AN SDE APPROACH TO LEAFWISE DIFFUSIONS ON FOLIATED SPACES AND ITS APPLICATION TO A LIMIT THEOREM

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October 21, 2014

Abstract. This is the lecture note of my talk "An SDE approach to leafwise diffusions on foliated spaces and its application to a limit theorem" at the Boston University/Keio University Workshop 2014, September 15-19, 2014, at Boston University. We construct a leafwise diffusion on a compact foliated space with solving a stochastic differential equation. This construction enables us to prove a central limit theorem for a class of additive functionals of the leafwise diffusion.

1. Preliminaries

We introduce some notation and facts. Let Z be a locally compact, separable, metrizable space and U an open subset of $\mathbb{R}^d \times Z$. A function $f : U \to \mathbb{R}$ is said to be leafwise C^k (denoted by C^k_L) if $f(\cdot, z)$ is of C^k for any z and the partial derivative $(y, z) \mapsto \partial^\alpha_y f(y, z)$ is continuous for any multi-index α with $|\alpha| \leq k$. Similarly, a mapping $f : U \to \mathbb{R}^p$ is said to be C^k_L if each of the component functions is C^k_L. Now we give the definition of a foliated space by this smoothness.

Definition 1.1. Let M be a locally compact, separable, metrizable space. M is a d-dimensional foliated space (modeled transversely on Z) if there exist an open cover $\mathcal{U} = \{U_\alpha\}$ of M and homeomorphisms $\{\varphi_\alpha : U_\alpha \to B_{\alpha,1} \times B_{\alpha,2}\}$ such that if $U_\alpha \cap U_\beta \neq \emptyset$, then the coordinate change

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \ni (y_\alpha, z_\alpha) \mapsto (y_\beta(y_\alpha, z_\alpha), z_\beta(z_\alpha)) \in \varphi_\beta(U_\alpha \cap U_\beta)$$

satisfies that y_β is C^∞_L, z_β depends only on z_α and continuous, where $B_{\alpha,1}$ and $B_{\alpha,2}$ are open subsets of $\mathbb{R}^d$ and Z, respectively.

A subset of the form $\varphi_\alpha^{-1}(B_{\alpha,1} \times \{z\})$ is called a plaque. In addition the subset

$$L_x = \{ y \in M \ : \ \text{there exists a finite number of plaques } P_1, P_2, \dots, P_n \\ \text{such that } x \in P_1, y \in P_n \text{ and } P_i \cap P_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq n-1 \}$$

of M is called the leaf passing through x. Putting $\mathcal{L} = \{L_\lambda\}_{\lambda \in \Lambda}$ as the leaves of M, we note some facts:

(i) M is decomposed into the leaves, that is,

$$M = \bigsqcup_{\lambda \in \Lambda} L_\lambda.$$

(ii) L_λ is a d-dimensional smooth manifold for any $\lambda \in \Lambda$.

(iii) Foliated spaces are regarded as generalization of dynamical systems. For examples, a nonsingular flow on a manifold and a mapping torus induced by a topological dynamical system correspond to foliated spaces with one-dimensional leaves.

These basic facts for foliated spaces are available in [2] and [5].

Next we construct leafwise diffusions on M. In what follows, we assume that M is compact. We note that if M is compact, each leaf is not always compact. Let $C_L^k(M)$ be the totality of functions f on M satisfying that f is leafwise C^k in every chart and A_0, $A_1, \dots, A_r$ leafwise smooth vector fields on M. Each of them is a leafwise smooth section

$$A_\alpha \ : \ M \ni x = (y, z) \mapsto \sum_{i=1}^{d} A_\alpha^i(y, z) \frac{\partial}{\partial y^i} \in T_x(L_x)$$

of the tangent bundle of M. M does not always have a manifold structure. But we can regard the tangent space $T_x(L_x)$ of L_x at x as that of M at x. Also the leafwise elliptic differential operator A is defined by

$$A = \sum_{\alpha=1}^{r} A_\alpha A_\alpha + A_0.$$

Now we consider the stochastic differential equation

$$dX(t) = \sum_{\alpha=1}^{r} A_\alpha(X(t)) \circ dB^\alpha(t) + A_0(X(t))dt, \tag{1.1}$$

where the first term on the right-hand side is understood in the sense of the Fisk-Stratonovich integral.

2. Main theorem

For the equation (1.1), we can show the following.

Theorem 2.1 (Theorem 2.1 in [7]). (1) *The equation* (1.1) *has a unique strong solution. In particular, for any $x \in M$ there exists a solution $X^x = \{X^x(t)\}_{t\geq 0}$ of the equation* (1.1) *on the r-dimensional classical Wiener space (W_0^r, P^W) such that $X^x(0) = x$ P^W-a.s.*

(2) *X^x is stochastically continuous with respect to starting points. This means that*

$$X^x \to X^{x_0} \text{ in probability as } x \to x_0 \text{ in } M.$$

The family $X = \{X^x\}_{x\in M}$ of the solutions induces the diffusion on M generated by the operator A. We call $X = \{X^x\}_{x\in M}$ the A-leafwise diffusion.

Remark 2.2. (i) For any second order leafwise elliptic differential operator without zero order term on M, we obtain a leafwise diffusion generated by the operator with applying Theorem 2.1 to an SDE on the bundle of orthonormal frames of M. The proof of this fact is also given in [7].

(ii) Candel [1] constructed a leafwise diffusion generated by a leafwise elliptic differential operator via the Hille-Yosida theorem.

(iii) Our construction of leafwise diffusions provides a good approach to the limit problem as stated below.

Now we consider a class of Borel measures. A Borel measure m on M is said to be A-harmonic if

$$\int_M E\left[f(X^x(t))\right] m(dx) = \int_M f\, dm$$

for any $t \geq 0$ and any continuous function f on M. The notion of harmonic measures was introduced by Garnett [4] in the case of the leafwise Brownian motion on a compact foliated manifold. In addition, the basic results for harmonic measures can be found in [1], [3] and [8]. By the compactness of M and the 2nd result in Theorem 2.1, we see that there always exists an A-harmonic probability measure on M.

Finally we state a central limit theorem for the A-leafwise diffusion X.

THEOREM 2.3 (Theorem 4.4 in [7]). *Assume that there exists a unique A-harmonic probability measure m. For a real-valued function $g \in C^2_L(M)$ let $f = Ag$. Then for any $x \in M$, the stochastic processes defined by*

$$t \mapsto \frac{1}{\lambda} \int_0^{\lambda t} f(X^x(s))\, ds \tag{2.1}$$

converge in law to the Brownian motion with variance

$$\left(\int_M \sum_{\alpha=1}^{r} (A_\alpha g)^2 \, dm \right) \cdot t$$

for each time $t \geq 0$ as $\lambda \to \infty$.

REMARK 2.4. (i) If there are many A-harmonic probability measures on M, then we can show that the stochastic processes defined by (2.1) converge in law to a Brownian motion as $\lambda \to \infty$ for almost surely starting point with respect to any harmonic probability measure although the limiting variance may depend on the starting point (see [7]).

(ii) In the case of the leafwise Brownian motion on a mapping torus, the corresponding result to Theorem 2.3 is obtained and applied to the leafwise Brownian motion on a generalized Kronecker foliation in [6]. Moreover an elementary proof of the characterization of harmonic measures for the process is also given.

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