Existence of pearled patterns in the planar functionalized Cahn-Hilliard equation

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Outline

1. Introduction: pearled patterns and FCH model
2. Main result: existence of pearled bilayers in 2D FCH
3. Proof: spatial dynamics & degenerate 1:1 resonance
4. Outlook: multicomponent FCH systems

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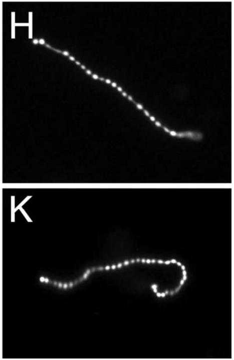
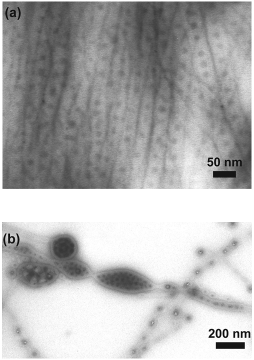
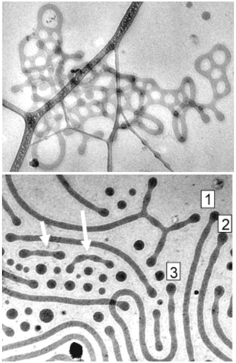
Pearling patterns



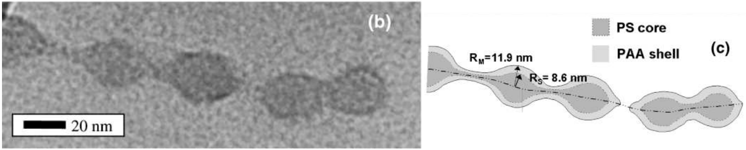
Primitive membranes[Szostak *et al*, 11']



Diblock copolymer[Hayward *et al*, 08']



Diblock copolymer[Bates *et al*, 04']



Copolymers[Bendejacq *et al*,05']

FCH: Cahn-Hilliard expansion

For amphiphilic mixtures: added higher derivatives to the classical Cahn-Hilliard energy [Teubner, Strey, 87'; Gompper, Schick, 90']

$$\mathcal{F}(u) := \int_\Omega f(u) + \varepsilon^2 A(u)|\nabla u|^2 + \varepsilon^2 B(u)\Delta u + \overbrace{C(u)}^{\geqslant 0} (\varepsilon^2 \Delta u)^2 \, dx.$$

For the primitive $\overline{A}$ of A, replace $A(u)\nabla u$ with $\nabla \overline{A}(u)$ and integrate by parts

$$\mathcal{F}(u) := \int_\Omega f(u) + (B(u) - \overline{A}(u))\varepsilon^2 \Delta u + C(u)(\varepsilon^2 \Delta u)^2 \, dx,$$

Complete the square

$$\mathcal{F}(u) := \int_\Omega \overbrace{C(u)}^{\frac{1}{2}} \Big(\varepsilon^2 \Delta u - \overbrace{\boxed{\frac{\overline{A} - B}{2C}}}^{W'(u)} \Big)^2 + \overbrace{\boxed{f(u) - \frac{(\overline{A} - B)^2}{C(u)}}}^{P(u)} \, dx.$$

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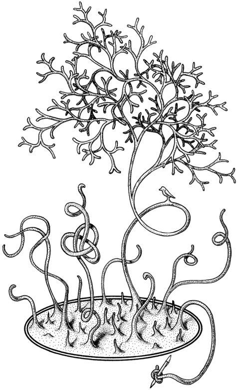
FCH: stabilization of equilibria of Cahn-Hilliard energy

Consider the functionalized Cahn-Hilliard energy

$$\mathcal{F}_{CH} = \int_\Omega \frac{1}{2}\boxed{\left(\varepsilon^2 \Delta u - W'(u)\right)^2} - \boxed{\varepsilon}\left(\frac{1}{2}\eta_1 \varepsilon^2 |\nabla u|^2 + \eta_2 W(u)\right) \mathrm{d}x, \quad (1)$$

in a large bounded domain Ω.



Unstable equilibrium in CH

$\Downarrow$

Potential stable equilibrium in FCH

- The square term stabilizes all the equilibria of CH energy, including the saddle points.
- The small functionalized term selects stable equilibria.

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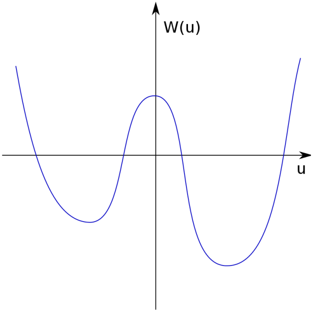
[Frank Morgan, Riemannian Geometry, 98']

Functionalized Cahn-Hilliard equation(FCH)

The FCH, $u_t = \Delta \frac{\delta \mathcal{F}_{CH}}{\delta u}$, that is,

$$u_t = \Delta\Big(\big(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1\big)\big(\varepsilon^2\Delta u - W'(u)\big) + \varepsilon\eta_d W'(u)\Big) \qquad (2)$$

is a gradient flow, preserving mass with zero-flux boundary condition.



ε–width of interfaces.
W–non-degenerate double-well potential.
η_1–interfacial parameter(amphiphilicity).
η_2–pressure parameter.
$\eta_d = \eta_1 - \eta_2$.

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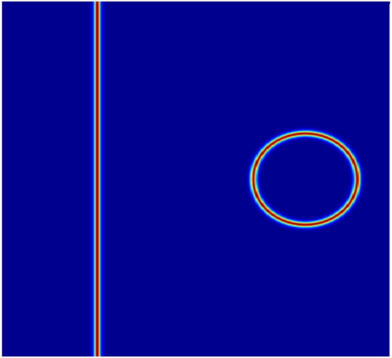
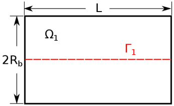
Pearling: bifurcation of bilayers along interfaces

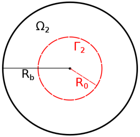
We look for pearled solutions to the stationary 2D FCH

$$(\Delta - W''(u) + \varepsilon\eta_1)(\Delta u - W'(u)) + \varepsilon\eta_d W'(u) = \varepsilon\gamma. \qquad (3)$$





Ω_1 and Γ_1



Ω_2 and Γ_2

Bilayers–symmetric pulse profiles along interfaces(single layer: front).
Pearled patterns–small amplitude modulations of bilayer width.

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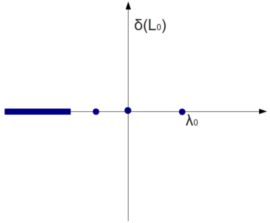
FCH: extended flat bilayer solutions

Lemma (Existence of extended flat bilayers)

Fix $\eta_1, \eta_d, \gamma = \mathcal{O}(1) \in \mathbb{R}$. Assume $\varepsilon > 0$ sufficiently small, there exists a flat bilayer solution–a homoclinic solution $u_b(r)$ to the ODE

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - W''(u) + \varepsilon\eta_1\right)\boxed{\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}u - W'(u)\right)} + \varepsilon\eta_d W'(u) = \varepsilon\gamma.$$

[A. Doelman, G. Hayrapetyan, K. Promislow, and B. Wetton, 14']



- $u_b(r) = u_0(r) + \mathcal{O}(\varepsilon)$, where u_0 is the homoclinic orbit to $u_{rr} - W'(u) = 0$.
- $\mathcal{L}_0 := \partial_r^2 - W''(u_0)$ admits only one positive eigenvalue, denoted as λ_0.

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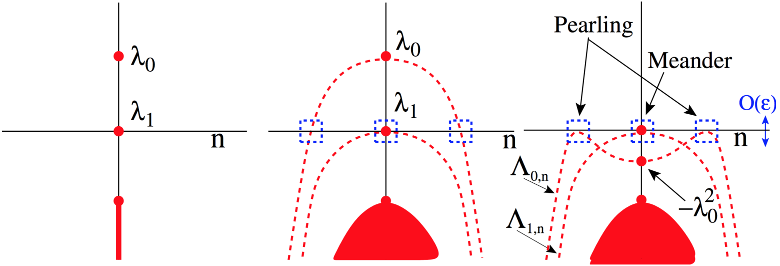
FCH: spectral analysis

Linearizing the rescaled stationary 2D FCH in the extended plane,

$$\left(\partial_r^2 - W''(u) + \varepsilon^2\partial_\tau^2 + \varepsilon\eta_1\right)\left(\partial_r^2 u - W'(u) + \varepsilon^2\partial_\tau^2 u\right) + \eta_d W'(u) = \varepsilon\gamma, \quad (4)$$

at the bilayer u_b gives the linearized operator

$$\mathbb{L} := \frac{\delta^2\mathcal{F}}{\delta u^2}(u_b) = \boxed{(\mathcal{L}_0 + \varepsilon^2\partial_\tau^2)^2} + \mathcal{O}(\varepsilon).$$



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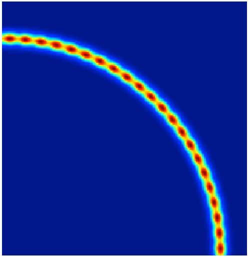
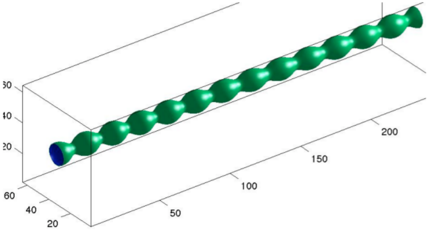
FCH: pearled solutions

Pearled solutions–a tangential periodic modulation of the flat bilayer solution u_{b} on the plane $\mathbb{R}^2$. More precisely, symmetric extended pearled solutions u_{p} are solutions to

$$\begin{cases} \left(\partial_r^2 - W''(u) + \varepsilon^2\partial_\tau^2 + \varepsilon\eta_1\right)\left(\partial_r^2 u - W'(u) + \varepsilon^2\partial_\tau^2 u\right) + \eta_d W'(u) = \varepsilon\gamma, \\ \lim_{r\to\pm\infty} |u(x,r) - u_\infty| = 0, \tau \in \mathbb{R}, \\ u(-\tau, r) = u(\tau, r), u(\tau + T_{\mathrm{p}}, r) = u(\tau, r), (\tau, r) \in \mathbb{R}^2, \end{cases} \tag{5}$$

where u_∞ and T_{p} are constants to be determined.





Assumption—existence of extended pearled solutions

Assumption

We assume that

(i) W is a fixed non-degenerate double well potential.

(ii) There are two primary parameters

$$\begin{aligned} \alpha_0 &:= \frac{1}{4\lambda_0^2}\int_{\mathbb{R}} \left(W'''(u_0)\mathcal{L}_0 u_0 - \eta_d W''(u_0)\right)\psi_0^2 \mathrm{d}r > 0, \\ \beta_0 &:= \frac{1}{4\lambda_0^2}\int_{\mathbb{R}} \left(W'''(u_0)\mathcal{L}_0 u_0 - \eta_d W''(u_0)\right)\psi_1^2 \mathrm{d}r \neq 0, \end{aligned} \tag{6}$$

where

$$\begin{cases} \mathcal{L}_0\psi_0 = \lambda_0\psi_0, \quad \mathcal{L}_0\psi_1 = 0, \\ \|\psi_0\|_{L^2} = \|\psi_1\|_{L^2} = 1. \end{cases}$$

Remark

Basically, $\alpha_0 = c_1\gamma + c_2\eta_d$, $\beta_0 = c_3\gamma + c_4\eta_d$, where c_j's depend only upon the shape of the well.

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Main result I–existence of extended pearled solutions

Theorem (existence of extended pearled flat bilayers)

Fix $\eta_1, \eta_d, \gamma = \mathcal{O}(1) \in \mathbb{R}$. *Under the Assumption, there exist* $\varepsilon_0, \kappa_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$*, up to translation, the stationary 2D-FCH,*

$$\left(\partial_r^2 - W''(u) + \varepsilon^2 \partial_\tau^2 + \varepsilon\eta_1\right)\left(\partial_r^2 u - W'(u) + \varepsilon^2 \partial_\tau^2 u\right) + \varepsilon\eta_d W'(u) = \varepsilon\gamma,$$

admits a smooth 1-parameter family of extended pearled solutions, $u_{\mathrm{p}}(\tau, r; \sqrt[4]{\varepsilon}, \sqrt{|\kappa|})$ *with period* $T_{\mathrm{p}}(\sqrt[4]{\varepsilon}, \sqrt{|\kappa|})$*, parameterized by* $\kappa \in [-\kappa_0, \kappa_0]$*. In fact, we have*

$$\begin{aligned}
u_{\mathrm{p}} &= u_{\mathrm{b}}(r;\gamma) + \boxed{2\frac{\sqrt{\varepsilon|\kappa|}}{\sqrt[4]{\alpha_0}} \cos\left(\frac{2\pi}{T_{\mathrm{p}}}\tau\right)\psi_0(r)} + \mathcal{O}\left(\varepsilon(\sqrt{\varepsilon} + \sqrt{|\kappa|})\right), \\
T_{\mathrm{p}} &= \frac{2\pi\varepsilon}{\sqrt{\lambda_0}}\left[1 - \sqrt{\alpha_0\varepsilon} + \mathcal{O}\left(\varepsilon(1+\sqrt{\kappa})\right)\right], \\
u_\infty &= \lim_{r\to\infty} u_{\mathrm{b}}(r;\gamma),
\end{aligned} \tag{7}$$

where the error is in the $L^\infty(\mathbb{R}^2)$*-norm.*

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Main result II–existence of extended pearled solutions

Theorem (existence of extended pearled circular bilayers)

Fix $\eta_1, \eta_d = \mathcal{O}(1)$, $R_0 \in \mathbb{R}$. *Under the Assumption, there exist* $\varepsilon_0, \kappa_0 > 0$ *and* $\gamma(\varepsilon)$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$, *up to translation, the stationary 2D-FCH in the infinite stripe* $(\theta, r) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$,

$$\left(\partial_r^2 - W''(u) + \frac{\varepsilon\partial_r}{R_0+\varepsilon r} + \frac{\varepsilon^2\partial_\theta^2}{(R_0+\varepsilon r)^2} + \varepsilon\eta_1\right)\left(\partial_r^2 u - W'(u) + \frac{\varepsilon\partial_r u}{R_0+\varepsilon r} + \frac{\varepsilon^2\partial_\theta^2 u}{(R_0+\varepsilon r)^2}\right) + \varepsilon\eta_d W'(u) = \varepsilon\gamma.$$

admits a discrete family of extended pearled solutions, $u_{\mathrm{p}}(\tau, r; \sqrt[4]{\varepsilon}, \sqrt{|\kappa_j|})$ *with period* $T_{\mathrm{p}}(\sqrt[4]{\varepsilon}, \sqrt{|\kappa_j|})$, *parameterized by* $\{\kappa_j\}_{j\in I} \subset [-\kappa_0, \kappa_0]$. *In fact, we have*

$$\begin{aligned}
u_{\mathrm{p}} &= u_{\mathrm{b}}(r;\gamma) + \boxed{2\frac{\sqrt{\varepsilon|\kappa|}}{\sqrt[4]{\alpha_0}}\cos\left(\frac{2\pi}{T_{\mathrm{p}}}\tau\right)\psi_0(r)} + \mathcal{O}\left(\varepsilon(\sqrt{\varepsilon} + \sqrt{|\kappa|})\right),\\
T_{\mathrm{p}} &= \frac{2\pi\varepsilon}{R_0\sqrt{\lambda_0}}\left[1 - \sqrt{\alpha_0\varepsilon} + \mathcal{O}\left(\varepsilon(1+\sqrt{\kappa})\right)\right] \in \left\{\frac{2\pi}{n} \mid n \in \mathbb{Z}\backslash\{0\}\right\},\\
u_\infty &= \lim_{r\to\infty} u_{\mathrm{b}}(r;\gamma),
\end{aligned} \tag{8}$$

where the error is in the $L^\infty(\mathbb{R}^2)$*-norm.*

Remark–existence of extended pearled solutions

- The results are in unbounded domains – both flat and circular bilayers.
- For the flat bilayers, for each value of γ, we get a one-parameter family of pearled solutions in terms of κ.
- For the circular bilayer, we have only one value of γ, and we get an countable family of pearled bilayers, parameterized by $\kappa_j = \kappa_j(\varepsilon, R_0)$.
- Note that for a physical system containing a single amphiphilic material, the parameter η_1, η_2 and ε are fixed.
  - According to the Theorem I, the FCH admits a two-parameter family of pearled flat bilayers, parameterized by γ and κ.
  - According to the Theorem II, there is an interesting tuning of the radius and the discrete parameter κ. For fixed $\mathbb{R}_0$, the family has distinct amplitudes, but quite similar periods, which arises from the degenerate 1:1 degeneracy.

Parameters–κ and γ

- κ– an $\varepsilon^{-3/2}$-scaled first integral in the *degenerate 1:1 resonance*.
- γ–the far-field state.

Large bounded domains retain the results[Sandstede et al, 11'].

- κ-tuned by period; γ–fixed by mass conservation.
- The influence of κ:
  - second order on the period–$\mathcal{O}(\varepsilon^2\sqrt{|\kappa|})$;
  - first-order on the amplitude–$\mathcal{O}(\sqrt{|\kappa|})$.
- For pearled circular bilayers, the periods satisfy

$$\frac{2\pi}{T_{\mathrm{p}}} = \mathcal{O}\left(1 + \boxed{\sqrt{|\kappa|} + \frac{R_0}{\varepsilon}}\right) \in \mathbb{Z}^+.$$

Thus, the tuning is $\mathcal{O}(\varepsilon)$ in R_0 and $\mathcal{O}(1)$ in κ. Consequently, small changes in R_0 have a huge influence on κ, and thus on the pearling amplitude but not the period.

Pearling bifurcation as $\alpha_0 \to 0$

Consider the FCH with fixed ε. Recall the pearled solution

$$u_{\mathrm{p}} = u_h(r) + \boxed{2\frac{\sqrt{\varepsilon|\kappa|}}{\sqrt[4]{\alpha_0}} \cos\left(\frac{2\pi}{T_{\mathrm{p}}}\tau\right)\psi_0(r)} + \mathcal{O}\left(\varepsilon(\sqrt{\varepsilon} + \sqrt{|\kappa|})\right).$$

The *degenerate* 1 : 1 *resonance* shows that

$$\sqrt{\varepsilon_0}\kappa_0 < \frac{\alpha_0}{2|\alpha_2|},$$

where α_2 is generically nonzero. Therefore, we have

$$\lim_{\alpha_0 \to 0} \boxed{2\frac{\sqrt{\varepsilon|\kappa|}}{\sqrt[4]{\alpha_0}} \cos\left(\frac{2\pi}{T_{\mathrm{p}}}\tau\right)\psi_0(r)} = 0.$$

That is to say, up to leading order, for fixed ε, the amplitude of the pearling dies out as $\alpha_0 \to 0$. This fact indicates that this degenerate bifurcation retains some supercritical characteristics.

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Idea of the proof

The proof can be summarized into the following steps:

- Rewrite the PDE (4),

$$\left(\partial_r^2 - W''(u) + \varepsilon^2\partial_\tau^2 + \varepsilon\eta_1\right)\left(\partial_r^2 u - W'(u) + \varepsilon^2\partial_\tau^2 u\right) + \eta_d W'(u) = \varepsilon\gamma,$$

  as an infinite-dimension dynamical system via spatial dynamics,
- Reduce the PDE (4) to an ODE system via center manifold reduction,
- Obtain the normal form of the reduced ODE system,
- Find transformed pearling solutions in the degenerate 1:1 resonance normal form,
- Show persistence of pearling solutions in the full ODE via an implicit-function-theorem argument on a Poincaré map.

Spatial dynamics

We rewrite (4) as an infinite-dimension dynamical system.

- View τ as the "time" variable and apply the rescaling $t = \frac{\sqrt{\lambda_0}}{\varepsilon}\tau$
- Let $U := (u, u_t, \mathcal{L}_b u + \lambda_0 u_{tt}, (\mathcal{L}_b u + \lambda_0 u_{tt})_t)$,
- Linearize the PDE (5) around the bilayer u_{b}.

$$\dot{U} = \mathbb{L}(\varepsilon)U + \mathbb{F}(U, \varepsilon), \tag{9}$$

where

$$\mathbb{L}(\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{\lambda_0}\mathcal{L}_b & 0 & \frac{1}{\lambda_0} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\lambda_0}V & 0 & -\frac{1}{\lambda_0}(\mathcal{L}_b + \varepsilon\eta_1) & 0 \end{pmatrix}, \mathbb{F}(U, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\lambda_0}\mathcal{F} \end{pmatrix}.$$

and $\mathcal{L}_b := \partial_{rr} + W''(u_{\mathrm{b}})$, $V := \varepsilon\eta_d W''(u_{\mathrm{b}}) - \left(\partial_r^2 u_{\mathrm{b}} - W'(u_{\mathrm{b}})\right) W'''(u_{\mathrm{b}})$, $\mathcal{F}$ is the nonlinear term. Note that $\mathbb{L} : \mathcal{Y} \to \mathcal{X}$ is colsed, where

$$\mathcal{X} = H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}), \quad \mathcal{Y} = H^4(\mathbb{R}) \times H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R}).$$

Center manifold reduction

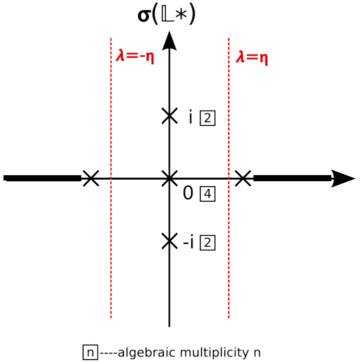
The center manifold reduction is based on the spectral analysis of $\mathbb{L}_\star := \mathbb{L}(0) \sim (\mathcal{L}_0 + \lambda_0 \partial_{tt})^2$.

Spectrum: $\sigma(\mathbb{L}_\star) = \{\lambda \in \mathbb{C} \mid d(\lambda, \mu) = (\mu + \lambda_0 \lambda^2)^2 = 0, \mu \in \sigma(\mathcal{L}_0)\}$

Imaginary spectrum: $\sigma(\mathbb{L}_\star) \cap i\mathbb{R} = \{0, \pm i\}$.

Center manifold reduction of (9) to a reversible ODE system:

$$\frac{\mathrm{d}U_c}{\mathrm{d}t} = \mathbb{P}_c \mathbb{L}(\varepsilon)(U_c + \Psi(U_c, \varepsilon)) + \mathbb{P}_c \mathbb{F}(U_c + \Psi(U_c, \varepsilon)). \tag{10}$$



- (10) is an 8th-order system.
- U_c is the projection of U onto the center subspace.
- $\|\Psi(U_c, \varepsilon)\|_{\mathcal{Y}} = \mathcal{O}((|\varepsilon| \|U_c\| + \|U_c\|^2)$.
- 0 is geo. 1 & alg. 4.
- $\pm$i is geo. 1 & alg. 2.

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Normal form

The normal form(NF) of the reduced ODE (10), up to cubic terms, is

$$\begin{cases} \dot{C}_1 = & \mathrm{i}(1+\omega_1\varepsilon)C_1 + C_2 + \mathcal{P}_{3,1}(\mathbf{C}), \\ \dot{C}_2 = & -\alpha_0\varepsilon C_1 + \mathrm{i}(1+\omega_1\varepsilon)C_2 + \mathcal{P}_{3,2}(\mathbf{C}) \\ \dot{D}_1 = & D_2, \\ \dot{D}_2 = & D_3, \\ \dot{D}_3 = & D_4, \\ \dot{D}_4 = & \omega_3\varepsilon D_1 + \omega_4\varepsilon D_3 + \mathcal{P}_{3,8}(\mathbf{C}). \end{cases} \tag{11}$$

with higher order terms

$$\mathcal{O}\left(\varepsilon^2\|\mathbf{C}\| + \varepsilon\|\mathbf{C}\|^2 + \|\mathbf{C}\|^4\right).$$

- $\mathcal{P}_{3,j}(\mathbf{C})$ are homogeneous polynomials of degree 3 in $\mathbf{C}$.
- Equations for $\overline{C_1}$ and $\overline{C_2}$ omitted.
- The normal form gains an extra symmetry–the rotational symmetry, related to time-translation invariance.

Invariant subspace & 1:1 resonance

Invariant subspace $S_I = \{\mathbf{C} \mid D_j = 0, j = 1, 2, 3, 4\}$
In S_I, the NF (11) becomes the pearling normal form (PNF)

$$\begin{cases} \dot{C}_1 = & \mathrm{i}(1+\omega_1\varepsilon)C_1 + C_2 + \mathrm{i}C_1\left[\alpha_7 C_1\bar{C}_1 + \alpha_8\mathrm{i}(C_1\bar{C}_2 - \bar{C}_1C_2)\right], \\ \dot{C}_2 = & \mathrm{i}(1+\omega_1\varepsilon)C_2 + \mathrm{i}C_2\left[\alpha_7 C_1\bar{C}_1 + \alpha_8\mathrm{i}(C_1\bar{C}_2 - \bar{C}_1C_2)\right] + \\ & C_1\left[-\alpha_0\varepsilon + \cancel{\alpha_1 C_1\bar{C}_1} + \mathrm{i}\alpha_2(C_1\bar{C}_2 - \bar{C}_1C_2)\right], \end{cases} \tag{12}$$

admitting a degenerate1:1 resonance[G. Iooss, M. Pérouème 93'].

- 1:1 resonance typically occurs in reversible and Hamiltonian systems, where the spectrum is symmetric with respect to both axies, thus a co-dim 1 bifurcation.
- Two symmetries $\Rightarrow$ two first integrals(Noether's theorem)

$$K = \frac{\mathrm{i}}{2}(C_1\overline{C_2} - \overline{C_1}C_2), \quad H = |C_2|^2 + (-\alpha_0\varepsilon + 2\alpha_2 K)|C_1|^2.$$

- For fixed K and H, ODE (12) $\Rightarrow$ a 2nd order ODE.

$$\left(\frac{\mathrm{d}u_1}{\mathrm{d}t}\right)^2 = 4f_{H,K}(u_1) := 4\left[(-\alpha_0\varepsilon + 2\alpha_2 K)u_1^2 + Hu_1 - K^2\right], \tag{13}$$

where $u_1 = |C_1|^2$.

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1:1 resonance dynamics

Lemma (degenerate 1:1 resonance)

For sufficiently small $\varepsilon > 0$*,*

(i) $\alpha_0 < 0$ *, the PNF system* (12) *has no periodic solutions.*

(ii) $\alpha_0 > 0$*, the PNF system* (12) *possesses a family of periodic orbits, depending on* $\kappa := \varepsilon^{-3/2}K$*, admitting the form*

$$\begin{aligned} C_1^p(t,\theta;\sqrt{\varepsilon},\sqrt{|\kappa|}) &= \sqrt{\varepsilon|\kappa|}\,r_1 \mathrm{e}^{\mathrm{i}(\omega t+\theta)}, \\ C_2^p(t,\theta;\sqrt{\varepsilon},\sqrt{|\kappa|}) &= \operatorname{sgn}(\kappa)\mathrm{i}\varepsilon\sqrt{|\kappa|}\,r_2 \mathrm{e}^{\mathrm{i}(\omega t+\theta)}, \end{aligned} \tag{14}$$

where

$$\left\{ \begin{array}{l} r_1(\sqrt{\varepsilon},\sqrt{|\kappa|}) = (\alpha_0 - 2\alpha_2\sqrt{\varepsilon}\kappa)^{-1/4}, \\ r_2 = \frac{1}{r_1}, \\ \omega = 1 + \omega_1\varepsilon + \operatorname{sgn}(\kappa)\sqrt{\varepsilon}r_2^2 + \alpha_7\varepsilon|\kappa|r_1^2 + 2\alpha_8\varepsilon^{3/2}\kappa. \end{array} \right. \tag{15}$$

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Outlook: Multicomponent FCH systems

Milnikov condition↔traveling speed↔intrinsic curvature

The multicomponent FCH energy is

$$\mathcal{F}(\mathbf{u}) = \int_\Omega |\varepsilon^2\Delta\mathbf{u} - \nabla_{\mathbf{u}} W(\mathbf{u}) + \varepsilon\mathbf{P}(\mathbf{u})|^2 + ...\mathrm{d}x \tag{16}$$

To make (16) of order $\mathcal{O}(\varepsilon^2)$, we want to solve the ODE system

$$\mathbf{u}_{rr} + \varepsilon H(s)\mathbf{u}_r - \nabla_{\mathbf{u}} W(\mathbf{u}) + \varepsilon\mathbf{P}(\mathbf{u}) = 0. \tag{17}$$

If the $\varepsilon = 0$ problem has an asymmetric homoclinic, we need to tune the Melnikov parameter b_0 to obtain persistence, that is,

$$\mathbf{u}_{rr} + \varepsilon b_0\mathbf{u}_r - \nabla_{\mathbf{u}} W(\mathbf{u}) + \varepsilon\mathbf{P}(\mathbf{u}) = 0. \tag{18}$$

Plugging the difference between (26) and (18) into (16) and take the sharp interface limit gives an intrinsic curvature type term

$$\int_\Gamma a(H - H_0)^2 + ...\mathrm{d}s$$

Canham-Helfrich sharp interface energy for a codim-one interface

$$\int_\Gamma a(H - H_0)^2 + b + cK\mathrm{d}s$$

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Thank You!