LYAPUNOV OPTIMIZING MEASURES FOR HÉNON-LIKE MAPS AT THE FIRST BIFURCATION

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ABSTRACT. We develop a thermodynamic formalism for a strongly dissipative Hénon-like map at the first bifurcation parameter at which the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set. For any $t \in \mathbb{R}$ we prove the existence of an invariant Borel probability measure which minimizes the free energy associated with a non continuous geometric potential $-t \log J^u$, where J^u denotes the Jacobian in the unstable direction. Under a mild condition, we show that any accumulation point of these measures as $t \to +\infty$ is a measure which minimizes the unstable Lyapunov exponent. We also show that the equilibrium measures converge as $t \to -\infty$ to a Dirac measure which maximizes the unstable Lyapunov exponent.

This is an excerpt from the paper [20].

1. INTRODUCTION

A basic problem in dynamics is to describe how structurally stable systems lose their stability through continuous modifications of the systems. The loss of stability of horseshoes through homoclinic bifurcations is modeled by a family of Hénon-like diffeomorphisms

$$f_a\colon (x,y) \in \mathbb{R}^2 \mapsto (1-ax^2,0) + b\cdot \Phi(a,b,x,y), \quad a\in\mathbb{R},\ 0<b\ll 1. \tag{1}$$

Here, Φ is bounded continuous in (a,b,x,y) and C^2 in (a,x,y). It is known [2, 8, 10, 18] that there is a *first bifurcation parameter* $a^* = a^*(b) \in \mathbb{R}$ with the following properties:

- $a^* \to 2$ as $b \to 0$;
- the non wandering set of f_a is a uniformly hyperbolic horseshoe for $a > a^*$;
- for $a = a^*$ there is a single orbit of homoclinic or heteroclinic tangency. If f_{a^*} preserves orientation, the tangency is homoclinic. Otherwise it is heteroclinic. The tangency is quadratic, and the family $\{f_a\}_{a\in\mathbb{R}}$ unfolds the tangency at $a = a^*$ generically.

The study of the map f_{a^*} opens the door to understanding the dynamics beyond uniform hyperbolicity in dimension two. In this paper we advance the thermodynamic formalism for f_{a^*} initiated in [15, 16]. We prove the existence of equilibrium measures for a family $\{\varphi_t\}_{t\in\mathbb{R}}$ of non continuous geometric potentials, and study accumulation points of these measures as $t \to \pm\infty$.

Write f for f_{a^*}. The non wandering set of f, denoted by Ω, is a compact f-invariant set. Let $\mathcal{M}(f)$ denote the space of f-invariant Borel probability measures endowed with the topology of weak convergence. For a potential function $\varphi\colon \Omega \to \mathbb{R}$ the minus of the free energy $F_\varphi\colon \mathcal{M}(f) \to \mathbb{R}$ is defined by

$$F_\varphi(\mu) = h(\mu) + \int \varphi d\mu,$$

where $h(\mu)$ denotes the entropy of μ. An *equilibrium measure* for the potential φ is a measure $\mu_\varphi \in \mathcal{M}(f)$ which maximizes F_φ, i.e.,

$$F_\varphi(\mu_\varphi) = \sup\{F_\varphi(\mu)\colon \mu \in \mathcal{M}(f)\}.$$

The existence and uniqueness of equilibrium measures depends upon the characteristics of the system and the potential. The family of potentials we are concerned with is

$$\varphi_t = -t \log J^u \quad t \in \mathbb{R},$$

where J^u denotes the Jacobian in the *unstable direction* which is defined as follows. For a point $x \in \mathbb{R}^2$ let E^u_x denote the one-dimensional subspace of $T_x\mathbb{R}^2$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^{-n}|E^u_x\| < 0. \tag{2}$$

Since f^{-1} expands area, the one-dimensional subspace of $T_x\mathbb{R}^2$ with this property is unique when it makes sense. We call E^u_x the *unstable direction at* x and define $J^u(x) = \|D_x f|E^u_x\|$. It was proved in [15, Proposition 4.1] that E^u_x makes sense for all $x \in \Omega$, and $x \in \Omega \mapsto E^u_x$ is continuous except at the fixed saddle near $(-1, 0)$ where it is merely measurable.

Since the chaotic behavior of f is created by the (non-uniform) expansion along the unstable direction, a good deal of information is obtained by studying the equilibrium measures for φ_t and the associated *pressure function* $t \in \mathbb{R} \mapsto P(t)$, where

$$P(t) = \sup\{F_{\varphi_t}(\mu)\colon \mu \in \mathcal{M}(f)\}.$$

The existence of equilibrium measures for φ_t was proved in [15] for all $t \leq 0$, and for those $t > 0$ such that $P(t)/t$ is slightly bigger than $-\log 2$. However, the arguments and the result in [15] do not cover sufficiently large $t > 0$. Our first theorem complements this point.

Theorem A. *Assume f preserves orientation. For any $t \in \mathbb{R}$ there exists an equilibrium measure for φ_t.*

For t in a large bounded interval, the uniqueness and some geometric/statistical properties of equilibrium measures were established in [16]. It would be nice to prove the uniqueness for all $t \in \mathbb{R}$, including the orientation reversing case.

Since t represents the inverse of the temperature in statistical mechanics, $t \to \pm\infty$ means that the temperature goes to zero. Hence, it is natural to study accumulation points of equilibrium measures for φ_t as $t \to \pm\infty$. They represent the lowest energy states, and may reflect the characteristics of the system.

The study of the behavior of the equilibrium measures as $t \to \pm\infty$ is also related to the ergodic optimization (See e.g. [5] and the references therein): given a continuous dynamical system T acting on a compact metric space X, and a real-valued function ϕ on X, one looks for T-invariant Borel probability measures which maximize the integral of ϕ. One way to do this is by freezing the system: to consider a family $\{t\phi\}_{t\in\mathbb{R}}$ of potentials and an associated family $\{\nu_t\}_{t\in\mathbb{R}}$ of equilibrium measures, and to let $t \to +\infty$. If the topological entropy is finite and the potential is continuous, then any accumulation point as $t \to +\infty$ maximizes the integral of ϕ. For uniformly hyperbolic systems (or the subshift of finite type), the convergence has been established for certain locally constant potentials [4, 12] as well as for a residual set of continuous potentials [9, 11]. However, little is known for non hyperbolic systems.

An *unstable Lyapunov exponent* of a measure $\mu \in \mathcal{M}(f)$ is a number $\lambda^u(\mu)$ defined by

$$\lambda^u(\mu) = \int \log J^u d\mu.$$

Of interest to us are measures which optimize the unstable Lyapunov exponent. Since the unstable Lyapunov exponent is not continuous as a function of measures, the existence of such measures is an issue. We show that any accumulation point of the equilibrium measures for $\varphi_t = -t \log J^u$ as $t \to \pm\infty$ optimizes the unstable Lyapunov exponent.

Set

$$\lambda^u_m = \inf\{\lambda^u(\mu)\colon \mu \in \mathcal{M}(f)\}.$$

A measure $\mu \in \mathcal{M}(f)$ is called *Lyapunov minimizing* if $\lambda^u(\mu) = \lambda^u_m$. Let Q denote the fixed point of f near $(-1, 0)$, and δ_Q the Dirac measure at Q.

Theorem B. *Assume f preserves orientation. Let $\{\mu_t\}_{t\in\mathbb{R}}$ be such that μ_t is an ergodic equilibrium measure for φ_t for all $t \in \mathbb{R}$. Then any accumulation point of $\{\mu_t\}_{t\in\mathbb{R}}$ as $t \to +\infty$ is δ_Q, or a Lyapunov minimizing measure. If $(1/2)\lambda^u(\delta_Q) \neq \lambda^u_m$, then any accumulation point as $t \to +\infty$ is Lyapunov minimizing.*

Since $\lambda^u(\delta_Q) \to \log 4$ and $\lambda^u_m \to \log 2$ as $b \to 0$, it is not easy to verify $(1/2)\lambda^u(\delta_Q) \neq \lambda^u_m$. However, from a given family (1) of Hénon-like diffeomorphisms one can construct another satisfying this condition by slightly perturbing the reminder term Φ.

It is worthwhile to compare Theorem B with the results of Leplaideur [13]. In this paper, he studied an orientation preserving non-uniformly hyperbolic horseshoe map with three symbols, with a single orbit of homoclinic tangency, introduced in [14]. Although this map is similar to our f at a first glance, its equilibrium measures converge as $t \to +\infty$ to a Dirac measure which maximizes the unstable Lyapunov exponent. He also proved the nonexistence of a measure which minimizes the unstable Lyapunov exponent.

Since the uniqueness of Lyapunov minimizing measures of f is not known, it is important to give a criterion for which one is "selected" in the limit $t \to +\infty$. The next theorem gives a criterion in terms of entropy. A Lyapunov minimizing measure $\mu \in \mathcal{M}(f)$ is called *entropy maximizing* if

$$h(\mu) = \{h(\nu)\colon \nu \in \mathcal{M}(f), h(\nu) \text{ is entropy maximizing}\}.$$

Theorem C. *Let f and $\{\mu_t\}_{t\in\mathbb{R}}$ be the same as in Theorem B. If $(1/2)\lambda^u(\delta_Q) \neq \lambda^u_m$, then any accumulation point as $t \to +\infty$ is entropy maximizing.*

We now turn to the case $t \to -\infty$. The next theorem holds regardless of the orientation of the map f.

Theorem D. *Let $\{\mu_t\}_{t\in\mathbb{R}}$ be such that μ_t is an ergodic equilibrium measure for φ_t for all $t \in \mathbb{R}$. Then μ_t converges to δ_Q as $t \to -\infty$.*

It follows from a proof of Theorem D that δ_Q is the unique measure which maximizes the unstable Lyapunov exponent. Apart from the uniqueness, the existence of such maximizing measures follows from the result in [7].

A key ingredient for proofs of the theorems is a control of the derivatives in the unstable direction. To this end we develop Benedicks-Carleson's critical point approach [3] further. The same strategy has been taken already in [15, 16], but substantial improvements are necessary to treat all $t > 0$. The assumption on the orientation of the map f will be used to construct measures with small unstable Lyapunov exponent, and to estimate the pressure $P(t)$ from below.

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