Measures with maximum total exponent of C^1 diffeomorphisms with basic sets

YUSUKE TOKUNAGA

We study ergodic optimization. For any diffeomorphism, measures with maximum total exponent can be defined. In order to define this notion, we begin with introducing the notation. Let M be a compact smooth Riemannian manifold without boundary. It is also assumed to be connected. Let T be a C^1-diffeomorphism. Consider a compact T-invariant set Λ. We denote by $\mathcal{M}(T,\Lambda)$ the space of all T-invariant Borel probability measures supported on Λ equipped with the weak-$*$ topology. We say that Λ is a basic set of T if Λ is isolated and hyperbolic for T and $T|_\Lambda : \Lambda \to \Lambda$ is topologically transitive. We recall that Λ is isolated for T if there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{i\in\mathbb{Z}} T^i(U)$ holds. Let $DT(x)$ be the derivative of T at $x \in M$. We denote by $J(T)(x)$ the Jacobian of T at $x \in M$, that is, the absolute value of the determinant of $DT(x)$.

In ergodic optimization, invariant Borel probability measures maximizing the integral of a given function are mainly considered. In our study, those of a specific function are investigated. Now we define measures with maximum total exponent.

Definition 1 *A measure $\nu \in \mathcal{M}(T,\Lambda)$ is called a measure with maximum total exponent on Λ for T if*

$$\int \log J(T)(x)d\nu(x) \geq \int \log J(T)(x)d\mu(x)$$

holds for any measure $\mu \in \mathcal{M}(T,\Lambda)$. *Let* $\mathcal{L}(T,\Lambda)$ *denote the set of all measures with maximum total exponent on* Λ *for* T.

That is, we are interested in T-invariant Borel probability measures maximizing the integral of the function $\log J(T)$. By virtue of the Oseledec theorem, we see that $\int \log J(T)(x) d\mu(x)$ is equal to the integral of the sum of all Lyapunov exponents of T with respect to μ. This is the reason why we use the term 'measure with maximum total exponent'. T is a C^1-diffeomorphism, so the function $\log J(T)$ is continuous on M. In addition, the space $\mathcal{M}(T,\Lambda)$ is compact, so the set $\mathcal{L}(T,\Lambda)$ is not empty, that is, there exists at least one measure with maximum total exponent on Λ for T.

The following theorem is the main result.

Theorem 1 *Let* $T: M \to M$ *be a* C^1*-diffeomorphism with a basic set* Λ. *Then there exists a* C^1*-neighborhood* $\mathcal{U}$ *of* T *such that a generic element* S *in* $\mathcal{U}$ *satisfies the following properties.*

(1) S *has a unique measure with maximum total exponent on the continuation* $\Lambda_S = \bigcap_{i\in\mathbb{Z}} S^i(U)$ *of* Λ *for* S.

(2) *Any measure with maximum total exponent on* Λ_S *for* S *has zero entropy.*

(3) *Any measure with maximum total exponent on* Λ_S *for* S *is fully supported on* Λ_S.

Next, we state one more result obtained from Theorem 1. To this end, we give further definitions. Let $\Omega(T)$ be the nonwandering set of T. We say that T is C^1-Ω-stable if for any element S in some C^1-neighborhood of T, there exists a conjugacy map from $\Omega(S)$ to $\Omega(T)$. From the definition, we see that the totality of C^1-Ω-stable C^r-diffeomorphisms is open in the space of all C^r-diffeomorphisms. It is known that every C^1-Ω-stable diffeomorphism satisfies Axiom A. Therefore, by virtue of Smale's spectral decomposition theorem, if T is C^1-Ω-stable, then there exist a finite number of disjoint basic sets $\Lambda_1, \dots, \Lambda_n$ of T such that $\Omega(T) = \bigcup_{i=1}^{n} \Lambda_i$. We call each Λ_i a Smale basic set. Then, applying Theorem 1 to each Smale basic set, we can obtain local properties about measures with maximum total exponent not only on a Smale basic set but also on M. But, in fact, we can obtain the following theorem.

Theorem 2 *Each of the following properties is generic in C^1-Ω-stable C^1-diffeomorphisms.*

(1) *There exists a unique measure with maximum total exponent on M.*

(2) *Any measure with maximum total exponent on M has zero entropy.*

(3) *Any measure with maximum total exponent on M is fully supported on one of the Smale basic sets.*

That is, we can obtain not only local properties but also global properties about measures with maximum total exponent on M. On the other hand, for C^1-Ω-stable diffeomorphisms with higher regularity, we have the following theorem.

Theorem 3 *Let $r \geq 2$. Then any measure with maximum total exponent on M for a generic C^1-Ω-stable C^r-diffeomorphism is not fully supported on every Smale basic set unless the basic set itself is a periodic orbit.*

That is, a generic C^1-Ω-stable C^r-diffeomorphism has never measures with maximum total exponent on M satisfying the properties in Theorem 2.

Now, we consider these theorems from a viewpoint of ergodic optimization. In 2006, O. Jenkinson have introduced the following definition.

Definition 2 *Let X denote a compact metrizable space. Let $T : X \to X$ be a continuous map. Let $f : X \to \mathbb{R}$ be a continuous function. Then a measure $\nu \in \mathcal{M}(T, X)$ is called an f-maximizing measure if*

$$\int f d\nu \geq \int f d\mu$$

holds for any measure $\mu \in \mathcal{M}(T, X)$.

Under this definition, O. Jenkinson proved the following theorem.

Theorem 4 *Let $T : X \to X$ be a transitive and hyperbolic continuous map with local product structure. Then a generic continuous function has a unique maximizing measure of full support.*

Although Theorem 4 holds, the following problem raised by O. Jenkinson is an open problem.

Problem Let $T : X \to X$ be any transitive and hyperbolic continuous map with local product structure. Find an explicit example of a continuous function with a unique maximizing measure of full support.

That is, O. Jenkinson says that it is difficult to give a concrete function satisfying the property in Theorem 4. In this problem, if X is a compact manifold and T is a C^1-diffeomorphism, then Theorem 2 gives a partial answer. In fact, a compact manifold M itself is a basic set for any transitive Anosov diffeomorphism, and the totality of transitive Anosov diffeomorphisms is an open subset of C^1-Ω-stable diffeomorphisms. So we have the following corollary.

Corollary 1 *For a generic transitive Anosov diffeomorphism $T : M \to M$, $\log J(T)$ is a function required in Problem.*

But, from the proof of Theorem 3, we see that not only for a generic but also for an arbitrary transitive Anosov C^2-diffeomorphism $T : M \to M$, $\log J(T)$ has never a unique maximizing measure of full support. So we have no answer to Problem for maps with higher regularity. Finally, we state a theorem giving a partial answer to Problem for expanding maps with higher regularity on the circle.

Theorem 5 *Let $r \geq 1$. Then for a generic C^r-expanding map on the circle, the r-th differential is a function required in Problem.*