Open problems on the mean stability of polynomial semiroups and random complex dynamical systems *

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We consider random dynamical systems. One of the purposes to study dynamical systems is to describe nature. Since nature has a lot of random terms, it seems natural to consider random dynamical systems. Recently, many new phenomena of random dynamical systems which cannot hold in the usual iteration dynamical systems are found and investigated. Such phenomena are called "**randomness-induced phenomena**" or "**noise-induced phenomena**".

In this presentation, we consider the dynamics of semigroups of complex polynomial maps on the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$, where the semigroup operation being the functional composition, and the associated random dynamical systems on $\hat{\mathbb{C}}$. The motivations to study random complex dynamical systems are as follows. (1) One of the most important and popular subjects in the study of dynamical systems is the polynomial dynamics on $\mathbb{R}$. In polynomial dynamics, it is important to consider not only real initial values but also complex initial values, to investigate the detail of polynomial dynamics. Combining this idea and that of random dynamical systems, it seems natural to consider "**random complex dynamical systems**". (2) The second motivation to study random complex dynamics is "Newton's method" to find roots of complex polynomial $f(z)$. In this case, we consider the iteration of rational function $R(z) = z - \frac{f(z)}{f'(z)}$. In Newton's method, we sometimes use computers, and since computers have error terms, it seems natural to consider random iterations of rational functions. (3) Third motivation to study random complex dynamics is the study on group actions on complex manifolds. It is related to algebraic geometry, low dimensional topology (e.g. the actions of mapping class groups of Riemann surfaces on some complex manifolds), Painlevé equations, etc. Random complex dynamics on the Riemann sphere might be a prototype of these kind of subjects.

Let $\mathcal{P} := \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f \text{ is a polynomial map}, \deg(f) \geq 2\}$ endowed with distance η which is defined as $\eta(f,g) := \sup_{z\in\hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}} \cong S^2$. Also, we have $\mathcal{P} \cong \cup_{d=2}^{\infty} \mathcal{P}_d$ (disjoint union), where $\mathcal{P}_d := \{f \in \mathcal{P} \mid \deg(f) = d\} \cong (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^d$. (Note that a complex polynomial $f : \mathbb{C} \to \mathbb{C}$ is regarded as a polynomial map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $f(\infty) := \infty$.)

Note that $\mathcal{P}$ is a semigroup where the semigroup operation being functional composition. We say that a non-empty subset G of $\mathcal{P}$ is a **polynomial semigroup** if G is a subsemigroup of $\mathcal{P}$.

Let $\mathrm{Cpt}(\mathcal{P})$ be the space of all non-empty compact subsets of $\mathcal{P}$ endowed with Hausdorff distance d_H, which is defined as

$$d_H(K,L) := \max\{\sup_{a\in K} \inf_{b\in L} \eta(a,b),\ \sup_{b\in L} \inf_{a\in K} \eta(a,b)\}.$$

*Date: September 17, 2014. 2010 Mathematics Subject Classification. 37F10, 37H10. Keywords: Random dynamical systems; Random complex dynamics; Rational semigroups; Fractal geometry; Cooperation principle; Noise-induced order; Randomness-induced phenomena

Definition 0.1 ([1])**.** We say that an element $\Gamma \in \mathrm{Cpt}(\mathcal{P})$ is **mean stable** if there exist non-empty open subsets U, V of $\hat{\mathbb{C}}$ with $\sharp(\hat{\mathbb{C}} \setminus U) \geq 3$ and a positive integer n such that

- $\overline{V} \subset U$,
- for each $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$, we have $\gamma_n \circ \cdots \gamma_1(U) \subset V$, and
- for each $z \in \hat{\mathbb{C}}$, there exists an element $(\alpha_1, \dots, \alpha_m) \in \Gamma^m$ for some $m \in \mathbb{N}$ such that $\alpha_m \circ \cdots \circ \alpha_1(z) \in U$.

Also, we say that a polynomial semigroup G_Γ generated by Γ (i.e., $G_\Gamma := \{\gamma_1 \circ \cdots \circ \gamma_m \mid m \in \mathbb{N}, \forall \gamma_j \in \Gamma\}$ is **mean stable** if Γ is mean stable.

For each Borel probability measure τ on $\mathcal{P}$, we consider the i.i.d. random dynamical system on $\hat{\mathbb{C}}$ associated with τ (i.e., the random dynamical system on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \mathcal{P}$ according to τ). This gives us a Markov chain with phase space $\hat{\mathbb{C}}$ such that the transition probability $p(x, A)$ from $x \in \hat{\mathbb{C}}$ to a Borel subset A of $\hat{\mathbb{C}}$ is equal to $\tau(\{h \in \mathcal{P} \mid h(x) \in A\})$. We have the following.

Theorem 0.2 ([1, 2])**.** *Let $\Gamma \in \mathrm{Cpt}(\mathcal{P})$ be mean stable. Let τ be a Borel probability measure on $\mathcal{P}$ with $\mathrm{supp}\, \tau = \Gamma$, where $\mathrm{supp}\, \tau$ denotes the topological support of τ (Note that for each $A \in \mathrm{Cpt}(\mathcal{P})$, there exists a Borel probability measure ρ on $\mathcal{P}$ with $\mathrm{supp}\, \rho = A$.) We consider the i.i.d. random dynamical system on $\hat{\mathbb{C}}$ associated with τ. Then we have all of the following statements.*

(1) (**Negativity of Lyapunov exponents**) *There exists a* **negative** *constant c with the following property* $(*)$.

   $(*)$ *For* **each** *$z \in \hat{\mathbb{C}}$, there exists a subset $\mathcal{A}_z$ of $\mathcal{P}^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty} \tau)(\mathcal{A}_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots,) \in \mathcal{A}_z$, the Lyapunov exponent*

   $$\chi(\gamma, z) := \lim_{n \to \infty} \frac{1}{n} \log \| D(\gamma_n \circ \cdots \circ \gamma_1)(z) \|$$

   along the sequence γ of polynomials starting with the initial value z exists and

   $$\chi(\gamma, z) \leq c < 0.$$

   Here, $Df(z)$ denote the complex derivative of f at z and $\| \cdot \|$ denotes the norm of the derivative with respect to the spherical metric on $\hat{\mathbb{C}} \cong S^2$.

(2) *There exists a subset $\mathcal{B}$ of $\mathcal{P}^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty} \tau)(\mathcal{B}) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots,) \in \mathcal{B}$, the Julia set J_γ of the sequence γ of polynomials, which is defined as*

   $$J_\gamma := \{z \in \hat{\mathbb{C}} \mid \forall \text{ nbd } U \text{ of } z, \{\gamma_n \circ \cdots \circ \gamma_1 : U \to \hat{\mathbb{C}}\}_{n=1}^{\infty} \text{ is not equicontinuous on } U \text{ w.r.t. } d\},$$

   satisfies $Leb_2(J_\gamma) = 0$, where Leb_2 denotes the 2-dimensional Lebesgue measure on $\hat{\mathbb{C}}$.

(3) *The i.i.d. random dynamical system on $\hat{\mathbb{C}}$ associated with τ* **behaves well** *on the Banach space $C(\hat{\mathbb{C}})$ of all continuous complex-valued functions on $\hat{\mathbb{C}}$ endowed with supremum norm $\| \cdot \|_\infty$. More precisely, there exist a non-empty finite dimensional subspace U_τ of $C(\hat{\mathbb{C}})$ and a continuous projection $\pi_\tau : C(\hat{\mathbb{C}}) \to U_\tau$ such that for each $\varphi \in C(\hat{\mathbb{C}})$, we have that $\|M_\tau^n(\varphi - \pi_\tau(\varphi))\|_\infty \to 0$ as $n \to \infty$, where $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ is an operator defined by $M_\tau(\psi)(z) := \int_{\mathcal{P}} \psi(h(z)) d\tau(h)$ for each $\psi \in C(\hat{\mathbb{C}})$ and $z \in \hat{\mathbb{C}}$.*

Theorem 0.3 ([2])**.** *Let $\mathcal{MS} := \{\Gamma \in \mathrm{Cpt}(\mathcal{P}) \mid \Gamma$ is mean stable$\}$. Then $\mathcal{MS}$ is an* **open dense** *subset of* $\mathrm{Cpt}(\mathcal{P})$.

The following is an open problem.

Problem 0.4. (**Open Problem.**) Let $m \in \mathbb{N}$ with $m \geq 2$. Let

$$\mathcal{MS}_m := \{(f_1, \ldots, f_m) \in \mathcal{P}^m \mid \{f_1, \ldots, f_m\} \text{ is mean stable}\}.$$

Then, is $\mathcal{MS}_m$ an **open dense** subset of $\mathcal{P}^m$? (Remark: at least we know that $\mathcal{MS}_m$ is an open subset of $\mathcal{P}^m$.)

Remark 0.5. If $\mathcal{MS}_m$ is an open dense subset of $\mathcal{P}^m$ for each $m \in \mathbb{N}$ with $m \geq 2$, then it gives us another proof of Theorem 0.3.

Remark 0.6. For each $f \in \mathcal{P}$, the set $\{f\}$ is not mean stable, since we always have that the Julia set $J(f)$ of f is not empty, and the dynamical system generated by f on $J(f)$ is **chaotic**. Thus Theorems 0.2, 0.3 illustrate **randomness-induced phenomena** of random dynamical systems (phenomena in random dynamical systems which cannot hold in the usual iteration dynamics of a single map). Note that in random complex dynamical systems, there are a lot of randomness-induced phenomena (see [1, 2]). The phenomena in Theorems 0.2, 0.3 are due to the (automatic) cooperation of many kinds of maps in one system so that they make the chaos of the averaged system disappear, even though the iteration dynamical system generated by each map of the system has a chaotic part (and each sequence of maps in the system has a chaotic part) . This is called the "**cooperation principle**".

Problem 0.7. (1) Find many randomness-induced phenomena in random complex dynamical systems.

(2) Find many randomness-induced phenomena in random real dynamical systems.

(3) Compare the difference between the phenomena between random real dynamical systems and random complex dynamical systems.

Note that in random real dynamical systems, there are many examples of robust chaoticity.

Remark 0.8. We also remark that regarding the random complex dynamical systems, even if the chaos disappears in the C^0 sense, the chaos (or some kind of complexity) may remain in the C^1 sense. More precisely, under certain conditions, there exists a constant $0 < \alpha_0 < 1$ such that

(1) for each α with $0 < \alpha < \alpha_0$, the system behaves well on the Banach space $C^\alpha(\hat{\mathbb{C}})$ of all α-Hölder continuous functions on $\hat{\mathbb{C}}$ endowed with α-Hölder norm $\| \cdot \|_\alpha$ (i.e., the iterations of M_τ on $C^\alpha(\hat{\mathbb{C}})$ satisfy a similar situation as in statement (3) of Theorem 0.2 with $(C(\hat{\mathbb{C}}), \| \cdot \|_\infty)$ replaced by $(C^\alpha(\hat{\mathbb{C}}), \| \cdot \|_\alpha)$, but

(2) for each α with $\alpha_0 < \alpha < 1$, the system does not behave well on the Banach space $C^\alpha(\hat{\mathbb{C}})$ (e.g. there exists an element $\varphi \in C^\alpha(\hat{\mathbb{C}})$ such that $\|M_\tau^n(\varphi)\|_\alpha \to \infty$ as $n \to \infty$).

Thus, regarding random (complex) dynamical systems, we have

gradation between chaos and order.

This is a new concept regarding random dynamical systems and it seems me very important to study it.

Problem 0.9. Study gradation between chaos and order regarding random dynamical systems which look "mild" and find the hidden complexity of the systems. Also, classify such systems in terms of the quantities which indicate the gradation between chaos and order (like α_0 in the above).

The following example illustrates the gradation between chaos and order.

Example 0.10 ([1, 2])**.** (**Devil's coliseums**.) Let $g_1(z) = z^2 - 1, g_2(z) = \frac{z^2}{4}$ and let $f_1 := g_1 \circ g_1, f_2 := g_2 \circ g_2$. Let $\tau = \sum_{j=1}^{2} \frac{1}{2}\delta_{f_j}$ and we consider the random dynamical system on $\hat{\mathbb{C}}$ associated with τ. That is, we consider the random dynamical system on $\hat{\mathbb{C}}$ such that at every step we choose f_1 with probability $1/2$ and we choose f_2 with probability $1/2$. It turns out that $\{f_1, f_2\}$ is mean stable. For each initial value $z \in \hat{\mathbb{C}}$, let $T_\infty(z)$ be the probability of tending to ∞ starting with the initial value z. Then, for each $\varphi \in C(\hat{\mathbb{C}})$ such that φ is 1 around ∞ and 0 in $\{z \in \mathbb{C} \mid |z| \leq 4\}$, we have $T_\infty(z) = \lim_{n\to\infty} M_\tau^n(\varphi)(z)$ (uniform convergence), $T_{\infty,\tau}$ is a Hölder continuous function on $\hat{\mathbb{C}}$ and $T_{\infty,\tau}$ varies precisely on the Julia set of the polynomial semigroup $G = \{f_{i_n} \circ \cdots \circ f_{i_1} \mid n \in \mathbb{N}, \forall i_j \in \{1, 2\}\}$ generated by $\{f_1, f_2\}$, which is a thin fractal set. It turns out that in this case, there exists an α_0 with $0 < \alpha_0 \leq 1/2$ such that the statements of Remark 0.8 hold. The function $T_{\infty,\tau}$ can be regarded as a **complex analogue of the devil's staircase (the Cantor function) (see Figure 1) or Lebesgue's singular functions (see Figure 2)**. The function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0, 1]$ is called a "**devil's coliseum**". For the figures of the Julia set of semigroup $G = \langle f_1, f_2 \rangle$ generated by $\{f_1, f_2\}$ and the graph of $T_{\infty,\tau}$, see Figures 3, 4, 5.

The reason why we can regard $T_{\infty.\tau}$ as a complex analogue of the devil's staircase or Lebesgue's singular functions is as follows. For each $x \in \mathbb{R}$, let $h_1(x) = 3x, h_2(x) = 3x - 2$ and we consider the random dynamical system on $\mathbb{R}$ (or $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$) such that at every step we choose h_1 with probability $1/2$ and we choose h_2 with probability $1/2$. Let $T_{+\infty}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then it turns out that $T_{+\infty}|_{[0,1]}$ is equal to the devil's staircase. For the figure of the graph of $T_{+\infty}|_{[0,1]}$, see Figure 1. Similarly, let $k_1(x) = 2x, k_2(x) = 2x - 1$ and and $p \in (0, 1)$ with $p \neq 1/2$. We consider the random dynamical system on $\mathbb{R}$ such that at every step we choose k_1 with probability p and choose k_2 with probability $1 - p$. Let $T_{+\infty,p}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then it turns out that $T_{+\infty,p}|_{[0,1]}$ is equal to the Lebesgue's singular function with parameter p. For the graph of $T_{+\infty,p}|_{[0.1]}$, see Figure 2.

Figure 1: The graph of the devil's staircase (Cantor function)

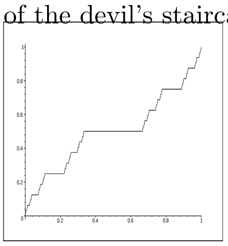


Figure 2: The graph of Lebesgue's singular function

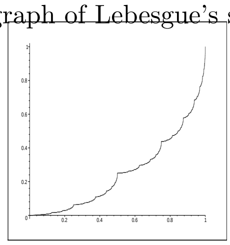


Figure 3: The Julia set of polynomial semigroup $G = \langle f_1, f_2 \rangle$ generated by $\{f_1, f_2\}$, where $g_1(z) := z^2 - 1, g_2(z) := z^2/4, f_1 := g_1^2, f_2 := g_2^2$. We have that $\{f_1, f_2\}$ is mean stable and $\dim_H(J(G)) < 2$.

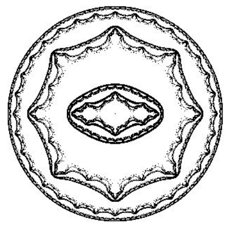


Figure 4: The graph of $T_{\infty,\tau}$, where $\tau = \sum_{i=1}^{2} \frac{1}{2}\delta_{f_i}$ with the same f_i as in Figure 3. $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$. The set of varying points of $T_{\infty,\tau}$ is equal to $J(G)$ in Figure 3. A"devil's coliseum" (A complex analogue of the devil's staircase).

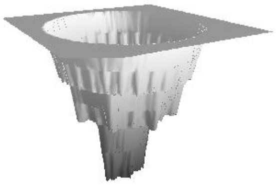
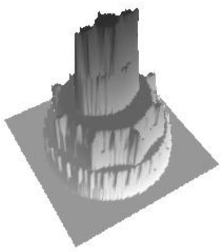


Figure 5: Figure 4 upside down. A "fractal wedding cake".



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