}$ (the closure in $\Omega(\mathbb{R}^d)$ with respect to the tiling metric defined above).

**Proposition 2.12.** *If a tiling $\mathcal{T}$ has FLC, then its continuous hull $\Omega_{\mathcal{T}}$ is compact.*

We then introduce cylinder sets, which form a basis for the relative topology of the metric topology on certain continuous hulls.

**Definition 2.13.** Take a tiling $\mathcal{T}$, a $\mathcal{T}$-legal patch $\mathcal{P}$ and an open neighborhood $U$ of 0 in $\mathbb{R}^d$. Set

$$C_{\mathcal{T}}(U, \mathcal{P}) = \{\mathcal{S} \in \Omega_{\mathcal{T}} \mid \text{there is } x \in U \text{ such that } \mathcal{P} + x \subset \mathcal{S}\}.$$

**Lemma 2.14.** *For any tiling $\mathcal{T}$, the topology generated by*

$$\{C_{\mathcal{T}}(U, \mathcal{P}) \mid \mathcal{P}\text{: finite } \mathcal{T}\text{-legal patch and } U\text{: open neighborhood of } 0 \text{ in } \mathbb{R}^d\}$$

*is weaker than the relative topology of the metric topology.*

**Lemma 2.15.** *If a tiling $\mathcal{T}$ has finite tile type, then the set*

$$\{C_{\mathcal{T}}(U, \mathcal{P}) \mid \mathcal{P}\text{: finite } \mathcal{T}\text{-legal patch and } U\text{: open neighborhood of } 0 \text{ in } \mathbb{R}^d\}$$

*generates the relative topology of metric topology on $\Omega_{\mathcal{T}}$.*

Next we introduce a sufficient condition for the tiling dynamical system to be minimal. Recall that a dynamical system $\mathbb{R}^d \curvearrowright \Omega$, where $\Omega$ is a compact Hausdorff space, is minimal if any of its orbits is dense in $\Omega$.

**Definition 2.16.** A subset $S \subset \mathbb{R}^d$ is relatively dense in $\mathbb{R}^d$ if there is $R > 0$ such that for any $x \in \mathbb{R}^d$, we have $B(x, R) \cap S \neq \emptyset$.

**Definition 2.17.** A tiling $\mathcal{T}$ is repetitive if for any finite patch $\mathcal{P} \subset \mathcal{T}$, the set $\{x \in \mathbb{R}^d \mid \mathcal{P} + x \subset \mathcal{T}\}$ is relatively dense in $\mathbb{R}^d$.

**Remark 2.18.** In literature repetitivity is defined in various ways. For example, a tiling $\mathcal{T}$ may defined to be repetitive if for any compact $K \subset \mathbb{R}^d$, there is a compact set $K' \subset \mathbb{R}^d$ such that whenever $x_1, x_2 \in \mathbb{R}^d$ there is $y \in K'$ with $(\mathcal{T} + x_1) \cap K = (\mathcal{T} + x_2 + y) \cap K$. For FLC tilings of finite tile type, this condition and our definition of repetitivity are equivalent. Definition 2.17 also coincides with the definition in [8].

**Lemma 2.19.** *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ of finite tile type. If $\mathcal{T}$ is repetitive, then the associated tiling dynamical system is minimal.*

## 2.3 Eigenfunctions for tiling dynamical systems

Recall that for locally compact abelian group $G$ and $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, a continuous group homomorphism $\chi\colon G \to \mathbb{T}$ is called a character.

**Definition 2.20.** Let $G$ be a locally compact abelian group and $X$ be a compact space. Assume $G$ acts on $X$ continuously. Then we call a continuous function $f\colon X \to \mathbb{C}$ a continuous eigenfunction if there is a character $\chi\colon G \to \mathbb{T}$ such that $f(g \cdot x) = \chi(g) f(x)$ holds for any $g \in G$ and $x \in X$. We call this character $\chi$ the eigencharacter.

**Definition 2.21.** For $a \in \mathbb{R}^d$, let $\chi_a$ be the character of $\mathbb{R}^d$ defined by $\chi_a(x) = e^{2\pi i \langle a, x \rangle}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^d$. Every character of $\mathbb{R}^d$ is of this form. If $G = \mathbb{R}^d$ in Definition 2.20, and $\chi_a$ is the eigencharacter of an eigenfunction $f$, $a$ is called the eigenvalue for $f$.

### 2.4 Substitution Rules

Here we introduce substitution rule, which generates interesting examples of tilings.

**Definition 2.22.** A substitution rule is a triple $(\mathcal{A}, \lambda, \omega)$ where,

- $\mathcal{A}$ is a finite set of tiles that contain the origin,
- $\lambda$ is a real number greater than 1, and
- $\omega$ is a map from $\mathcal{A}$ to
$$\{\mathcal{P} \mid \mathcal{P} \text{ is a patch and any } T \in \mathcal{P} \text{ is a translate of a member of } \mathcal{A}\}$$
such that
$$\operatorname{supp} \omega(P) = \lambda \overline{P}$$
for each $P \in \mathcal{A}$.

Elements of $\mathcal{A}$ are called proto-tiles of the substitution. The number $\lambda$ is called the expansion factor of substitution. The map $\omega$, called substitution map, is a map obtained by first expanding each proto-tile and then decomposing it to obtain a patch consisting of translates of proto-tiles.

Definition 2.22 is for substitution with the group $\mathbb{R}^d$. One can also consider a substitution rule for a closed subgroup of $\mathbb{R}^d \rtimes \mathrm{O}(d)$ bigger than $\mathbb{R}^d$, for example Radin's pinwheel tiling ([6]). We do not deal with such substitutions in this article. One can also consider substitution rules with expansion maps, in place of expansion factors (see for example [7]). An expansion map is a linear transformation of $\mathbb{R}^d$ the eigenvalues of which lie all outside the closed unit disc of $\mathbb{C}$. We do not deal with such substitutions either.

We extend the substitution map $\omega$ to translates of proto-tiles by
$$\omega(P + x) = \omega(P) + \lambda x \tag{2}$$
for each $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$. For any patch $\mathcal{P}$ consisting of translates of proto-tiles, set $\omega(\mathcal{P})$ by
$$\omega(\mathcal{P}) = \bigcup_{T \in \mathcal{P}} \omega(T).$$
The above extension (2) is justified by the fact that $\omega(\mathcal{P})$ is again a patch.

**Definition 2.23.** Let $(\mathcal{A}, \lambda, \omega)$ be a substitution rule. A tiling $\mathcal{T}$ consisting of proto-tile $\mathcal{A}$ is called a fixed point of a substitution $(\mathcal{A}, \lambda, \omega)$ if $\omega(\mathcal{T}) = \mathcal{T}$.

**Definition 2.24.** A substitution rule $(\mathcal{A}, \lambda, \omega)$ is repetitive if there is $K > 0$ such that for any $P, P' \in \mathcal{A}$ there is $x \in \mathbb{R}^d$ with $P' + x \in \omega^K(P)$.

**Definition 2.25.** A substitution rule has FLC if
$$\{\omega^n(P) \cap B(x, R) \mid P \in \mathcal{A}, n \in \mathbb{Z}_{>0}, x \in \mathbb{R}^d\}/{\sim_t}$$
is finite for all $R > 0$.

## 3 Main Results

**Theorem 3.1** ([7]). *Let $\mathcal{T}$ be a repetitive fixed point of a primitive substitution. Assume $\mathcal{T}$ has FLC. Then the associated tiling dynamical system $(\Omega_{\mathcal{T}}, \mathbb{R}^d)$ is not mixing.*

The proof of this theorem uses a fact that, given any $\mathcal{T}$-legal patch $\mathcal{P}$, a return vector $x$ and large $n \in \mathbb{Z}_{>0}$, there are "many" translates of the patch $\mathcal{P} \cup (\mathcal{P} + \lambda^n \mathcal{P})$ in the tiling $\mathcal{T}$, in a certain sense. Here a return vector is a vector $x \in \mathbb{R}^d$ such that there exists $T \in \mathcal{T}$ with $T + x \in \mathcal{T}$, and $\lambda > 1$ is the expansion factor of the substitution. For any tile $T \in \mathcal{T}$, the number

$$\lim_{N\to\infty} \frac{\mathrm{card}\{\text{translate of } \mathcal{P} \cup (\mathcal{P} + \lambda^n x) \text{ in } \lambda^N T\}}{\text{Lebesgue measure of } \lambda^N(T)}$$

is estimated from below. Here a property of distribution of patches implies a property of the dynamical system $(\Omega_{\mathcal{T}}, \mathbb{R}^d)$.

In this talk we show a converse implication; we show from a property of the dynamical system $(\Omega_{\mathcal{T}}, \mathbb{R}^d)$ a property of distribution of patches.

**Theorem 3.2** ([8]). *Let $\mathcal{T}$ be a repetitive, non-periodic fixed point of a primitive FLC substitution. Let the expansion factor of the substitution rule be $\theta$. Then there exists a basis $\{b_1, b_2, \dots, b_d\}$ of $\mathbb{R}^d$ such that any vectors from*

$$\mathbb{Z}[\theta^{-1}]b_1 + \cdots + \mathbb{Z}[\theta^{-1}]b_d$$

*are eigenvalues.*

Using this we can prove the following.

**Theorem 3.3.** *Let $\mathcal{T}$ be a non-periodic repetitive fixed point of a primitive FLC substitution the expansion factor of which is irrational Pisot. Then for any $\mathcal{T}$-legal non-empty finite patch $\mathcal{P}$ and $x \in \mathbb{R}^d \setminus \{0\}$, there exists $N \in \mathbb{Z}_{>0}$ such that, the patch $\bigcup_{n=0}^N (\mathcal{P} + nx)$ is* not *$\mathcal{T}$-legal.*

*Proof.* We only sketch the idea of the proof when $\mathcal{P}$ is large enough. The proof for the general case is obtained by combining this with self-similarity of the tiling. For $x \neq 0$ take an eigenfunction $f$ with eigenvector $a$ such that $\langle x, a\rangle \in \mathbb{R} \setminus \mathbb{Q}$. For this $f$, if $\mathcal{P}$ is large enough, $\mathbb{T} \setminus \overline{f(C_{\mathcal{T}}(\{0\}, \mathcal{P}))}$ is an nonempty open set. For this $\mathcal{P}$ we prove the theorem. There is $N > 0$ such that, for any $z \in \mathbb{T}$ there exists $n \in \{0, 1, \dots, N\}$ with $e^{2\pi i\langle nx, a\rangle} z \in \mathbb{T} \setminus \overline{f(C_{\mathcal{T}}(\{0\}, \mathcal{P}))}$. If $\mathcal{S} \in \Omega_{\mathcal{T}}$, there is $n \in \{0, 1, \dots, N\}$ such that $f(\mathcal{S} - nx) \notin f(C_{\mathcal{T}}(\{0\}, \mathcal{P}))$. This means that $\mathcal{P} + nx \not\subset \mathcal{S}$ and we have proved $\bigcup_{n=0}^N (\mathcal{P} + nx)$ is not $\mathcal{T}$-legal. $\square$

The situation is contrary to the one in limit-periodic tilings.

**Definition 3.4.** A patch $\mathcal{P}$ of $\mathbb{R}^d$ is translational-symmetric if there is a lattice $\Lambda$ in $\mathbb{R}^d$ such that $\mathcal{P} + x = \mathcal{P}$ for any $x \in \Lambda$. A tiling $\mathcal{T}$ is said to be limit-periodic if there is a translational-symmetric patches $\mathcal{P}_1, \mathcal{P}_2, \dots$ such that $\mathcal{T} = \lim_{N\to\infty} \bigcup_{n=1}^{N} \mathcal{P}_n$.

For limit-periodic tilings, there are an abundance of infinite sequence of patches. For examples of limit-periodic tilings, see [2].

Note also that by the following theorem there are arbitrary long finite sequences in tilings.

**Theorem 3.5** ([4]). *Let $\mathcal{T}$ be a FLC tiling with local isomorphism property. Let $\mathcal{P}$ be a subset of $\mathcal{T}$ such that $\operatorname{supp}\mathcal{P}$ is bounded. Take a finite subset $F$ of $\mathbb{R}^d$ and an open neighborhood $U$ of $0$ in $\mathbb{R}^d$. Then there is $n \in \mathbb{Z}_{>0}$ and $z_f \in U$ for each $f \in F$ such that, the patch $\bigcup_{f\in F}(\mathcal{P} + nf + z_f)$ is $\mathcal{T}$-legal.*

**Corollary 3.6.** *Let $\mathcal{T}$ be a repetitive fixed point of a primitive FLC substitution the expansion constant of which is a Pisot number. Let $\mathcal{P}$ be a $\mathcal{T}$-legal patch, $x \in \mathbb{R}^d$, $n \in \mathbb{Z}_{>0}$ and $U$ be an open neighborhood of $0 \in \mathbb{R}^d$. Then there are $y \in \mathbb{R}^d$ and $m \in \mathbb{Z}_{>0}$ such that, $mx - y \in U$ and $\bigcup_{i=0}^{n}(\mathcal{P} + iy)$ is $\mathcal{T}$-legal.*

*Proof.* This follows from Meyer property of the tiling. $\square$

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