Local risk-minimization for Barndorff-Nielsen and Shephard models

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This talk is based on a joint work with Yuto Imai (Waseda) and Ryoichi Suzuki (Keio)

Outline

1. Local risk-minimization
2. Barndorff-Nielsen and Shephard models
3. Main results
4. Numerical experiments

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2. Barndorff-Nielsen and Shephard models
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Discrete time model

A self-financing strategy satisfies the following at each trading time n:

$$\eta_n + \xi_n S_n = \eta_{n+1} + \xi_{n+1} S_n (= V_n(\varphi))$$

In particular, we have $V_n(\varphi) = V_0(\varphi) + \sum_{k=1}^n \xi_k \Delta S_k$.

Introduction to pricing theory (cont'd)

A market is called complete, if, for any F, we can find a $c \in \mathbb{R}$ and a predictable process ξ satisfying

$$F = c + \int_0^T \xi_t dS_t.$$

The pair (c, ξ) is called the perfect hedge of F.
Note that (c, ξ) has a one-to-one corresponding to φ for self-financing strategies.

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In this talk, we focus on hedging strategies for incomplete markets.
Instead of the perfect hedge, we consider a replicating strategy which is not self-financing.

Risk minimization

Consider a strategy $\varphi := (\xi, \eta)$, which is not necessarily self-financing.
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φ is said risk-minimizing if φ satisfies $F = V_T(\varphi)$ and

$$R_t(\varphi) \leq R_t(\widetilde{\varphi}) \ \ \mathbb{P}\text{-}a.s. \text{ for every } t \in [0, T]$$

for any strategy $\widetilde{\varphi}$ satisfying $F = V_T(\widetilde{\varphi})$.

Local risk-minimization

Assumption 1

1. S is a special semimartingale with the canonical decomposition $S = S_0 + M + A$.
2. We can find a predictable process Λ such that $dA = \Lambda d\langle M\rangle$.
3. The mean-variance trade-off process $K_t := \int_0^t \Lambda_s^2 d\langle M\rangle_s$ is finite, that is, K_T is finite $\mathbb{P}$-a.s.

Local risk-minimization (cont'd)

Definition

1. Θ_S denotes the space of all $\mathbb{R}$-valued predictable processes ξ satisfying $\mathbb{E}[\int_0^T \xi_t^2 d\langle M\rangle_t + (\int_0^T |\xi_t dA_t|)^2] < \infty$.

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2. An L^2-strategy is given by a pair $\varphi = (\xi, \eta)$, where $\xi \in \Theta_S$ and η is an adapted process such that $V(\varphi) := \xi S + \eta$ is a right continuous process with $\mathbb{E}[V_t^2(\varphi)] < \infty$ for every $t \in [0, T]$.
   Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the riskfree asset) an investor holds at time t.

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   Note that ξ_t (resp. η_t) represents the amount of units of the risky asset (resp. the riskfree asset) an investor holds at time t.
3. For $F \in L^2(\mathbb{P})$, the process $C^F(\varphi)$ defined by

$$C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$$

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4. An L^2-strategy φ is said locally risk-minimizing strategy for F if $V_T(\varphi) = 0$ and $C^F(\varphi)$ is a martingale orthogonal to M, that is, $C^F(\varphi)M$ is a martingale.

Föllmer-Schweizer decomposition

An $F \in L^2(\mathbb{P})$ admits an FS decomposition if it can be described by

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \quad (1)$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ and L^F is a square-integrable martingale orthogonal to M with $L_0^F = 0$.

[1] Schweizer, M.: Local Risk-Minimization for Multidimensional Assets and Payment Streams. Banach Center Publ. 83, 213–229 (2008)

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Proposition 5.2 of Schweizer[1]
Under Assumption 1, LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition, and its relationship is given by

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F\mathbf{1}_{\{t=T\}} - \xi_t^F S_t.$$

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We define $Z := \mathcal{E}(-\int \Lambda dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y.
Note that Z is a solution to the SDE $dZ_t = -\Lambda_t Z_{t-} dM_t$.

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Note that Z is a solution to the SDE $dZ_t = -\Lambda_t Z_{t-} dM_t$.

Under Assumption 1, if Z is a positive square integrable martingale, then an MMM $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$.

Precedence research[2]

We consider a Lévy market as follows:

[2] Arai, T., Suzuki, R.: Local risk minimization for Lévy markets. International Journal of Financial Engineering, 2, 1550015 (2015)

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Assume that S is given by a solution to the following SDE:

$$dS_t = S_{t-}\left[\alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z}\widetilde{N}(dt, dz)\right], \quad S_0 > 0,$$

where α, β and γ are predictable processes.

Here, W is a 1-dimensional Brownian motion, N is a Poisson random measure, ν is its Lévy measure and $\widetilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$.

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Precedence research (cont'd)

Under some assumptions, A. and Suzuki gave the following expression of locally risk minimizing(LRM) strategy ξ^F for claim F:

$$\xi^F_t := \frac{\Lambda_t}{\alpha_t}\{h^0_t\beta_t + \int_{\mathbb{R}_0} h^1_{t,z}\gamma_{t,z}\nu(dz)\},$$

where $\Lambda_t := \dfrac{\alpha_t}{S_{t-}(\beta^2_t + \int_{\mathbb{R}_0}\gamma^2_{t,z}\nu(dz))}$, $u_t := \Lambda_t S_{t-}\beta_t$, $\theta_{t,z} := \Lambda_t S_{t-}\gamma_{t,z}$.

Moreover,

$$h^0_t := \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\left[\int_0^T D_{t,0}u_s dW^{\mathbb{P}^*}_s + \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}}\widetilde{N}^{\mathbb{P}^*}(ds,dx)\right]\Big|\mathcal{F}_{t-}\right],$$

$$h^1_{t,z} := \mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z} - 1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_{t-}],$$

where $H^*_{t,z} := \exp\{zD_{t,z}\log Z_T - \log(1-\theta_{t,z})\}$.

Precedence research (cont'd)

However, we need to assume the following:

Assumption (A)

1. u, $u^2 \in \mathbb{L}_0^{1,2}$; and $2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2 \in L^2(q \times \mathbb{P})$ for a.e. $s \in [0, T]$.
2. $\theta + \log(1 - \theta) \in \widetilde{\mathbb{L}}_1^{1,2}$, and $\log(1 - \theta) \in \mathbb{L}_1^{1,2}$
3. For q-a.e. $(s, x) \in [0, T] \times \mathbb{R}_0$, $\exists \varepsilon_{s,x} \in (0, 1)$ such that $\theta_{s,x} < 1 - \varepsilon_{s,x}$.
4. $Z_T \left\{ D_{t,0} \log Z_T 1_{\{0\}}(z) + \frac{e^{zD_{t,z} \log Z_T} - 1}{z} 1_{\mathbb{R}_0}(z) \right\} \in L^2(q \times \mathbb{P})$.
5. $F \in \mathbb{D}^{1,2}$; and $Z_T D_{t,z} F + F D_{t,z} Z_T + z D_{t,z} F \cdot D_{t,z} Z_T \in L^2(q \times \mathbb{P})$.
6. $FH^*_{t,z}$, $H^*_{t,z} D_{t,z} F \in L^1(\mathbb{P}^*)$ for q-a.e. $(t, z) \in [0, T] \times \mathbb{R}$,
   where $H^*_{t,z} := \exp\{zD_{t,z} \log Z_T - \log(1 - \theta_{t,z})\}$.

Precedence Research (cont'd)

Deterministic coeffcients case

In the case where α, β, and γ are given by deterministic functions satisfying the following three conditions, if condition 5 in Assumption (A) and $Z_T F \in L^2(\mathbb{P})$ are satisfied, then ξ^F is given as

$$\xi^F_t = \frac{\beta_t \mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}.$$

Conditions on α, β, and γ

1. $\gamma_{t,z} > -1$, $dt\nu(dz)$-a.e.
2. $\sup_{t\in[0,T]}(|\alpha_t| + \beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)) < C$ for some $C > 0$.
3. We can find a positive number ε such that

$$\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz) > \varepsilon \text{ and } \frac{\alpha_t\gamma_{t,z}}{\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)} < 1 - \varepsilon.$$

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BNS models

In this talk, we calculate locally risk-minimizing (LRM) strategies for Barndorff-Nielsen and Shephard (BNS) models [3]:

$$S_t = S_0 \exp\left\{\int_0^t \left(\mu - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s + \rho H_{\lambda t}\right\}.$$

where $S_0 > 0$, $\rho \leq 0$, $\mu \in \mathbb{R}$, $\lambda > 0$, H is a subordinator without drift, and

[3] Barndorff-Nielsen, O.E., Shephard, N.: Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial econometrics. J.R. Statistic. Soc. 63, 167–241 (2001)

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$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dH_{\lambda t}, \quad \sigma_0^2 > 0.$$

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$\rho H_{\lambda t}$ represents leverage effect.

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where $\alpha = \mu + \int_0^\infty (e^{\rho x} - 1)\nu(dx)$.
Here,

$$(J_t :=) H_{\lambda t} = \int_0^\infty x N([0, t], dx) \text{ and } \widetilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt,$$

where ν is the Lévy measure of J.

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Note that σ^2 is represented as

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dJ_s.$$

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Assumption

Denote $L_t := \log(S_t/S_0)$ for $t \in [0, T]$, that is,

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Assumption (BNS)

1. $\int_0^1 x\nu(dx) + \int_1^\infty \exp\{2(\mathcal{B}(T) \vee |\rho|)x\}\nu(dx) < \infty$, where $\mathcal{B}(t) := \frac{1-e^{-\lambda t}}{\lambda}$ for $t \in [0, T]$.
2. $\frac{\alpha}{e^{-\lambda T}\sigma_0^2 + C_\rho} > -1$, where $C_\rho := \int_0^\infty (e^{\rho x} - 1)^2 \nu(dx)$.

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1. Item 1 ensures $\int_0^\infty x^2 \nu(dx) < \infty$.
2. By $|e^{\rho x} - 1| \leq -\rho x$, we have $\int_0^\infty (e^{\rho x} - 1)^2 \nu(dx) \leq \int_0^\infty \rho^2 x^2 \nu(dx) < \infty$.

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2. By $|e^{\rho x} - 1| \leq -\rho x$, we have $\int_0^\infty (e^{\rho x} - 1)^2\nu(dx) \leq \int_0^\infty \rho^2 x^2 \nu(dx) < \infty$.
3. Item 2 ensures $\frac{\alpha}{\sigma_t^2 + C_\rho} > -1$ for any $t \in [0, T]$.

Representative examples of σ^2

IG-OU

The first is the case where the Lévy measure ν^H of the subordinator H is given as

$$\nu^H(dx) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1 + b^2 x)e^{-\frac{1}{2}b^2 x}\mathbf{1}_{(0,\infty)}(x)dx$$

where $a > 0$ and $b > 0$.
In this case, the invariant distribution of the squared volatility process σ^2 follows an inverse-Gaussian distribution with parameters $a > 0$ and $b > 0$.
σ^2 is called an IG-OU process.

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$$\nu^H(dx) = \frac{a}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1 + b^2x)e^{-\frac{1}{2}b^2x}\mathbf{1}_{(0,\infty)}(x)dx$$

where $a > 0$ and $b > 0$.
In this case, the invariant distribution of the squared volatility process σ^2 follows an inverse-Gaussian distribution with parameters $a > 0$ and $b > 0$.
σ^2 is called an IG-OU process.

If $\frac{b^2}{2} > 2(\mathcal{B}(T) \vee |\rho|)$, then item 1 of Assumption (BNS) is satisfied.

Representative examples of σ^2 (cont'd)

Gamma-OU

The second example is what we call Gamma-OU case, that is, the case where the invariant distribution of σ^2 is given by a Gamma distribution with parameter $a > 0$ and $b > 0$.
In this case, ν^H is described as

$$\nu^H(dx) = abe^{-bx}1_{(0,\infty)}(x)dx.$$

where $a > 0$ and $b > 0$.

Representative examples of σ^2 (cont'd)

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If $b > 2(\mathcal{B}(T) \vee |\rho|)$, then item 1 of Assumption (BNS) is satisfied.

Minimal martingale measure

Now, we consider the following SDE:

$$dZ_t = -Z_{t-}\Lambda_t dM_t, \quad Z_0 = 1,$$

where $\Lambda_s := \frac{1}{S_{s-}}\frac{\alpha}{\sigma_s^2 + C_\rho}$ and $C_\rho := \int_0^\infty (e^{\rho x} - 1)^2 \nu(dx)$.

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$$u_s := \Lambda_s S_{s-}\sigma_s = \frac{\alpha\sigma_s}{\sigma_s^2 + C_\rho} \quad \text{and} \quad \theta_{s,x} := \Lambda_s S_{s-}(e^{\rho x} - 1) = \frac{\alpha(e^{\rho x} - 1)}{\sigma_s^2 + C_\rho}$$

for $s \in [0, T]$ and $x \in (0, \infty)$, we have $\Lambda_t dM_t = u_t dW_t + \int_0^\infty \theta_{t,z}\widetilde{N}(dt, dz)$; and

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$$Z_t = \exp\left\{-\int_0^t u_s dW_s - \frac{1}{2}\int_0^t u_s^2 ds + \int_0^t \int_0^\infty \log(1 - \theta_{s,x})\widetilde{N}(ds, dx) + \int_0^t \int_0^\infty (\log(1 - \theta_{s,x}) + \theta_{s,x})\nu(dx)ds\right\}.$$

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Proposition

The process Z is a martingale with $Z_T \in L^2(\mathbb{P})$.

LRM for put options

Theorem

For $K > 0$, LRM $\xi^{(K-S_T)^+}$ of put option $(K - S_T)^+$ is represented as

$$\xi_t^{(K-S_T)^+} = \frac{1}{S_{t-}(\sigma_t^2 + C_\rho)} \Big\{ \sigma_t^2 \mathbb{E}_{\mathbb{P}^*}[-1_{\{S_T<K\}} S_T | \mathcal{F}_{t-}] + \int_0^\infty \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+ (H^*_{t,z} - 1) + zH^*_{t,z} D_{t,z} (K - S_T)^+ | \mathcal{F}_{t-}] \times (e^{\rho z} - 1)\nu(dz) \Big\},$$

where $H^*_{t,z} := \exp\{zD_{t,z} \log Z_T - \log(1 - \theta_{t,z})\}$ for $(t, z) \in [0, T] \times (0, \infty)$.

Reminder

Assumption (A)

1. u, $u^2 \in \mathbb{L}_0^{1,2}$; and $2u_s D_{t,z} u_s + z(D_{t,z} u_s)^2 \in L^2(q \times \mathbb{P})$ for a.e. $s \in [0, T]$.
2. $\theta + \log(1 - \theta) \in \widetilde{\mathbb{L}}_1^{1,2}$, and $\log(1 - \theta) \in \mathbb{L}_1^{1,2}$
3. For q-a.e. $(s, x) \in [0, T] \times \mathbb{R}_0$, $\exists \varepsilon_{s,x} \in (0, 1)$ such that $\theta_{s,x} < 1 - \varepsilon_{s,x}$.
4. $Z_T \left\{ D_{t,0} \log Z_T 1_{\{0\}}(z) + \frac{e^{zD_{t,z} \log Z_T} - 1}{z} 1_{\mathbb{R}_0}(z) \right\} \in L^2(q \times \mathbb{P})$.
5. $F \in \mathbb{D}^{1,2}$; and $Z_T D_{t,z} F + F D_{t,z} Z_T + z D_{t,z} F \cdot D_{t,z} Z_T \in L^2(q \times \mathbb{P})$.
6. $FH^*_{t,z}$, $H^*_{t,z} D_{t,z} F \in L^1(\mathbb{P}^*)$ for q-a.e. $(t, z) \in [0, T] \times \mathbb{R}$.

LRM for call options

Corollary

LRM for call option $(S_T - K)^+$ is given as $\xi^{(S_T-K)^+} = 1 + \xi^{(K-S_T)^+}$.

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LRM for call option $(S_T - K)^+$ is given as $\xi^{(S_T-K)^+} = 1 + \xi^{(K-S_T)^+}$.

Proof

$$
\begin{aligned}
(S_T - K)^+ &= S_T - K + (K - S_T)^+ \\
&= S_0 + \int_0^T dS_t - K + \mathbb{E}_{\mathbb{P}^*}[(K - S_T)^+] + \int_0^T \xi_t^{(K-S_T)^+} dS_t + L_T^{(K-S_T)^+} \\
&= \mathbb{E}_{\mathbb{P}^*}[S_T - K + (K - S_T)^+] + \int_0^T \left(1 + \xi_t^{(K-S_T)^+}\right) dS_t + L_T^{(K-S_T)^+} \\
&= \mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+] + \int_0^T \left(1 + \xi_t^{(K-S_T)^+}\right) dS_t + L_T^{(K-S_T)^+}.
\end{aligned}
$$

This is an FS-decomposition of $(S_T - K)^+$ since $1 \in \Theta_S$.

Proof of Theorem

In order to see condition 4, we need to show $Z_T \in \mathbb{D}^{1,2}$.

Condition 4: $Z_T \left\{ D_{t,0} \log Z_T \mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z} \log Z_T} - 1}{z} \mathbf{1}_{\mathbb{R}_0}(z) \right\} \in L^2(q \times \mathbb{P})$.

Reminder:

$$dZ_t = -Z_{t-} \left\{ u_t dW_t + \int_0^\infty \theta_{t,z} \widetilde{N}(dt, dz) \right\}, \quad Z_0 = 1,$$

where $u_s = \frac{\alpha \sigma_s}{\sigma_s^2 + C_\rho}$, $\theta_{s,x} = \frac{\alpha(e^{\rho x} - 1)}{\sigma_s^2 + C_\rho}$ and $C_\rho := \int_0^\infty (e^{\rho x} - 1)^2 \nu(dx)$.

Proof of Theorem (cont'd)

For $t \in [0, T]$, we define $Z_t^{(0)} := 1$ and

$$Z_t^{(n+1)} := 1 - \int_0^t Z_{s-}^{(n)} u_s dW_s - \int_0^t \int_0^\infty Z_{s-}^{(n)} \theta_{s,x} \widetilde{N}(ds, dx)$$

for $n \geq 0$.

Proof of Theorem (cont'd)

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for $n \geq 0$.
Besides, we denote, for $n \geq 0$,

$$\phi_n(t) := \mathbb{E}\left[\int_{[0,t]\times[0,\infty)} \left(D_{r,z} Z_t^{(n)}\right)^2 q(dr, dz)\right].$$

Note that $\phi_0(t) \equiv 0$.

Proof of Theorem (cont'd)

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Note that $\phi_0(t) \equiv 0$.

Lemma 1

We have $Z_t^{(n)} \in \mathbb{D}^{1,2}$ for every $n \geq 0$ and any $t \in [0,T]$. Moreover, there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$\phi_{n+1}(t) \leq k_1 + k_2 \int_0^t \phi_n(s) ds$$

for every $n \geq 0$ and any $t \in [0,T]$.

1 Local risk-minimization

2 Barndorff-Nielsen and Shephard models

3 Main results

4 Numerical experiments

Setting

We treat Gamma-OU model: $\nu(dx) = ab\lambda e^{-bx}\mathbf{1}_{(0,\infty)}(x)dx$, where $a > 0$, $b > 0$.

[4] Schoutens, W.: Lévy Processes in Finance: Pricing Financial Derivatives. John Wiley & Sons, Hoboken (2003)

Setting

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Fix $T = 1$, $r = 0.019$ and $q = 0.012$.
The asset price and the squared volatility at time t are fixed to $S_t = 1124.47$ and $\sigma_t^2 = 0.0145$, respectively.

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$\rho = -1.2606$, $\lambda = 0.5783$, $a = 1.4338$, $b = 11.6641$.

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$\rho = -1.2606$, $\lambda = 0.5783$, $a = 1.4338$, $b = 11.6641$.

Suppose that the discounted asset price process $e^{-(r-q)t}S_t$ is a martingale.
Hence, μ is given as

$$\mu = r - q + \int_0^\infty (1 - e^{\rho x})\nu(dx) = r - q - \frac{a\lambda\rho}{b - \rho}.$$

[4] Schoutens, W.: Lévy Processes in Finance: Pricing Financial Derivatives. John Wiley & Sons, Hoboken (2003)

Setting (cont'd)

A., Imai and Suzuki [5] developed a numerical scheme of LRM for exponential Lévy models using the Carr-Madan approach [6], which is a numerical method for option prices based on the fast Fourier transform (FFT).

We consider a call option with strike price K. Since $H^*_{t,z} = 1$ and $Z_T = 1$, we have

$$\begin{aligned}\xi_t^{(S_T-K)^+} &= \frac{e^{-(r-q)(T-t)}}{S_{t-}(\sigma_t^2 + C_\rho)}\Big(\sigma_t^2 \mathbb{E}[S_T \mathbf{1}_{\{S_T \geq K\}} | \mathcal{F}_{t-}] \\ &\quad + \int_0^\infty \mathbb{E}\left[\left(S_T e^{zD_{t,z}L_T} - K\right)^+ - (S_T - K)^+ | \mathcal{F}_{t-}\right](e^{\rho z} - 1)\nu(dz)\Big) \\ &=: \frac{\sigma_t^2 I_1 + I_2}{S_t(\sigma_t^2 + C_\rho)}.\end{aligned}$$

[5] Arai, T., Imai, Y., Suzuki, R.: Numerical analysis on local risk-minimization for exponential Levy models, International Journal of Theoretical and Applied Finance vol.19, 1650008 (2016)

[6] Carr, P., D. Madan: Option valuation using the fast Fourier transform. Journal of Computational Finance, 2, 61–73 (1999)

Reminder: main result

$$\xi_t^{(K-S_T)^+} = \frac{1}{S_{t-}(\sigma_t^2 + C_\rho)}\Bigg\{\sigma_t^2 \mathbb{E}_{\mathbb{P}^*}[-1_{\{S_T<K\}} S_T | \mathcal{F}_{t-}]$$

$$+ \int_0^\infty \mathbb{E}_{\mathbb{P}^*}[(K-S_T)^+(H^*_{t,z}-1) + zH^*_{t,z} D_{t,z}(K-S_T)^+ | \mathcal{F}_{t-}]$$

$$\times (e^{\rho z}-1)\nu(dz)\Bigg\},$$

where $H^*_{t,z} := \exp\{zD_{t,z}\log Z_T - \log(1-\theta_{t,z})\}$ for $(t,z) \in [0,T]\times(0,\infty)$.

$\xi^{(S_T-K)^+} = 1 + \xi^{(K-S_T)^+}$.

Characteristic function of L_T

$$\begin{aligned}\phi(\vartheta) &:= \mathbb{E}[\exp(i\vartheta L_T)|S_t, \sigma_t^2] \\ &= \exp\left(i\vartheta\left(L_t + \mu(T-t)\right) - (\vartheta^2 + i\vartheta)\frac{\mathcal{B}(T-t)}{2}\sigma_t^2 \right. \\ &\qquad \left. + \frac{a}{b-f_2}\left[b\log\left(\frac{b-f_1}{b-i\vartheta\rho}\right) + f_2\lambda(T-t)\right]\right)\end{aligned}$$

for $\vartheta \in \mathbb{C}$, where

$$f_1 := i\vartheta\rho - \frac{1}{2}(\vartheta^2 + i\vartheta)\lambda\mathcal{B}(T-t) \text{ and } f_2 := i\vartheta\rho - \frac{1}{2}(\vartheta^2 + i\vartheta).$$

Recall that $\mathcal{B}(t) = \frac{1-e^{-\lambda t}}{\lambda}$ for $t \in [0, T]$.

I_1

$$I_1 = e^{-(r-q)(T-t)}\mathbb{E}[S_T \mathbf{1}_{\{S_T \geq K\}}|\mathcal{F}_{t-}] = \frac{e^{-(r-q)(T-t)}}{\pi}\int_0^\infty K^{-i\zeta+1}\frac{\phi(\zeta)}{i\zeta-1}dv, \qquad (2)$$

where $\zeta := v - i\delta$, and δ is a real number satisfying

$$\sup_{t\leq s<T}\left\{\frac{1}{2}-\frac{\rho}{\mathcal{B}(T-s)}-\sqrt{D_s}\right\} < \delta < \inf_{t\leq s<T}\left\{\frac{1}{2}-\frac{\rho}{\mathcal{B}(T-s)}+\sqrt{D_s}\right\}.$$

Here,

$D_s := \left(-\frac{1}{2}+\frac{\rho}{\mathcal{B}(T-s)}\right)^2 + \frac{2\hat{\vartheta}}{\mathcal{B}(T-s)}$ and $\hat{\vartheta} := \sup\left\{\vartheta \,\middle|\, \int_0^\infty (e^{\vartheta x}-1)\nu(dx) < \infty\right\}$.

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$$I_1 = e^{-(r-q)(T-t)}\mathbb{E}[S_T \mathbf{1}_{\{S_T \geq K\}}|\mathcal{F}_{t-}] = \frac{e^{-(r-q)(T-t)}}{\pi}\int_0^\infty K^{-i\zeta+1}\frac{\phi(\zeta)}{i\zeta-1}dv, \qquad (2)$$

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$$\sup_{t\leq s<T}\left\{\frac{1}{2} - \frac{\rho}{\mathcal{B}(T-s)} - \sqrt{D_s}\right\} < \delta < \inf_{t\leq s<T}\left\{\frac{1}{2} - \frac{\rho}{\mathcal{B}(T-s)} + \sqrt{D_s}\right\}.$$

Here,

$D_s := \left(-\frac{1}{2} + \frac{\rho}{\mathcal{B}(T-s)}\right)^2 + \frac{2\hat{\vartheta}}{\mathcal{B}(T-s)}$ and $\hat{\vartheta} := \sup\left\{\vartheta \middle| \int_0^\infty (e^{\vartheta x} - 1)\nu(dx) < \infty\right\}$.

Note that the RHS of (2) is independent of the choice of δ.
As a result, since the integrand of (2) is given by the product of $K^{-i\zeta+1}$ and a function of ζ, we can compute I_1 through the FFT.

I_2

Reminder:

$$I_2 = e^{-(r-q)(T-t)} \int_0^\infty \mathbb{E}\left[\left(S_T e^{zD_{t,z}L_T} - K \right)^+ - (S_T - K)^+ | \mathcal{F}_{t-} \right] (e^{\rho z} - 1)\nu(dz).$$

Note that $\mathbb{E}[(S_T - K)^+ | S_t, \sigma_t^2] = \frac{1}{\pi} \int_0^\infty \frac{K^{-i\zeta+1}\phi(\zeta)}{(i\zeta-1)i\zeta} dv$

I_2

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$$\begin{aligned}
&\frac{S_T}{S_t}\exp(zD_{t,z}L_T) \\
&= \exp\Bigg(\mu(T-t) - \frac{1}{2}\int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s + \rho \int_t^T dJ_s \\
&\quad -\frac{z}{2}\mathcal{B}(T-t) + \int_t^T \left(\sqrt{\sigma_s^2 + ze^{-\lambda(s-t)}} - \sigma_s\right) dW_s + \rho z\Bigg) \\
&= \exp\left(\mu(T-t) - \frac{1}{2}\int_t^T \sigma_{s,z}^2 ds + \int_t^T \sigma_{s,z} dW_s + \rho \int_t^T dJ_s + \rho z\right)
\end{aligned}$$

where $\sigma_{s,z}^2 := \sigma_s^2 + ze^{-\lambda(s-t)}$ for $(s,z) \in [t,T] \times (0,\infty)$.

I_2 (cont'd)

Denoting

$$L_s^{(z)} := \int_t^s \left(\mu - \frac{1}{2}\sigma_{u,z}^2\right) du + \int_t^s \sigma_{u,z} dW_u + \rho \int_t^s dJ_u$$

for $(s, z) \in [t, T] \times (0, \infty)$, we have

$$S_T \exp(zD_{t,z}L_T) = S_t \exp(L_T^{(z)} + \rho z).$$

I_2 (cont'd)

Denoting

$$L_s^{(z)} := \int_t^s \left(\mu - \frac{1}{2}\sigma_{u,z}^2\right) du + \int_t^s \sigma_{u,z} dW_u + \rho \int_t^s dJ_u$$

for $(s, z) \in [t, T] \times (0, \infty)$, we have

$$S_T \exp(zD_{t,z}L_T) = S_t \exp(L_T^{(z)} + \rho z).$$

$(\sigma_{s,z}^2)_{t \le s \le T}$ is a solution to the same SDE as σ^2, that is, $d\sigma_t^2 = -\lambda\sigma_t^2 dt + dJ_t$ with initial condition $\sigma_{t,z}^2 = \sigma_t^2 + z$.
We denote

$$\begin{aligned}\phi^{(z)}(\vartheta) &:= \mathbb{E}\left[\exp(i\vartheta L_T^{(z)})|S_t, \sigma_t^2\right] S_t^{i\vartheta} = \mathbb{E}\left[\exp(i\vartheta L_T)|S_t, \sigma_t^2 + z\right] \\ &= \phi(\vartheta)\exp\left(-(\vartheta^2 + i\vartheta)\frac{\mathcal{B}(T-t)}{2}z\right).\end{aligned}$$

I_2 (cont'd)

$$
\begin{aligned}
& e^{(r-q)(T-t)} I_2 \\
= & \int_0^\infty \mathbb{E}\left[\left(S_T e^{zD_{t,z}L_T} - K\right)^+ - (S_T - K)^+ | \mathcal{F}_{t-}\right](e^{\rho z} - 1)\nu(dz) \\
= & \int_0^\infty \mathbb{E}\left[\left(S_t \exp\left(L_T^{(z)} + \rho z\right) - K\right)^+ - (S_T - K)^+ \middle| S_t, \sigma_t^2\right](e^{\rho z} - 1)\nu(dz) \\
= & \int_0^\infty \left(\frac{e^{\rho z}}{\pi}\int_0^\infty (Ke^{-\rho z})^{-i\zeta+1}\frac{\phi^{(z)}(\zeta)}{(i\zeta - 1)i\zeta}dv - \frac{1}{\pi}\int_0^\infty \frac{K^{-i\zeta+1}\phi(\zeta)}{(i\zeta-1)i\zeta}dv\right)(e^{\rho z} - 1)\nu(\\
= & \int_0^\infty \frac{1}{\pi}\int_0^\infty \frac{K^{-i\zeta+1}\phi(\zeta)}{(i\zeta-1)i\zeta}\left(e^{i\rho z\zeta}\exp\left(-(\zeta^2 + i\zeta)\frac{\mathcal{B}(T-t)}{2}z\right) - 1\right)dv(e^{\rho z} - 1)\nu(\\
= & \int_0^\infty \frac{1}{\pi}\frac{K^{-i\zeta+1}\phi(\zeta)}{(i\zeta-1)i\zeta}\int_0^\infty (e^{\eta z} - 1)(e^{\rho z} - 1)\nu(dz)dv,
\end{aligned}
$$

where $\eta := i\rho\zeta - (\zeta^2 + i\zeta)\frac{\mathcal{B}(T-t)}{2}$, which is a function of ζ.

I_2 (cont'd)

Note that $\Re(\eta) \leq 0$ when $0 < \delta < 1 - \frac{2\rho}{\mathcal{B}(T)}$.
Therefore, taking such an δ, we have

$$\int_0^\infty (e^{\eta z} - 1)(e^{\rho z} - 1)\nu(dz) = ab\lambda\left(\frac{1}{b-\eta-\rho} - \frac{1}{b-\eta} - \frac{1}{b-\rho} + \frac{1}{b}\right),$$

from which we can compute I_2 using the FFT.

I_2 (cont'd)

Note that $\Re(\eta) \leq 0$ when $0 < \delta < 1 - \frac{2\rho}{\mathcal{B}(T)}$.
Therefore, taking such an δ, we have

$$\int_0^\infty (e^{\eta z} - 1)(e^{\rho z} - 1)\nu(dz) = ab\lambda\left(\frac{1}{b - \eta - \rho} - \frac{1}{b - \eta} - \frac{1}{b - \rho} + \frac{1}{b}\right),$$

from which we can compute I_2 using the FFT.

Reminder: $\sup_{t \leq s < T}\left\{\frac{1}{2} - \frac{\rho}{\mathcal{B}(T-s)} - \sqrt{D_s}\right\} < \delta < \inf_{t \leq s < T}\left\{\frac{1}{2} - \frac{\rho}{\mathcal{B}(T-s)} + \sqrt{D_s}\right\}$.

Delta-hedging strategy

Next, we discuss delta-hedging strategy $\Delta_t^{(S_T-K)^+}$ for a call option with strike price K, which is given as the partial derivative of the option price with respect to S_t,

Delta-hedging strategy

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Noting that

$$\mathbb{E}[(S_T - K)^+ | S_t, \sigma_t^2] = \frac{1}{\pi} \int_0^\infty K^{-i\zeta+1} \frac{\phi(\zeta)}{(i\zeta - 1)i\zeta} dv,$$

we have

$$\begin{aligned} \Delta_t^{(S_T-K)^+} &= \frac{e^{-(r-q)(T-t)}}{\pi} \int_0^\infty \frac{K^{-i\zeta+1}}{(i\zeta-1)i\zeta} \frac{\partial \phi(\zeta)}{\partial S_t} dv \\ &= \frac{e^{-(r-q)(T-t)}}{\pi} \int_0^\infty K^{-i\zeta+1} \frac{\phi(\zeta) S_t^{-1}}{i\zeta - 1} dv = \frac{I_1}{S_t}. \end{aligned}$$

Numerical experiments

We show numerical results on LRM strategies $\xi_t^{(S_T-K)^+}$ and delta-hedging strategies $\Delta_t^{(S_T-K)^+}$.

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Reminder:
we take $T = 1$, $r = 0.019$, $q = 0.012$, $S_t = 1124.47$, $\sigma_t^2 = 0.0145$, $\rho = -1.2606$, $\lambda = 0.5783$, $a = 1.4338$, $b = 11.6641$.

Moreover, we take $\delta = 1.75$.

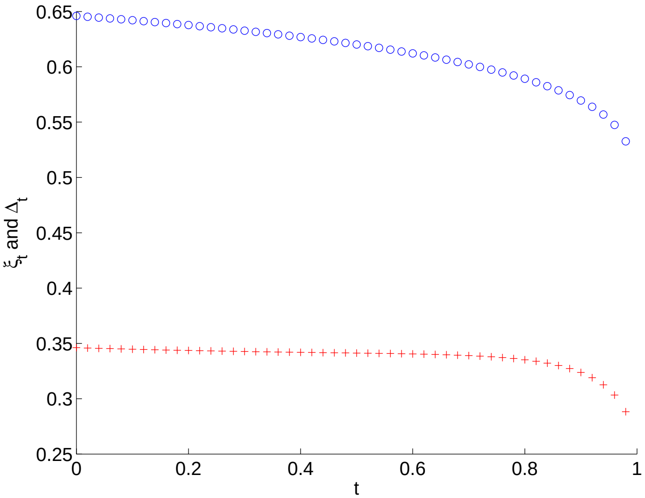

Figure: Values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$ when K is fixed to 1124.47(ATM) vs. times $t = 0, 0.02, \ldots, 0.98$. In this case, the option is in the money at time t. Red crosses and blue circles represent the values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$, respectively.

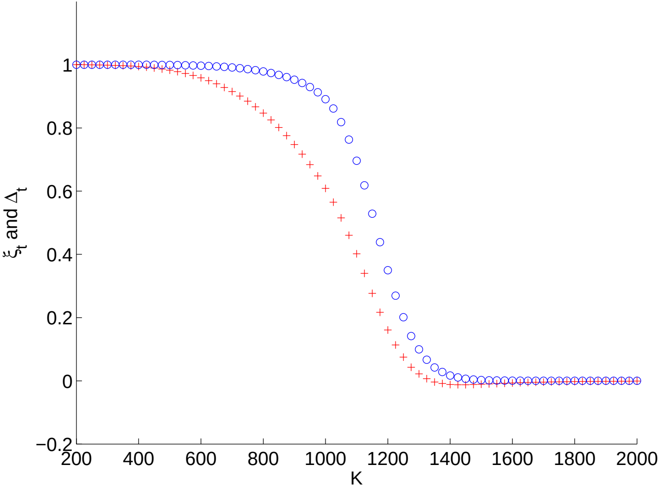

Figure: Values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$ at $t = 0.5$ vs. strike price K from 200 to 2000 at steps of 25. Red crosses and blue circles represent the values of $\xi_t^{(S_T-K)^+}$ and $\Delta_t^{(S_T-K)^+}$, respectively.

Thank you for your attention!