Patterns in infinite domains

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1 Introduction

Patterns are everywhere in nature [9, 5, 3, 4] and have yet been systematically studied for barely a century [25]. Turing's pioneering work [26] established a profound mathematical basis for pattern forming phenomena, featuring the start of the systematical and still on-going research on the dynamics of pattern forming systems within the mathematical community. The application of dynamical systems techniques in evolutionary PDEs, initially developed as methods to solve problems arising in delayed differential equations and generalized to be applicable to general evolutionary PDEs by mathematicians including J. K. Hale and D. Henry in 1970's-1980's [18, 14], has become one of the main tools in this field.

The mathematical illustration of the formation mechanisms of patterns is concerned primarily with the existence of solutions representing patterns, their defects and their interfaces, together with their qualitative properties, such as linear and nonlinear stability, instabilities, bifurcations, etc. In this particular lecture notes, we focus on the existence of patterns, their defects and interfaces in infinite domains. As for the stability aspect, we refer to the excellent pieces [22, 6] and also Margaret's lecture notes for this workshop.

2 Infinite domains VS finite domains

A natural question is: In real world, almost all the physical systems are of finite scales, then why do we bother to even talk about infinite domains? Well, for patterns "away from boundary", the intrinsic structure is the "ideal patterns" in the corresponding infinite domain, which gives rise to the existence of patterns in bounded domains; see [27] for a rigorous treatment. To illustrate the intuition, a prototype example is the second order ODE,

$$u_{tt} = u(1-u),$$

which admits a homoclinic orbit and a family of periodic orbits within the homoclinic orbit, corresponding respectively to the "ideal pattern" on the whole real line and the periodic patterns in periodic finite intervals,

3 Examples

In this section, we utilize two examples to show the procedure of extracting "ideal patterns" from finite-domain patterns spotted in nature and science.

Example 3.1 (grain boundaries in the Rayleigh-Bénard convection [23]) *The Rayleigh-Bénard convection is the phenomenon when the temperature difference between two plates overcomes the viscosity of the fluid in between, there is an onset of instability, yielding convection rolls, together with its defects. One of the defects is the so-called grain boundaries; see Figure 3.1.*

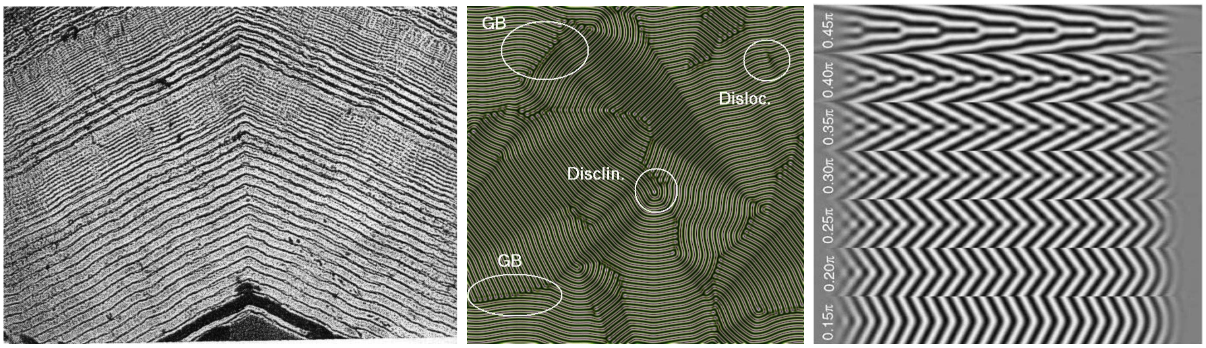

Figure 3.1: As a terminology initially from solid state physics(left), grain boundaries(middle, right) are one of the basic defects(middle) observed in the Rayleigh-Bénard convection.

To study grain boundaries, the model here is the Swift-Hohenberg equation on the whole plane,

$$\partial_t u = -(1+\Delta)^2 u + \mu u - u^3, \tag{3.1}$$

where $u(t,x,y)$ depends on $(x,y) \in \mathbb{R}^2$ and time $t \in \mathbb{R}$, and μ is a real parameter. Simple bifurcation analysis shows the existence of solutions $u_{\mathrm{r}}(kx;k,\mu)$ which are spatially periodic $u_{\mathrm{r}}(\xi;k,\mu) = u_{\mathrm{r}}(\xi + 2\pi;k,\mu)$, and even in ξ for $\mu > 0$, small. We refer to these stationary periodic patterns as roll solutions and denote rotated roll patterns as

$$u_{\mathrm{r}}^{\varphi}(x,y;k) := u_{\mathrm{r}}(k(x\cos\varphi - y\sin\varphi);k,\mu), \tag{3.2}$$

with $\varphi \in [0, 2\pi)$.

Grain boundaries are stationary solutions to (3.1), that are asymptotic to roll solutions of different orientation as $x \to \pm\infty$. In the simplest case that we shall be interested in, here, they possess an additional reflection symmetry $x \mapsto -x$ and periodic in y. This can be seen as a maximal symmetry assumption for a grain boundary, since the pattern imposed by asymptotic roll solutions with different angles accomodates such a reflection symmetry and periodicity. In all, assuming y-direction wavenumber k and rescaling y, the grain boundary solutions satisfy

$$\begin{cases} -(1+\partial_x^2 + k^2\partial_y^2)^2 u + \mu u - u^3 = 0, \\ u(x, y+2\pi) = u(x,y), \quad u(-x,y) = u(x,y), \\ \lim_{x\to\pm\infty} |u_{\mathrm{gb}}(x,y) - u_{\mathrm{r}}^{\pm\varphi}(x,y;k)| = 0. \end{cases} \tag{3.3}$$

Example 3.2 (pearling in the amphiphilic morphology [21]) *Amphiphilic materials are typically small molecules which contain both hydrophilic and hydrophobic components. This class of materials includes surfactants, lipids, and block copolymers. Their propensity to spontaneously assemble network morphologies, such as bilayers, pores, micelles, pearled partterns, end-caps and junctions, has drawn scientific attention for more than a century, [1]. We study pearled patterns here; see Figure 3.2.*

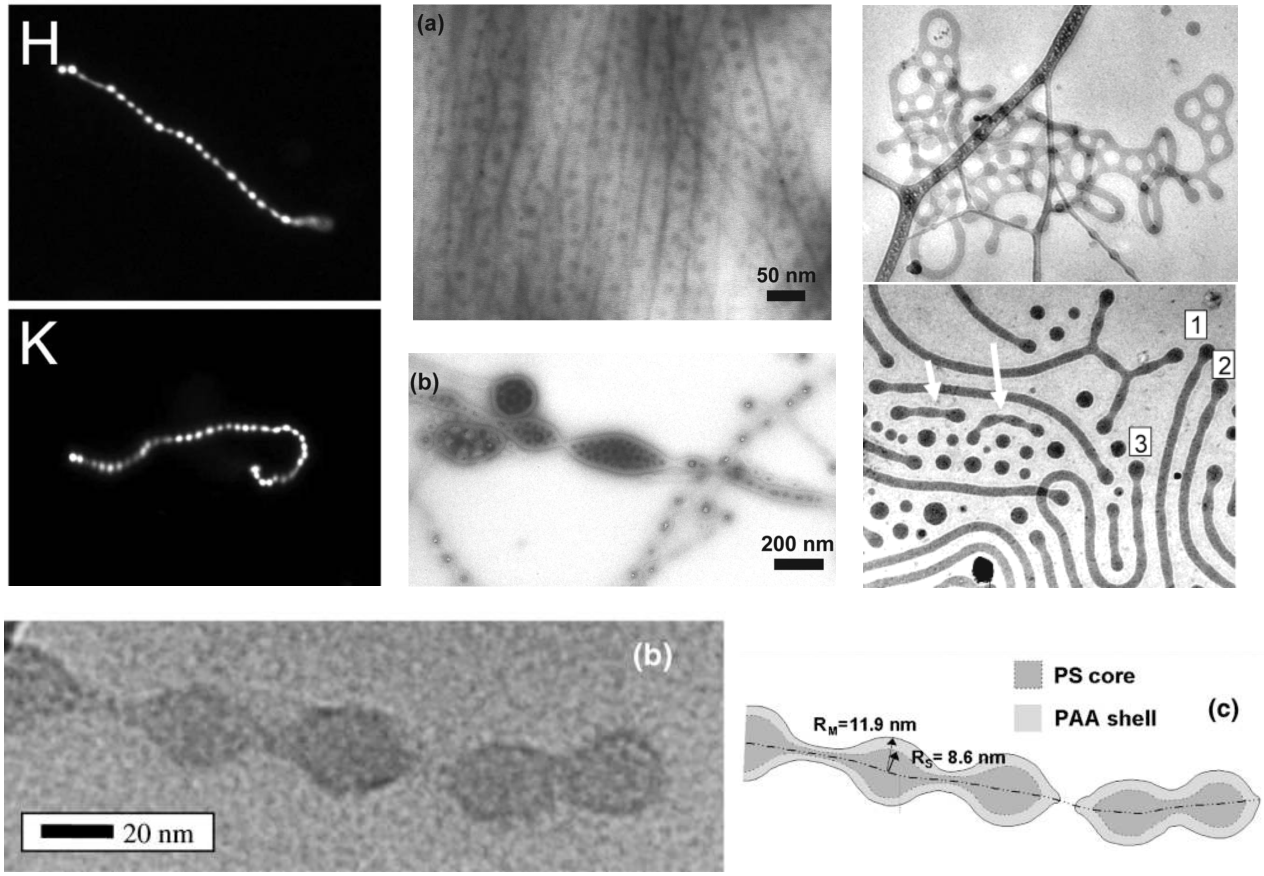

Figure 3.2: (top left) primitive membranes [8]; (top middle) diblock copolymer [28]; (top right) diblock copolymer [19]; (bottom) Copolymers [7]

Models of amphiphilic mixtures, such as [24] and [13], have been proposed. The functionalized Cahn-Hilliard (FCH) free energy; see [20, 12, 10], is a special case of these earlier models that supports stable network morphologies. In FCH model, extended pearled solutions can be viewed as small-amplitude modulations to stationary extended bilayers; see [21, 11, 17] for details.

4 Toolbox

To rigorously prove the existence of patterns in infinite domains, we briefly introduce the "spatial dynamics" toolbox. The primary idea is to recast the original PDE into an infinite-dimensional dynamical system by viewing one of the spatial variable as the new "time variable", which is reduced to an ODE system by a center manifold reduction. A normal form analysis usually follows to reveal the local dynamics of the reduced ODE system near the bifurcation. Pattern solutions typically corresponds to particular solutions such as equilibria, periodic solutions, etc, in the normal form system. Persistence arguments may be applied to push the results back to the whole reduced ODE system. For spatial dynamics, see [23, 21, 16] for concrete examples; For center-manifold-reduction and normal form analysis, see the excellent book [15] and the fantastic paper [2].

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