Dynamical charts for irrationally indifferent fixed points of holomorphic functions

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Complex Dynamics

$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ Rational map or $f : \mathbb{C} \to \mathbb{C}$ Polynomial

$$z_0 \mapsto z_1 = f(z_0) \mapsto z_2 = f^2(z_1) \mapsto \ldots \mapsto z_n = f^n(z_0) = \overbrace{f \circ \cdots \circ f}^{n}(z_0)$$

Dichotomy in Phase space (z-plane)

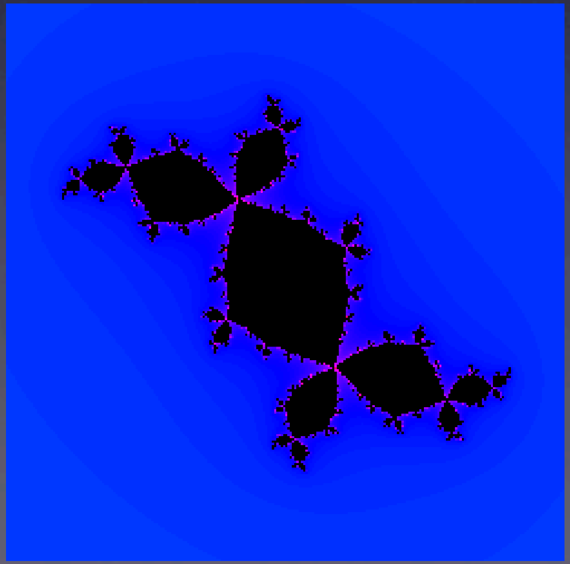
$$\widehat{\mathbb{C}} = F(f) \cup J(F)$$

where $F(f)$ = Fatou set (tame part), $J(f)$ = Julia set (chaotic part)

For f polynomial,

$K(f) := \{z \in \mathbb{C} : \{f^n(z)\}_{n=0}^{\infty}$ bounded $\}$ filled Julia set

$J(f) := \partial K(f) = \{z : \{f^n\}_{n=0}^{\infty}$ is not equicontinuous near $z\}$

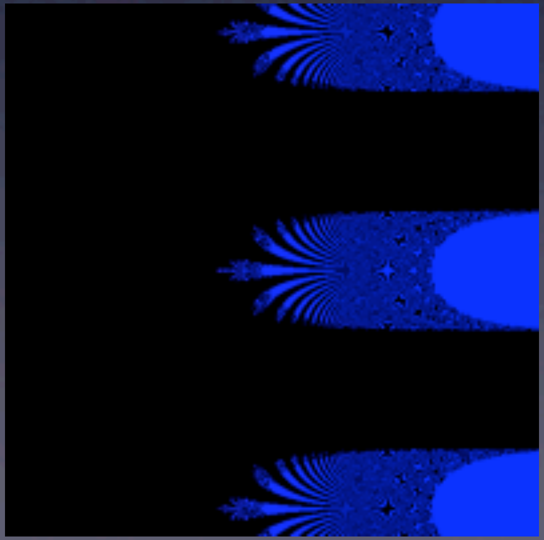


Other dynamics, e.g.

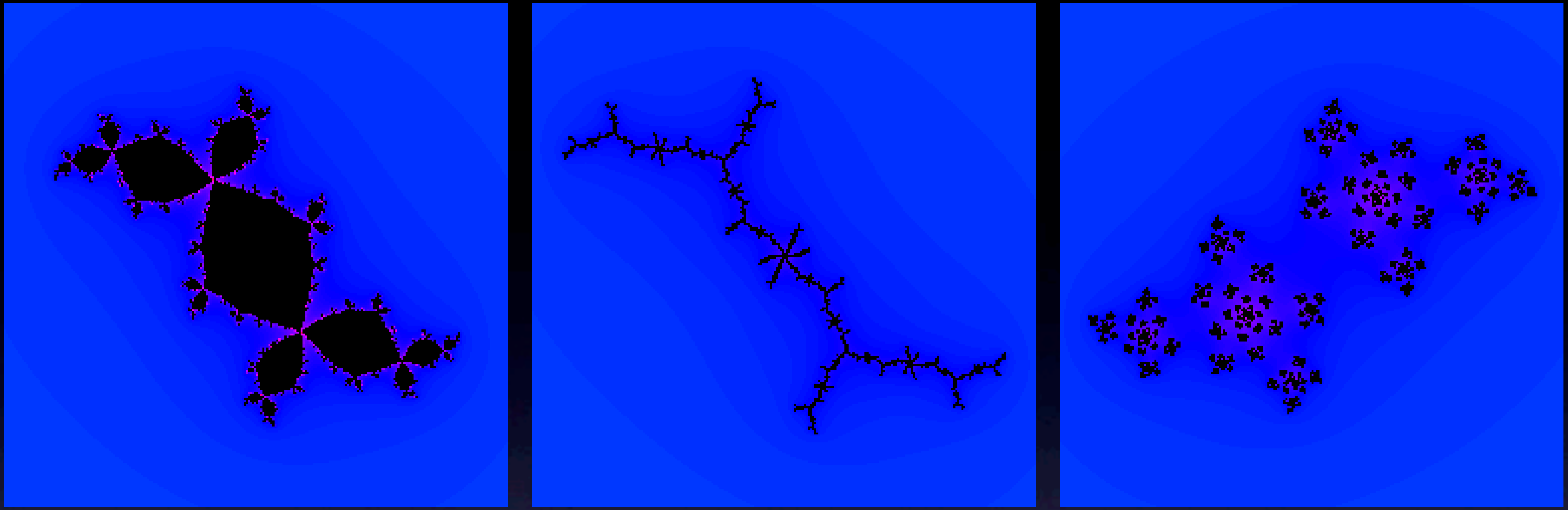
$E_\lambda(z) = \lambda e^z$

(cf. Devaney's talk:

Cantor Bouquet and hairs)



A few examples ($f_c(z) = z^2 + c$ with various c's)



Well-understatood if f is *hyperbolic*: all critical points are attracted to attracting periodic orbits, or equivalently f is expanding on $J(f)$. Then the Julia set is locally connected and the dynamics of $J(f)$ can be described as a quotient (factor) of $z \mapsto z^d$ on S^1.

There are other cases where the topology of Julia sets (and other invariant sets) are not well-understood.
An example is the maps with non-linearizable irrationally indifferent fixed points.

Goal of this talk is to understand the local dynamics of such a map via *renormalization*.

Local dynamics near irrationally indifferent fixed points

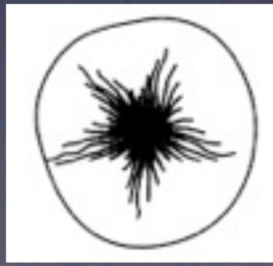
Irrationally indifferent fixed point of a holomorphic function

$$f(z) = e^{2\pi i\alpha} z + z^2 + \ldots, \qquad \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

The fixed point is *linearizable* if and only if it is locally (holomorphically) conjugate to a rotation.

Problem: What is the local dynamics when the fixed pt is not linearizable?

Perez-Marco's *Hedgehog*: $0 \in U$ topological disk in $\mathbb{C}$, $f : \overline{U} \to f(\overline{U})$ homeo, holomorphic in U, $f(z) = e^{2\pi i\alpha} z + O(z^2)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$
$\exists K \subset U$ maximal connected invariant set.



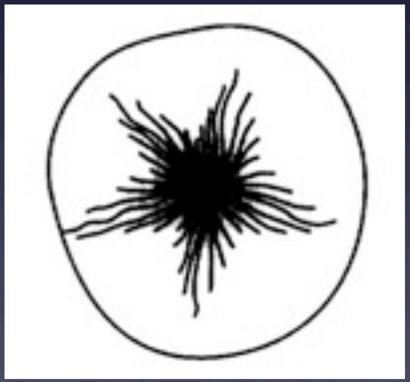
He constructed some examples of germs of hedgehogs.

Problem: What are possible hedgehogs for quadratic polynomials?

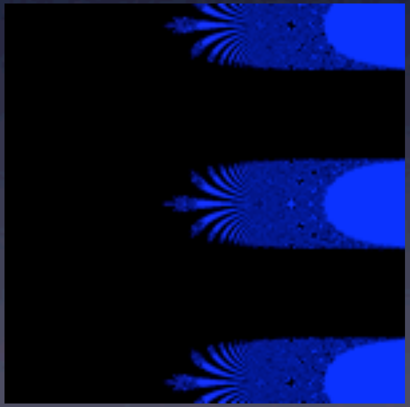
Theorem: If $f(z) = e^{2\pi i\alpha}z + z^2$ and the rotation number α is an irrational number of high type, i.e.

$$\alpha = \pm\cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots}}} \quad (a_i \in \mathbb{N},\ a_i \geq N \ \text{ large})$$

then the Julia set $J(f)$ contains an invariant subset Λ such that 0, critical pt $\in \Lambda$ and $\Lambda \setminus \{0\}$ consists of (disjoint) continuous hairs (like the Julia set of exponential map).

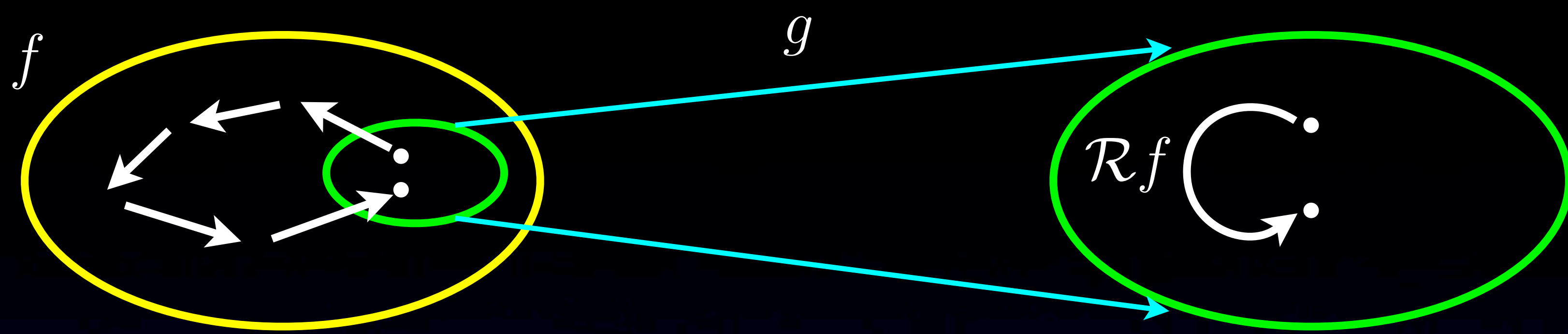


(maximal hedgehog)



The key idea is to use the *near-parabolic renormalization*, which is associated with the perturbation of parabolic fixed points (e.g. $z = 0$ for $z \mapsto z + z^2 + \ldots$).

Return map and renormalization



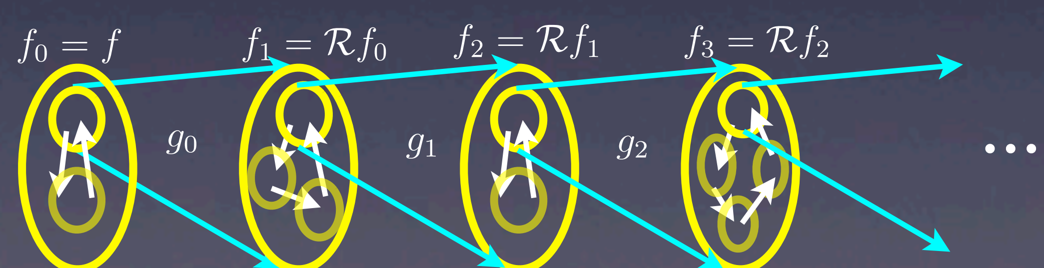
$$\mathcal{R}f = (\text{first return map of } f) \text{ after rescaling}$$
$$= g \circ f^k \circ g^{-1} \quad (\text{if return time} \equiv k)$$

Renormalization

high iterates of f $\longleftrightarrow$ fewer iterates of $\mathcal{R}f$

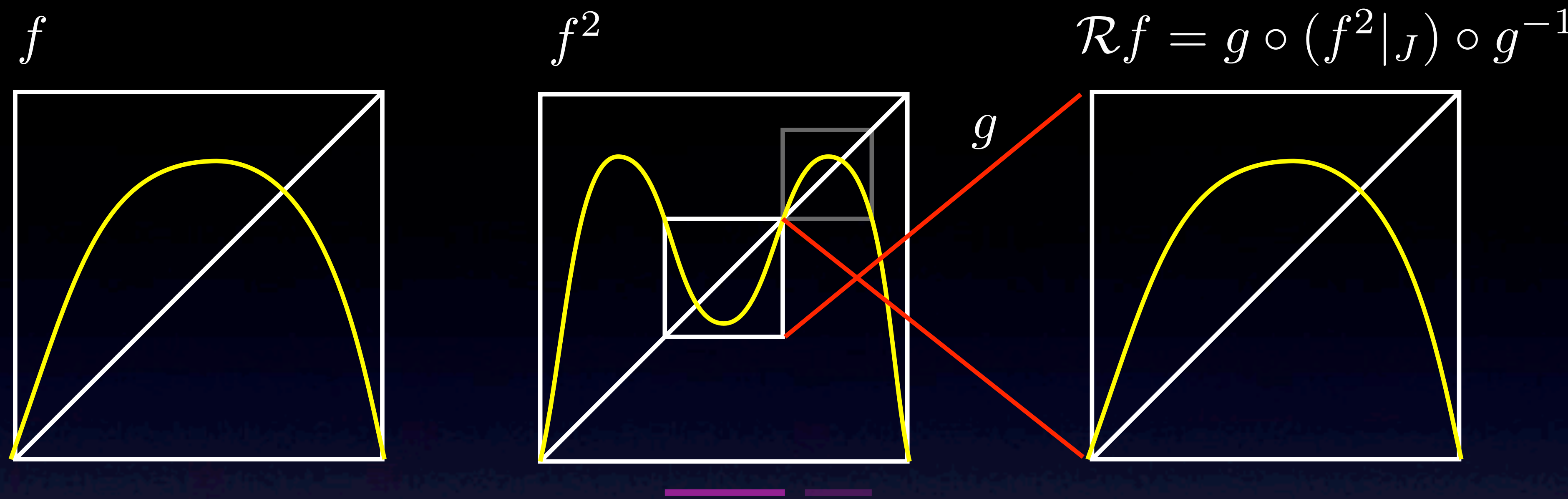
fine orbit structure for f $\longleftrightarrow$ large scale orbit structure for $\mathcal{R}f$

Successive construction of $\mathcal{R}f$, $\mathcal{R}^2 f$, $\ldots$, helps to understand the dynamics of f (orbits, invariant sets, rigidity, bifurcation, $\ldots$)



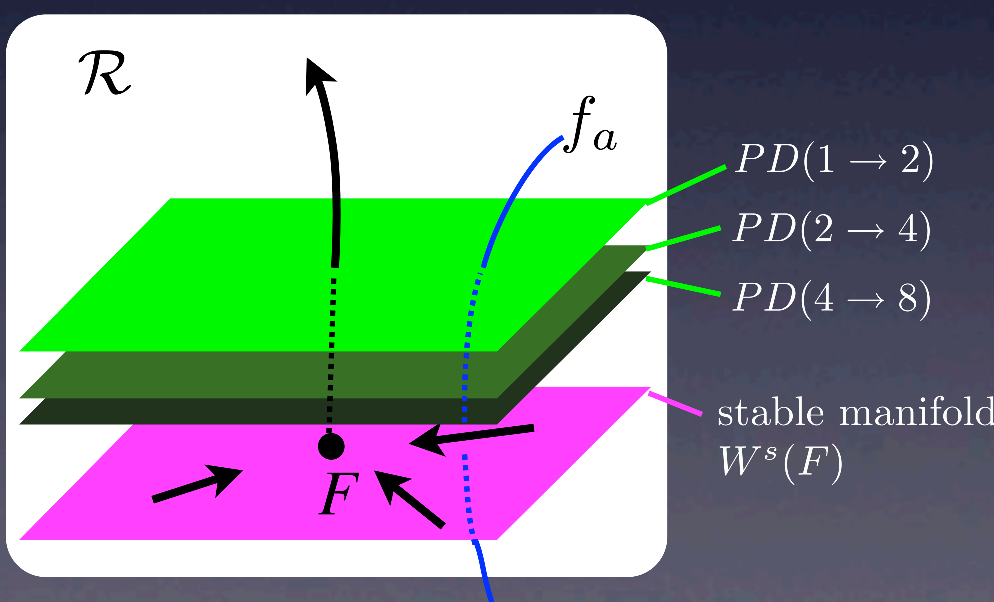
The renormalization $\mathcal{R} : f \mapsto \mathcal{R}f$ can be considered as a meta-dynamics on the space of dynamical systems of certain class.

Feigenbaum-Coullet-Tresser for unimodal maps



$I = [0, 1]$

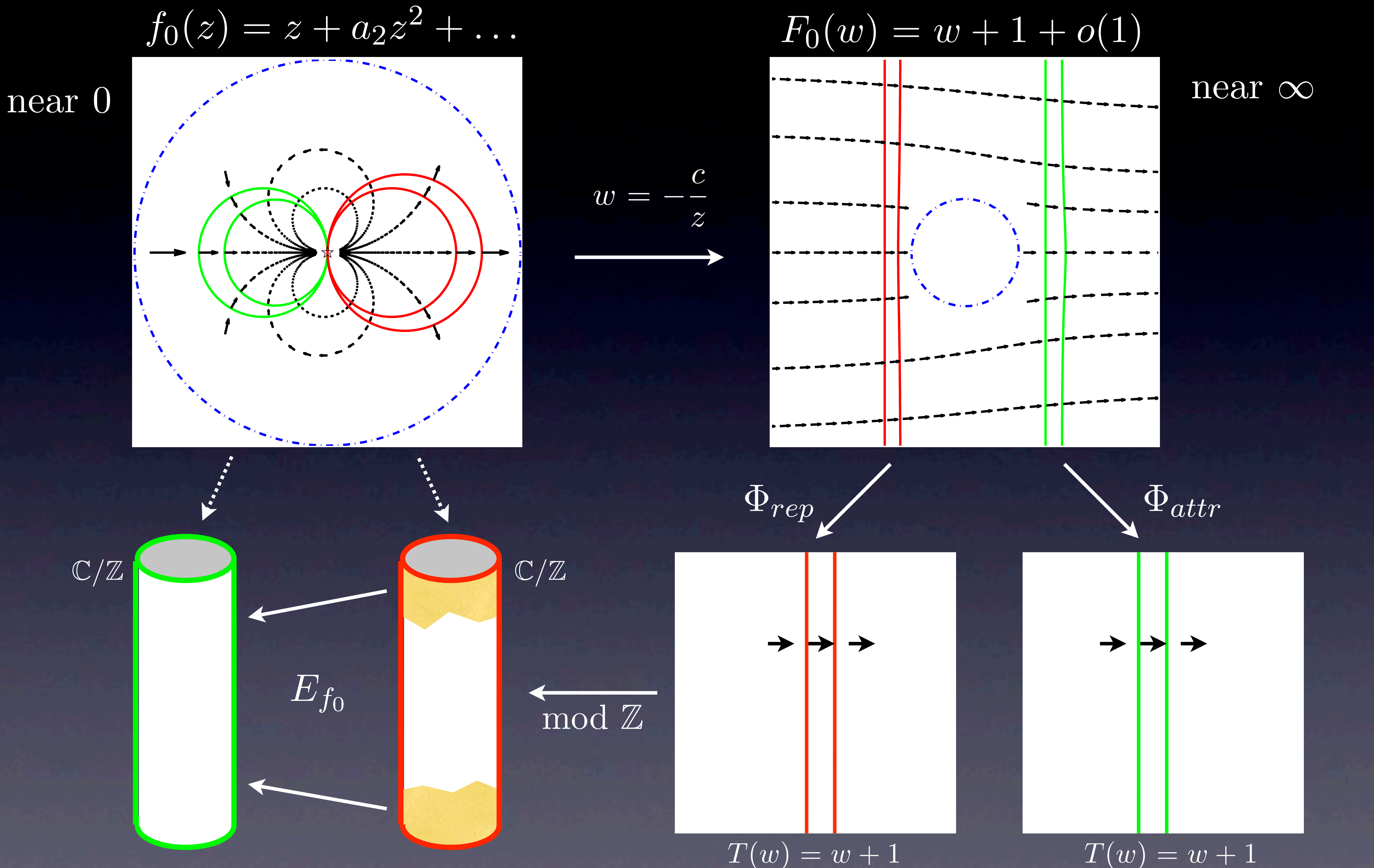
$J \subset I$ s.t. $f^2(J) \subset J$



Hyperbolic fixed point or hyperbolic horseshoe of the meta-dynamics imply conclusion on rigidity and structure of parameter space

The universality of ratios of bifurcating parameters

Fatou coordinates $\Phi_{attr/rep}$ and the Horn map E_{f_0} for $f_0(z) = z + z^2 + \ldots$

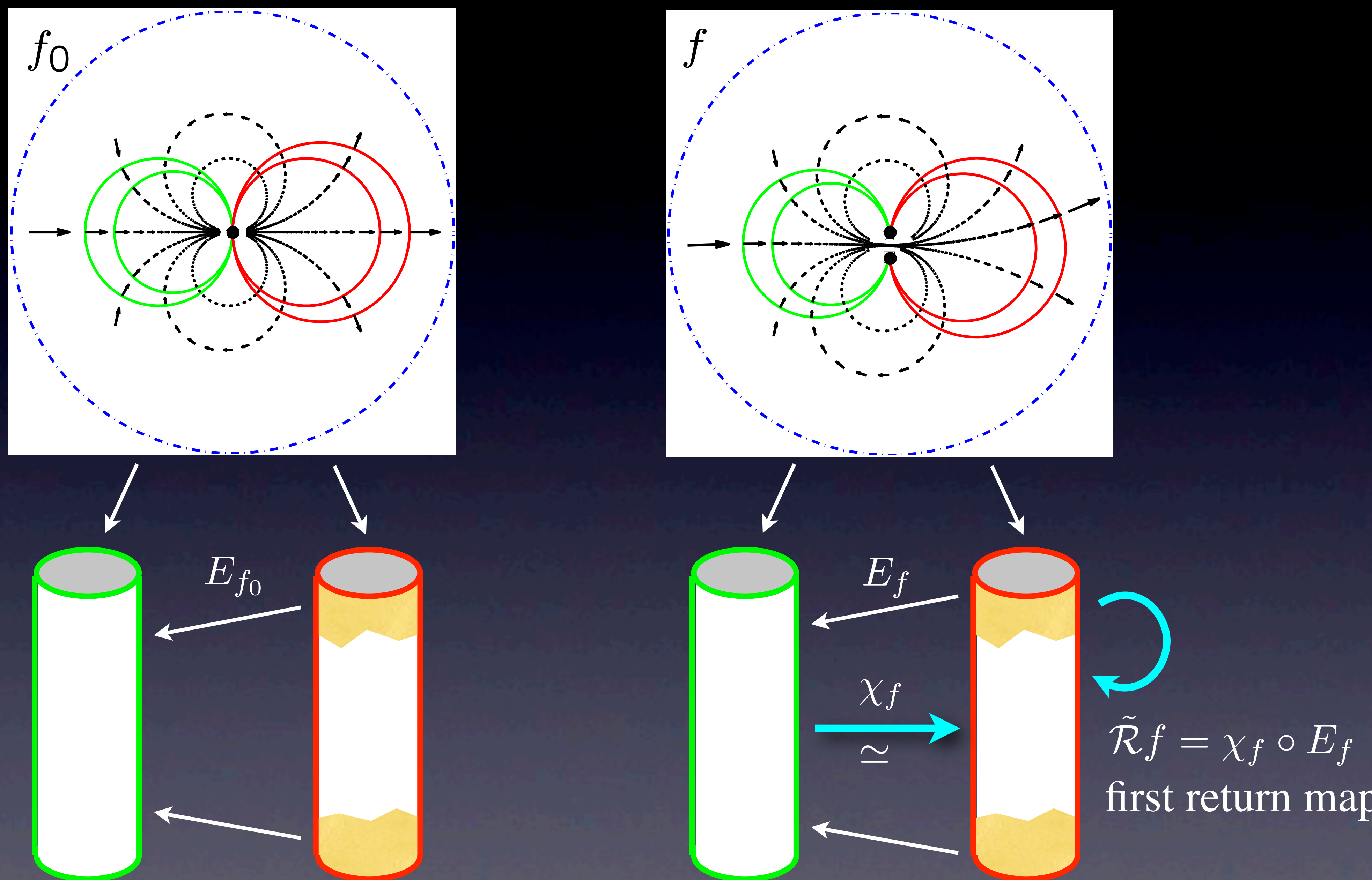


$E_{f_0}(z) = \Phi_{attr} \circ \Phi_{rep}^{-1}$

$\Phi_{\ldots}(f_0(z)) = \Phi_{\ldots}(z) + 1$

Perturbation

$f'(0) = e^{2\pi i \alpha}$, α small $|\arg \alpha| < \frac{\pi}{4}$

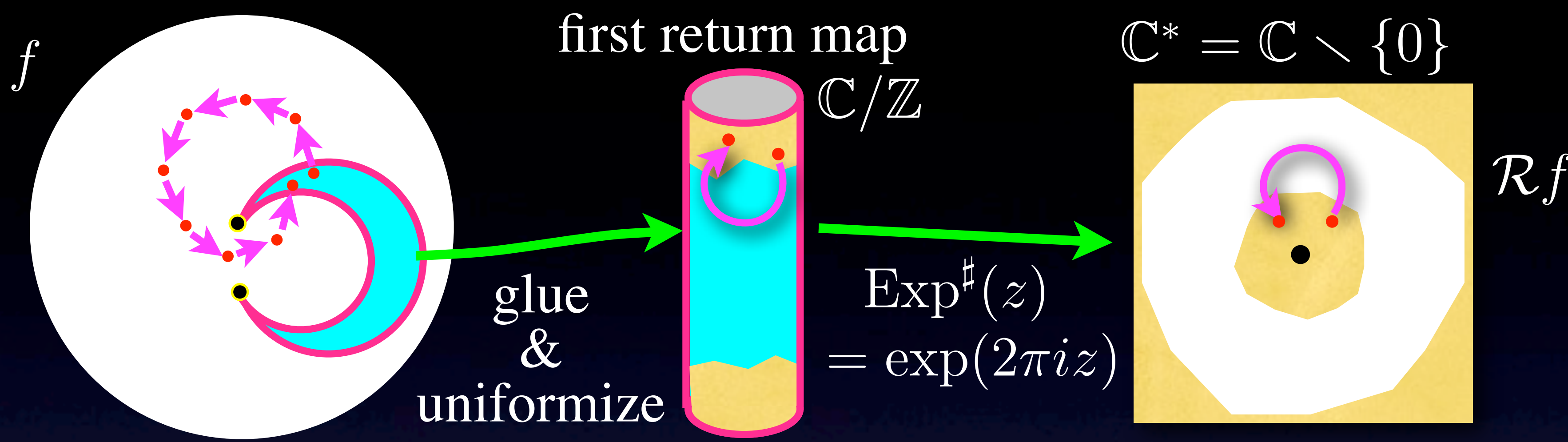


E_f depends continuously on f

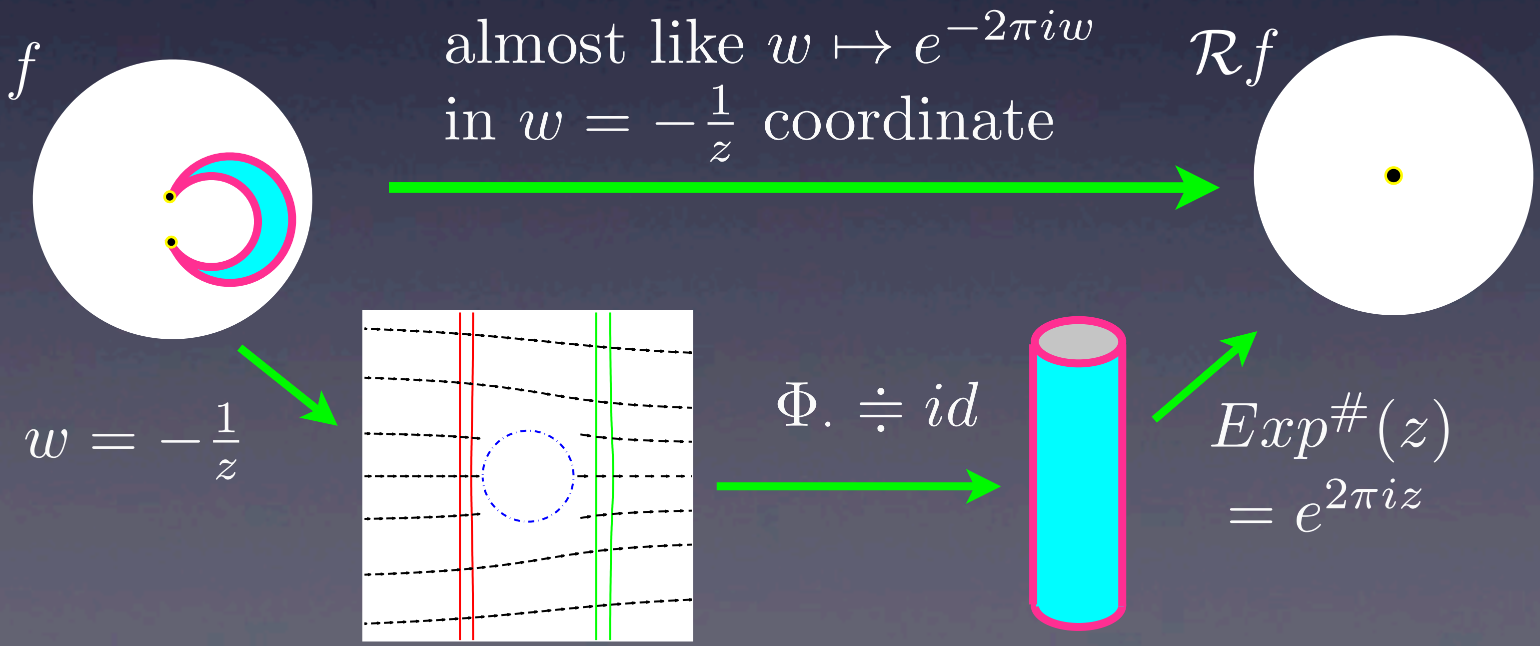
(after a suitable normalization)

$$\chi_f(z) = z - \frac{1}{\alpha}$$

Near-parabolic renormalization



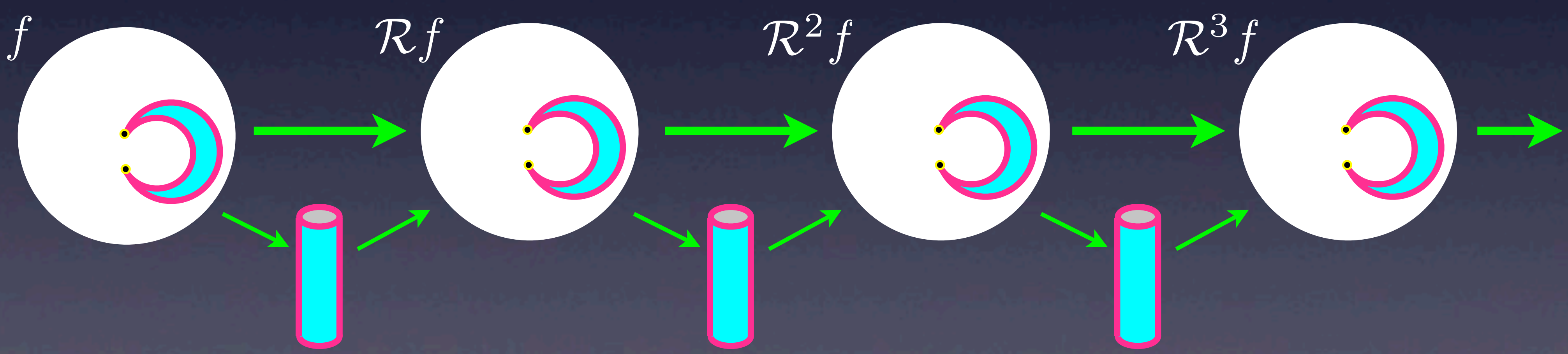
$\mathcal{R}f$ can be defined when $f(z) = e^{2\pi i\alpha} z + \dots$ is a small perturbation of $z + a_2 z^2 + \dots$ $(a_2 \neq 0)$ and $|\arg \alpha| < \pi/4$.



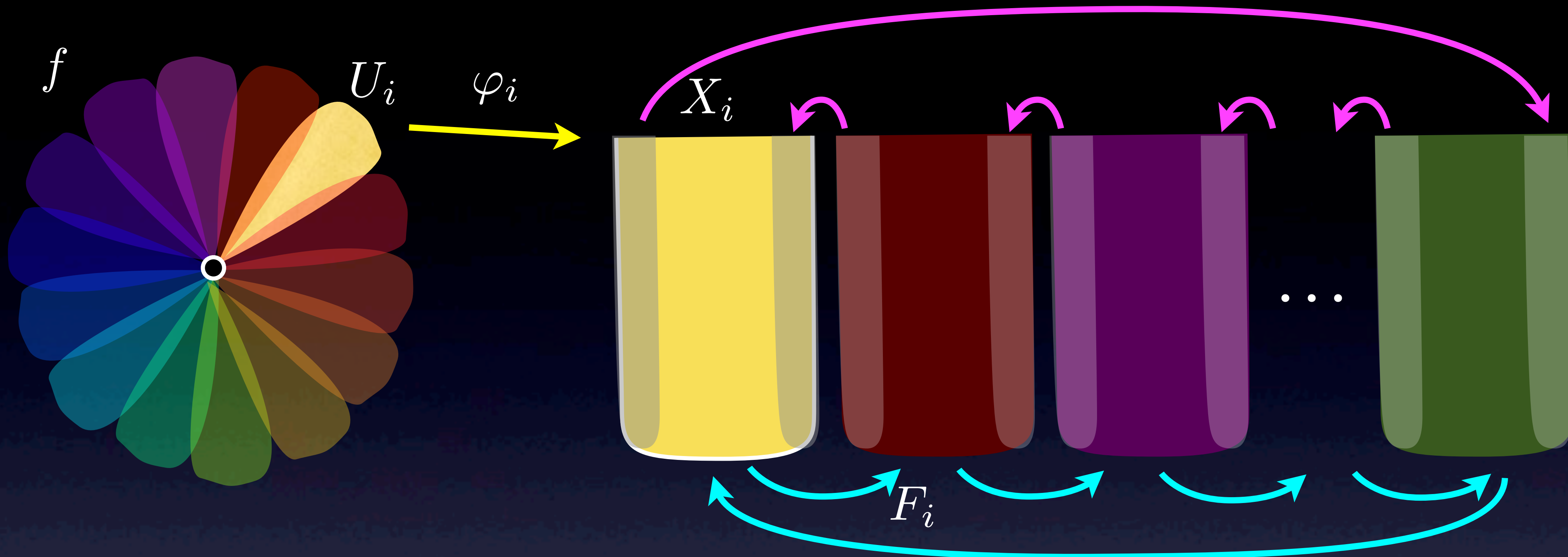
Theorem (Inou & S.): If α is an irrational number of sufficiently high type, then for $f(z) = e^{2\pi i\alpha}z + z^2$, the sequence of near-parabolic renormalizations

$$f,\ \mathcal{R}f,\ \mathcal{R}^2 f,\ \mathcal{R}^3 f, \ldots$$

are well-defined and these functions $\mathcal{R}^n f$ belong to a certain pre-compact class of functions. Each $\mathcal{R}^n f$ has a unique critical point in its domain of definition and the critical points are inherited between the levels of renormalization.



Dynamical charts



Open sets U_i $(i \in I)$, which cover a punctured nbd of fixed pt

Maps $\varphi_i : U_i \to X$, a small number of model spaces

A small number of model dynamics $F = \varphi_{r(i)} \circ f \circ \varphi_i^{-1}$
(for example, F_{can} and id)

Index set I with induced dynamics $r : I \to I$
(represents the combinatorics of the dynamics)

Gluing $\varphi_{i,j} = \varphi_i \circ \varphi_j^{-1}$ on overlaps, compatible with model dynamics
(absorbs the difference for particular maps)

Special for irrational dynamics: Refinements
one system of charts $\longrightarrow$ a new system of refined charts

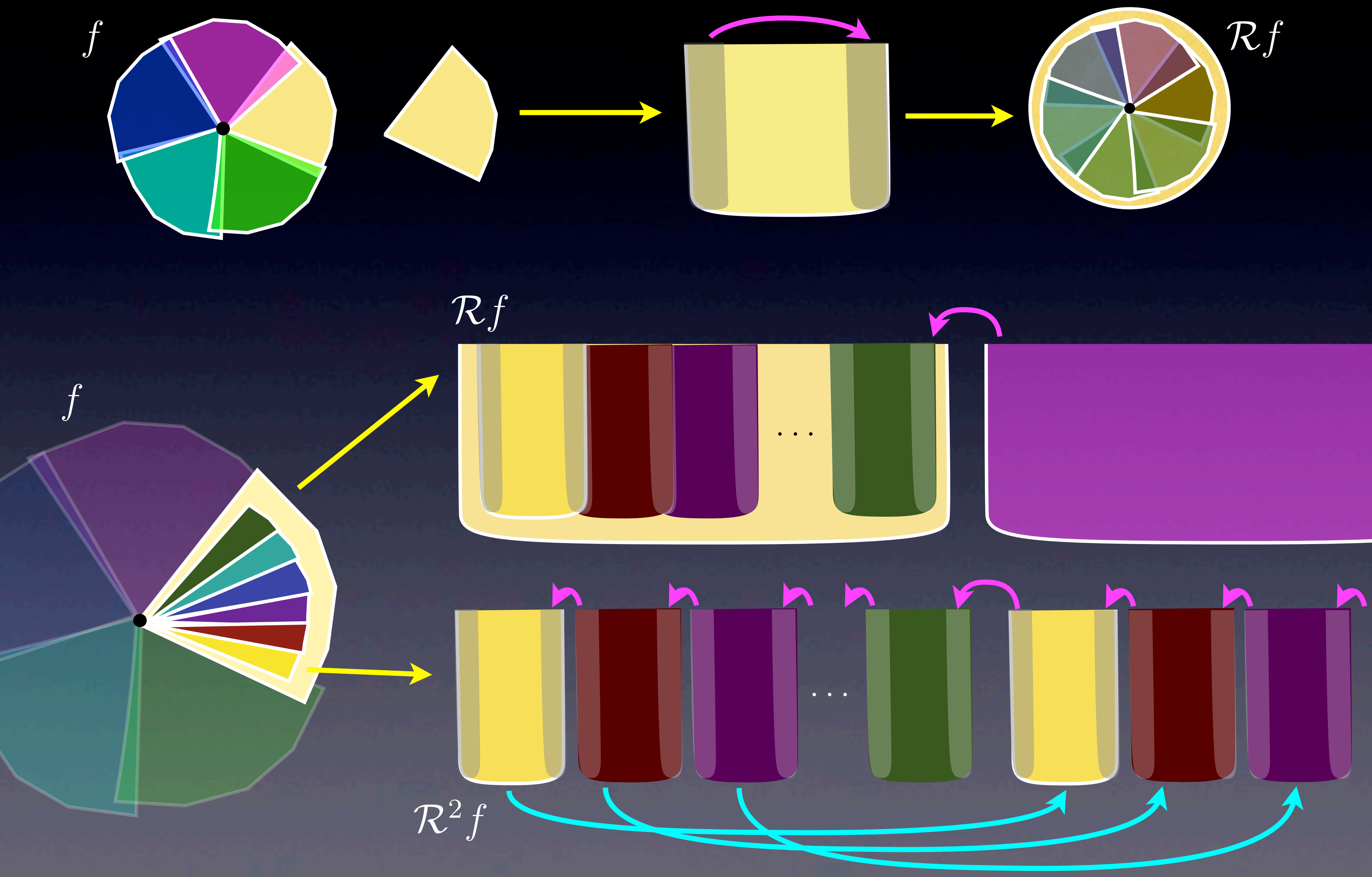
Convention

$z \to 0$ in U_i

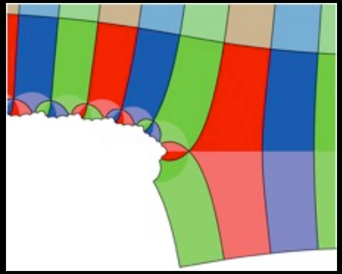
$\Updownarrow$

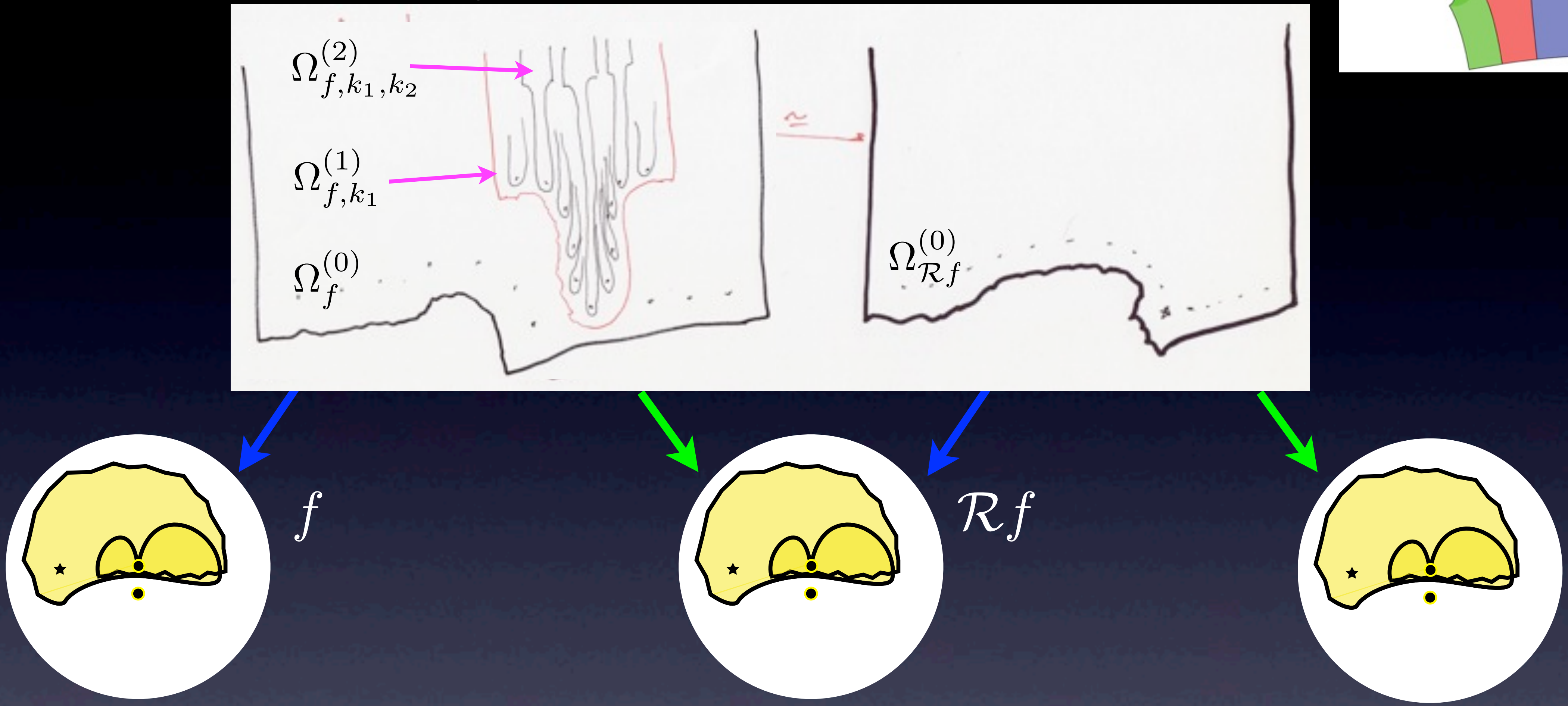
$Im\, w \to +\infty$ in X_i

Successive renormalizations = refinements of dynamical charts



Construction of dynamical charts



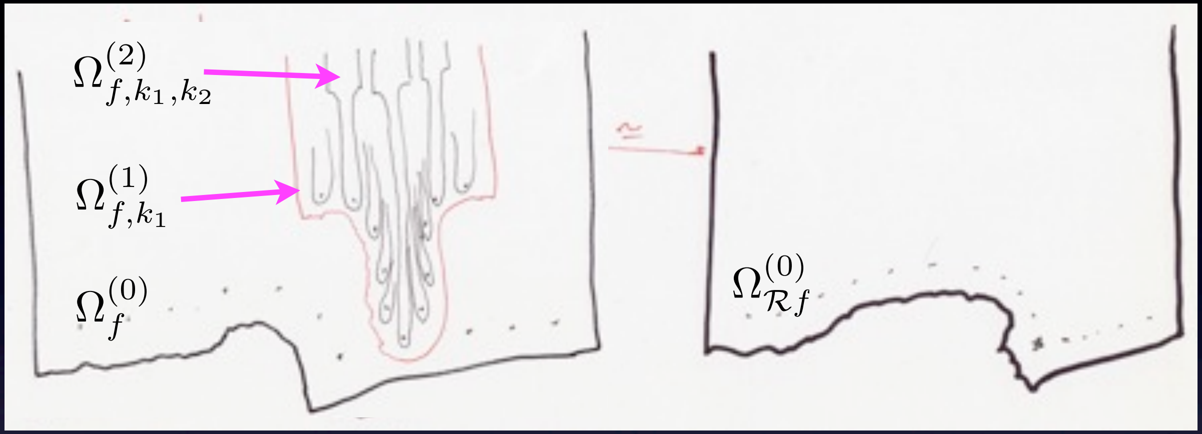


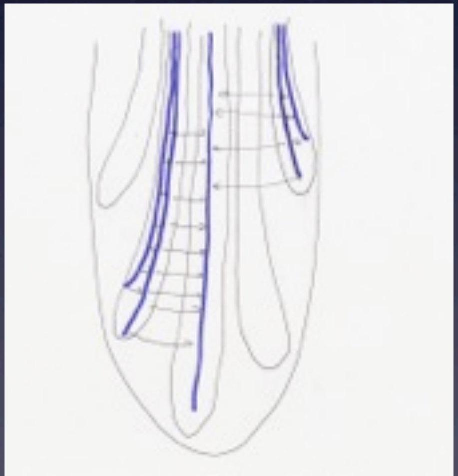
$$\Omega^{(0)} \supset \Omega^{(1)}_{k_1} \supset \cdots \supset \Omega^{(n)}_{k_1,k_2,\ldots,k_n} \supset \Omega^{(n+1)}_{k_1,k_2,\ldots,k_n,k_{n+1}} \supset \ldots$$

each $\Omega^{(n)}_{k_1,k_2,\ldots,k_n}$ is isomorphic to truncated checkerboard pattern $\Omega_{\mathcal{R}^n f}$ they are glued via $\theta_{\mathcal{R}^n f}$

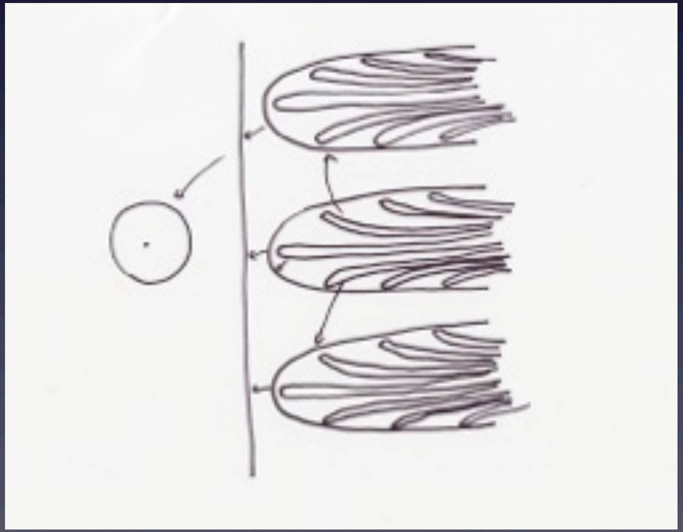
$$\Lambda_f = \bigcap_{n=0}^{\infty} \bigcup_{(k_1,\ldots,k_n)\in A_n} \Omega^{(n)}_{f,\,k_1,k_2,\ldots,k_n}$$ is an invariant set containing the critical orbit "maximal hedgehog"

Theorem For each $(k_1, k_2, \dots) \in A_\infty$ (set of admissible sequences of indices), the intersection $\bigcap_{n=1}^{\infty} \Omega^{(n)}_{f;k_1,\dots,k_n}$ is a simple arc ending at 0. (hair) The maximal hedgehog Λ_f is the union of arcs which are disjoint except at 0.





Proof by successive (cut-off) homotopies just like in the case of exponential maps



Iterating exponential(-like) maps is necessary to understand quadratic polynomials!

Thank you!