CENTRAL LIMIT THEOREM FOR A CLASS OF TRANSFORMATIONS WITH QUASI-COMPACT PERRON-FROBENIUS OPERATOR

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Abstract. We consider a class of nonsingular transformations of a probability space, which contains piecewise C^2 expanding maps on the interval as concrete examples. Given a transformation T in such a class and a real-valued function f satisfying an appropriate condition, we can show a central limit theorem of mixed type with nice convergence rate for the sum $\sum_{k=0}^{n-1} f \circ T^k$ as $n \to \infty$ provided that the limiting variance is nondegenerate.

1. Preliminaries

In this section, we give some notation and results to state the central limit theorem. First of all we introduce the Perron-Frobenius operator, which is key tool in the study of the central limit theorem for dynamical systems. In the sequel, functions are complex-valued unless otherwise stated. Let $(X, \mathcal{B}, m)$ be a probability space and $T : X \to X$ m-nonsingular transformation, i.e. T is a measurable transformation on $(X, \mathcal{B})$ satisfying $m(T^{-1}A) = 0$ for any $A \in \mathcal{B}$ with $m(A) = 0$. We write T^n as the n-fold iteration of T. For $1 \leq p \leq \infty$, $L^p(m)$ denotes the usual $L^p(m)$-space with respect to the the measure m endowed with the L^p-norm $\| \cdot \|_{p,m}$. Then the Perron-Frobenius operator $\mathcal{L}_{T,m} : L^1(m) \to L^1(m)$ for T with respect to m is defined as

$$\mathcal{L}_{T,m} f = \frac{d}{dm} \int_{T^{-1}(\cdot)} f \, dm$$

for $f \in L^1(m)$, where the right hand side in the above denotes the Radon-Nikodym derivative of the complex-valued measure $\mathcal{B} \ni A \mapsto \displaystyle\int_{T^{-1}(A)} g \, dm \in \mathbb{C}$ with respect to m. Also, we can define the Perron-Frobenius operator as the operator characterized by the identity

$$\int_X (\mathcal{L}_{T,m} f) g \, dm = \int_X f (g \circ T) \, dm$$

for any $f \in L^1(m)$ and for any $g \in L^\infty(m)$. The Perron-Frobenius operator is very useful for investigation into the existence of absolutely continuous invariant measures and

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their ergodic properties. We present [1, 7, 9, 12, 13] as references of basic facts on the Perron-Frobenius operator.

In what follows, we consider a nonsingular transformation satisfying following conditions (V) and (C).

(V) There exists a Banach space $(V, \|\cdot\|_V)$ of functions on X such that the following holds.

(V.1) V is continuously embedded in $L^\infty(m)$ and dense in $L^1(m)$.

(V.2) V contains the totality of constant functions and is closed under multiplication of functions. Moreover, there exists $C_0 > 0$ such that $\|fg\|_V \leq C_0\|f\|_V\|g\|_V$ holds for any f, $g \in V$.

(C) The Perron-Frobenius operator $\mathcal{L}_{T,m}$ is a bounded linear operator on V satisfying the following conditions.

(C.1) As a family of bounded linear operators acting on V, $\{\mathcal{L}^n_{T,m}\}_{n\in\mathbb{N}}$ is uniformly bounded, i.e. $\sup_{n\in\mathbb{N}} \|\mathcal{L}^n_{T,m}\|_V < \infty$.

(C.2) The Perron-Frobenius operator $\mathcal{L}_{T,m}$ is a quasi-compact operator on V, i.e. there exist $n_0 \in \mathbb{N}$ and a compact operator $K : V \to V$ such that $\|\mathcal{L}^{n_0}_{T,m} - K\|_V < 1$ holds.

Then we can show the following theorem on the existence of a so-called "ergodic component", which plays an important role to obtain the main result.

THEOREM 1.1. *Assume that an m-nonsingular transformation T satisfies conditions* (V) *and* (C). *Then there exist a positive integer $n(T)$ and a finite number of m-absolutely continuous T-invariant probability measures $\mu_1, \ldots, \mu_{n(T)}$ such that the following holds.*

(1) *For each $1 \leq i \leq n(T)$, the measure-theoretic dynamical system (T, μ_i) is ergodic.*
(2) *Let μ be an m-absolutely continuous T-invariant probability measure on the measurable space $(X, \mathcal{B})$. Then μ can be represented as a convex combination of μ_i's.*
(3) *For each $1 \leq i \leq n(T)$, we put $\Lambda_i = (d\mu_i/dm > 0)$ and $\Delta_i = \bigcup_{n=0}^{\infty} T^{-n}\Lambda_i$. Then we have $m\left(\bigcup_{i=1}^{n(T)} \Delta_i\right) = 1$ and $m(\Delta_i \cap \Delta_j) = 0$ if $i \neq j$.*

In this report, we call each of the sets Δ_i an ergodic component of T.

2. Main result

Let T be an m-nonsingular transformation. Assume that an m-nonsingular transformation T satisfies conditions (V) and (C). Given a real-valued element $f \in V$, then the limit

$$\sigma_i(f)^2 = \lim_{n\to\infty} \frac{1}{n} \int_X (S_n f - n a_i(f))^2 \, d\mu_i$$

exists for each $1 \leq i \leq n(T)$, where $S_n f = \sum_{k=0}^{n-1} f \circ T^k$ and $a_i(f) = \displaystyle\int_X f \, d\mu_i$. Now we are ready to state the central limit theorem of mixed type.

Theorem 2.1. *Let T, f, $a_i(f)$'s and $\sigma_i(f)$'s be the same as above, $\{\Delta_i\}_{i=1}^{n(T)}$ ergodic components of T and m_g an m-absolutely continuous probability measure with density $g \in V$. We assume that $\sigma_i(f) > 0$ for all $1 \leq i \leq n(T)$. Then there exists a positive constant C independent of choice of g and n such that for any $g \in V$ we have*

$$\sup_{y\in\mathbb{R}} \left| m_g\left(\frac{S_n f}{\sqrt{n}} \leq y\right) - \sum_{i=1}^{n(T)} \frac{m_g(\Delta_i)}{\sqrt{2\pi}\sigma_i(f)} \int_{-\infty}^{y} \exp\left(-\frac{(t-\sqrt{n}a_i(f))^2}{2\sigma_i(f)^2}\right) dt \right| \leq \frac{C}{\sqrt{n}} \|g\|_V$$

Theorem 2.1 is a generalization of a sort of limit theorem, which is proved by Hiroshi Ishitani for Lasota-Yorke type maps of the unit interval under some technical conditions in [7]. He assumes that each ergodic measure μ_i is weakly mixing. Then, he proves central limit theorem of mixed type with the order $O\left(1/n^{1/4}\right)$ for the rate of convergence. In addition, if an absolutely continuous probability measure m_g is an invariant measure, then he shows that the rate of convergence is $O\left(1/\sqrt{n}\right)$. However, in this study we shows that the rate of convergence is $O\left(1/\sqrt{n}\right)$ without such an assumption. Furthermore, we extend his result to the central limit theorem mixed type for a class of nonsingular transformations with quasi-compact Perron-Frobenius operator.

Remark 2.2. In [8], Ishitani announced that the rate of convergence of the central limit theorem of mixed type for Lasota-Yorke type maps is $O\left(1/\sqrt{n}\right)$ without the assumption that each ergodic measure μ_i is weakly mixing and an absolutely continuous probability measure m_g is an invariant measure.

3. Examples

In this section we give some examples of nonsingular transformations and a Banach space satisfying the the condition (V.1) and (V.2) such that the Perron-Frobenius operator of T with respect to m restricted to V fulfills the condition (C.1) and (C.2).

Example 3.1. (Lasota-Yorke type map) Let X be the unit interval $[0,1]$, $\mathcal{B}$ the σ- algebra of Borel subsets of $[0,1]$ and m the Lebesgue measure on $[0,1]$. We consider the Banach space $(BV, \|\cdot\|_{BV})$ as the totality of elements in $L^1(m)$ with versions of bounded variation endowed with the norm defined as $\|\cdot\|_V = \|\cdot\|_{1,m} + \bigvee(\cdot)$, where $\bigvee(\cdot)$ defined by $\bigvee g = \inf\{V\tilde{g}\ ;\ \tilde{g}$ is a function of bounded variation which is version of $g\}$ and $V\tilde{g}$ means the total variation of $\tilde{g}$. Then the Banach space $(BV, \|\cdot\|_{BV})$ satisfies the conditions (V.1) and (V.2).

Next we introduce the Lasota-Yorke type map. Let us consider an m-nonsingular transformation T with the properties (LY.1) and (LY.2) below:

(LY.1) $\mathcal{L}_{T,m}$ is a bounded linear operator on BV.

(LY.2) There exist $0 < \alpha < 1$, $\beta > 0$ and $n_0 \in \mathbb{N}$, such that

$$\bigvee \mathcal{L}_{T,m}^{n_0} f \leq \alpha \bigvee f + \beta \|f\|_{1,m} \tag{3.1}$$

holds for each $f \in BV$.

We call the m-nonsingular transformation T with the properties (LY.1) and (LY.2) a Lasota-Yorke type map and the inequality (3.1) is said to be the Lasota-Yorke type inequality. From the Lasota-Yorke type inequality, it is not hard to see that the family of bounded linear operators $\{\mathcal{L}_{T,m}^n\}_{n\geq 1}$ is uniformly bounded on BV. In addition, by virtue of the theorem of Ionescu-Tulcea and Marinescu [6], we can show the quasi-compactness of the Perron-Frobenius operator $\mathcal{L}_{T,m}$ as an operator on BV. Thus we have verified the validity of the conditions (C.1) and (C.2). As is well-known that various transformations on the interval have the properties (LY.1) and (LY.2), for example, piecewise C^2 expanding maps, the continued fraction map and piecewise convex maps and so on (see [7, 10, 14]).

Example 3.2. (Subshift of finite type)

First of all, we summarize some basic notions and results that we need. For a fixed integer $d \geq 2$, we consider the set $\mathcal{S} = \{1, 2, \ldots, d\}$. Let Σ^+ denote the set of all one-sided sequences i.e. $\Sigma^+ = \{(x_i)_{i\in\mathbb{Z}_{\geq 0}} : x_i \in \mathcal{S}$ for every $i \in \mathbb{Z}_{\geq 0}\}$. We endow Σ^+ with the product topology corresponding to the discrete topology on $\mathcal{S}$, which is a compact metrizable space. Given $0 < \theta < 1$, we introduce a metric d_θ on Σ^+ by setting

$$d_\theta(x,y) = \begin{cases} \theta^n & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for $x = (x_i)_{i\in\mathbb{Z}_{\geq 0}}$ and $y = (y_i)_{i\in\mathbb{Z}_{\geq 0}}$, where n is the smallest non-negative integer such that $x_n \neq y_n$. Then the topology induced by the metric d_θ on Σ^+ coincides with the product topology of Σ^+. Next, we consider the shift transformation $\sigma : \Sigma^+ \to \Sigma^+$ defined as $\sigma(x)_i = x_{i+1}$ for $x = (x_i)_{i\in\mathbb{Z}_{\geq 0}}$. It is seen that the shift transformation σ is a continuous surjective map on Σ^+. The dynamical system (Σ^+, σ) is called the full shift with alphabet $\mathcal{S}$. For a $d \times d$ zero-one matrix $A = (A(ij))_{i,j\in\mathcal{S}}$, the subshift of finite type defined by the transition matrix A is the restriction σ_A of the full shift $\sigma : \Sigma^+ \to \Sigma^+$ to the closed invariant subset $\Sigma_A^+ = \{x \in \Sigma^+ : A(x_i x_{i+1}) = 1 \text{ for every } i \in \mathbb{Z}_{\geq 0}\}$. Note that Σ_A^+ can be an empty set. In the sequel, we assume that the transition matrix A is irreducible i.e. for any i, $j \in \mathcal{S}$, there exists a positive integer $n_0 \geq 1$ such that $A^{n_0}(ij) > 0$. Then the set Σ_A^+ is not empty. As is well-known that the transition matrix A is irreducible if and only if the dynamical system (Σ_A^+, σ_A) is topologically transitive subshift of finite type.

Let $C(\Sigma_A^+)$ be the Banach space of all complex-valued continuous functions on Σ_A^+ endowed with the supremum norm $\|f\|_\infty = \sup_{x\in\Sigma_A^+} |f(x)|$. We define the Banach space $F_\theta(\Sigma_A^+)$ as the family of all complex-valued Lipschitz continuous functions on Σ_A^+ with respect to the metric d_θ, endowed with the norm $\|\cdot\|_\theta = \|\cdot\|_\infty + |\cdot|_\theta$, where $|\cdot|_\theta = \sup_{k\in\mathbb{N}} |\cdot|_{\theta,k}$ and $|\cdot|_{\theta,k}$ defined by $|\ f\ |_{\theta,k} = \sup\{|f(x) - f(y)|\,/d_\theta(x,y) : \text{ for } x, y \in \Sigma_A^+ \text{ such that } x \neq y \text{ and } d_\theta(x,y) \leq \theta^k\}$. In what follows, we often write as $F_\theta(\Sigma_A^+ \to \mathbb{R})$ to denote the set of all real-valued elements in $F_\theta(\Sigma_A^+)$.

Given a function $\phi \in F_\theta(\Sigma_A^+ \to \mathbb{R})$, let us consider a bounded linear operator $\mathcal{L}_\phi$ on $C(\Sigma_A^+)$ defined by

$$\mathcal{L}_\phi f(x) = \sum_{x\in\sigma_A^{-1}(y)} \exp(\phi(y)) f(y)$$

for $f \in C(\Sigma_A^+)$. Such an operator is called the transfer operator. One can easily verified that the transfer operator $\mathcal{L}_\phi$ is a bounded linear operator acting on $(F_\theta(\Sigma_A^+), \|\cdot\|_\theta)$. Without loss of generality, we may assume that $\mathcal{L}_\phi 1 = 1$. In order to apply our results to subshift of finite type, we recall so-called the Ruelle-Perron-Frobenius theorem, which concerns the spectral properties of the transfer operator (see Theorem 1.5 in [1]).

Consider the case when the probability space $(X, \mathcal{B}, m)$ is the space Σ_A^+ endowed with the σ-algebra of Borel subsets of Σ_A^+ and the Gibbs measure μ_ϕ with respect to the potential $\phi \in F_\theta(\Sigma_A^+ \to \mathbb{R})$ or the reference measure ν_ϕ. Here, the Banach space $(V, \|\cdot\|_V)$ is defined as the space $(F_\theta(\Sigma_A^+), \|\cdot\|_\theta)$. By using the standard functional calculus, we see that the conditions (V.1) and (V.2) are satisfied. Note that the subshift of finite type σ_A is an m-nonsingular transformation, so this will enable us to consider the Perron-Frobenius operator of σ_A with respect to m. It is easy to see that the transfer operator $\mathcal{L}_\phi$ can be extended to a bounded linear operator on $L^1(m)$ and it coincides with the Perron-Frobenius operator of σ_A with respect to m. Therefore the Ruelle-Perron-Frobenius theorem yields that the Perron-Frobenius operator $\mathcal{L}_{\sigma_A,m}$ acting on $F_\theta(\Sigma_A^+)$ is quasi-compact and the family of bounded linear operators $\{\mathcal{L}^n_{\sigma_A,m}\}_{n\geq 1}$ is uniformly bounded on $F_\theta(\Sigma_A^+)$. Hence the validity of the conditions (C.1) and (C.2) have been verified.

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