Default functions and Liouville type theorems

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[Plan of my talk]

§1 Default functions: definition and basic properties

§2 Submartingale properties of subharmonic functions : Symmetric diffusion cases

§3 L^1- Liouville properties of subharmonic functions

§4 Liouville theorems for holomorphic maps

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

Let M_t be a continuous local martingale.

Def. If M_t is not a true martingale, we say M_t is a strictly local martingale.

M_t is a (true) martingale
$\Leftrightarrow E[M_T] = E[M_0]$ for $\forall T$: bounded stopping time.

"Local" property of M_t :

$$\gamma_T(M) := E[M_0] - E[M_T].$$

is called a default function (Elworthy- X.M.Li-Yor('99)).

Default formula : **Assume that** $E[|M_T|] < \infty, E[|M_0|] < \infty$ **for a stopping time** T **and** $\{M^-_{T\wedge S}; S:$ **stopping times** $\}$ **is uniformly integrable. Set** $M^*_t := \sup_{0\le s\le t} M_s$**.**

$$E[M_T : M^*_T \le \lambda] + \lambda P(M^*_T > \lambda) + E[(M_0 - \lambda)_+] = E[M_0].$$

Letting $\lambda \to \infty$**,**

$$\gamma_T(M) = \lim_{\lambda\to\infty} \lambda P(\sup_{0\le t\le T} M_t > \lambda).$$

Another quantity: $\sigma_T(M)$

Def.

$$\sigma_T(M) := \lim_{\lambda\to\infty} \lambda P(\langle M\rangle_T^{1/2} > \lambda).$$

Theorem (Elworthy-Li-Yor, Takaoka('99)) Assume that $E[|M_T|] < \infty, E[|M_0|] < \infty$.

$$\exists \gamma_T(M) = \sqrt{\frac{\pi}{2}} \sigma_T(M).$$

Moreover $M_t^T := M_{T \wedge t}$ is a uniformly integrable martingale iff $\gamma_T(M) = \sigma_T(M) = 0$.

See also Azema-Gundy -Yor('80), Galtchouk-Novikov('97).

[Example]

R_t : d-dimensional Bessel process:

$dR_t = db_t + \frac{d-1}{2R_t} dt, \; R_0 = r.$

If $d > 2$, then R_t^{2-d} is a strictly local martingale.

As for default function, if $R_0 = r$,

$$\gamma_t(R^{2-d}) = \frac{1}{2^\nu \Gamma(\nu)} \int_0^t \frac{du}{u^{1+\nu}} \exp(-\frac{r^2}{2u}),$$

where $d = 2(1+\nu)$.

If $d = 2$, $\log R_t$ is a strictly local martingale.

$$\gamma_t(\log R) = \frac{1}{2} \int_0^t \frac{du}{u} \exp(-\frac{r^2}{2u}).$$

[submartingale case]

Let $X_t = X_0 + M_t + A_t$ where M is a local martingale and A is an adapted increasing process.

Lem.(Default function for submartingale)

If X is positive and $E[A_T] < \infty$,

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda P(\sup_{0\le t\le T} X_t > \lambda) &= \lim_{\lambda\to\infty} \lambda P(\sup_{0\le t\le T} M_t > \lambda) \\ &= E[X_0] - E[X_T] + E[A_T].\end{aligned}$$

Example (stochastic Jensen's formula).

Let Z_t : BM($\mathbb{C}$) with $Z_0 = o$, $\tau_r = \inf\{t > 0 : |Z_t| > r\}$ and f be a non-constant holomorphic function on $\mathbb{C}$, $f(o) \neq 0$.
Set $X_t := \log|f(Z_{\tau_r \wedge t}) - a|^{-2}$: a local martingale bounded below.

$$\lim_{\lambda\to\infty} \lambda P(\sup_{0<t<\tau_r} X_t > \lambda) = \sum_{f(\zeta)=a,\ |\zeta|<r} 2\log\frac{r}{|\zeta|}.$$

From this we can see an essential relationship between Nevanlinna theory and complex Brownian motion (Carne(86), A.(95)).

Our question:

When is a local submartingale $u(X_t)$ a submartingale ?

§2 Submartingale property of subharmonic functions.

[Settings]

Let $\mathcal{M}$: a smooth manifold, m a Radon measure on $\mathcal{M}$ with $\mathrm{supp} m = \mathcal{M}$, (X_t, P_x) be a symmetric diffusion process defined from the Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a core $\mathcal{C} \subset \mathcal{F} \cap C_o(\mathcal{M})$ where

$$\mathcal{E}(u,v) = \int_{\mathcal{M}} \Gamma(u,v) dm \quad (u, v \in \mathcal{C}).$$

We have there exists L a s.a.operator on $L^2(m)$ such that

$$\mathcal{E}(u,v) = -\int_{\mathcal{M}} uLvdm \text{ for } u, v \in \mathcal{C}.$$

L is the generator of the diffusion.

Assume that

- $(\mathcal{E}, \mathcal{F})$ **is a strongly local, irreducible regular Dirichlet form.**
- **the transition probability** $p(t, x, dy)$ **is absolutely continuous w.r.t.** m**.**
- **there exists a nonnegative exhaustion function** $r(x)$ **(i.e.** $\{r(x) < r\}$ **: rel.cpt for** $\forall r \geq 0$**) such that** $\Gamma(r(\cdot), r(\cdot))$ **is bounded a.e.**
- **there exists** $x_0 \in \mathcal{M}$ **and** $c_1(x_0), c_2(x_0) > 0$ **such that** $c_1(x_0)|\nabla u|^2 \geq \Gamma(u, u) \geq c_2(x_0)|\nabla u|^2$ **for** $\forall u \in \mathcal{C}$ **on a neighborhood of** x_0**.**

Note that the first assumption implies a diffusion process corresponds to the Dirichlet form.

Typical Example : Brownian motion on a complete, connected Riemannian manifold $\mathcal{M}$.

$L = \dfrac{1}{2}\Delta, \Gamma(u,u) = |\nabla u|^2, r(x) = d(o,x)$,

$m =$ Riemannian volume dv, $p(t,x,dy) = p(t,x,y)dv(y)$ where $p(t,x,y)$ is the heat kernel of $\partial/\partial t - \dfrac{1}{2}\Delta$.

$\mathcal{F} = H_0^1(\mathcal{M}) = \overline{C_0^\infty(\mathcal{M})}^{\mathcal{E}_1}$ where

$\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + ||u||^2_{L^2(m)}$. Note $\mathcal{C} = C_0^\infty(\mathcal{M})$.

[subharmonic function]

Def. u is (L-)subharmonic if $u \in \mathcal{F}_{loc}$ and $\mathcal{E}(\phi, u) \leq 0$ for $\forall \phi \geq 0, \phi \in \mathcal{F}$ with compact support.

It is well-known that $u(X_t)$ is a continuous local submartingale:

$$u(X_t) - u(x) = M_t^{[u]} + A_t^{[u]} \qquad P_x\text{-a.s.}$$

Def. Default function of $u(X_T)$

$$N_x(T, u) = \lim_{\lambda \to \infty} \lambda P_x(\sup_{0 \leq s \leq T} u(X_s) > \lambda).$$

As before if u is positive subharmonic and $E_x[A_t^{[u]}] < \infty$,

$$E_x[u(X_t)] - u(x) + N_x(t, u) = E_x[A_t^{[u]}].$$

We consider the condition for the default function to be vanishing. Let $\mathcal{U} := \{u :$ a positive subharmonic function $|E_x[u(X_t)] < \infty (\forall t > 0) a.e.x\}$.

Theorem. Let $B(r) := \{r(x) < r\}$.

If X is transient, $u \in \mathcal{U}$ and

$$\liminf_{r\to\infty} \frac{1}{r^2}\{\log \int_{B(r))} u^\alpha dm + \log m(B(r))\} < \infty$$

for some $\alpha > 2$, then $u(X_t)$ is a submartingale under P_x for a.e.x.

sketch of proof.

1°. Let $\tau_r = \inf\{t > 0 | X_t \notin B(r)\}$. If

$$\lim_{r\to\infty} E_x[u(X_{\tau_r}) : \tau_r < t] = 0,$$

then $N_x(t, u) = 0$.

2°. Estimate $E_x[u(X_{\tau_r})]$.

Lem. Let x_0 a point appearing in the assumption. If X is transient and u is a positive subharmonic function, there exists a constant $C(x_0)$ such that

$$E_{x_0}[u(X_{\tau_r})] \le C(x_0)\{(\int_{B(r+1)} u(x)^2 dm)^{1/2} + \int_{B(r)} u(x)dm\}.$$

3°. Estimate $P_x(\tau_r < t)$.

Lem. (Takeda's inequality) Fix $1 > r_0 > 0$. If $r > r_0$, there exists $c > 0$ such that

$$\int_{B(r_0)} P_y(\tau_r < t)dm(y) \le const.\frac{m(B(r+1))}{r}e^{-\frac{cr^2}{t}},$$

4°. $N_{x_0}(t_0, u) = 0$ for some x_0, t_0 implies $N_x(t, u) = 0$ for $\forall t > 0$ and a.e.x.

[Brownian motion case]

When $\mathcal{M}$ is a complete Riemannian manifold and (X_t, P_x) is Brownian motion on $\mathcal{M}$, the Ricci curvature controls the conditions in the above theorem.

Theorem. If there exists a constant $C > 0$ such that $Ric \geq -Cr(x)^2 - C$ and a positive subharmonic function u satisfies

$$\liminf_{r\to\infty} \frac{1}{r^2} \log \int_{B(r)} u(x)dv(x) < \infty,$$

then $u(X_t)$ is a submartingale.

§3. L^1 Liouville theorem.

[Known results]

1-1. L^p-Liouville theorem: (Yau '76, cf. P.Li-Schoen '84) If $\mathcal{M}$ is a complete Riemannian manifold and a positive Δ-subharmonic function u is L^p-integrable for some $p > 1$, u is constant.

1-2. Generalization in the context of Dirichlet form (T.Sturm '94). Under our setting, if a positive L-subharmonic u satisfies

$$\int^{\infty} \frac{r\,dr}{\int_{B(r)} u^p\,dm} = \infty$$

for some $p > 1$, then u is constant.

2. L^1**-Liouville theorem.** Let $\mathcal{M}$ be a complete Riemannian manifold and u a positive Δ-subharmonic function.

Ricci curvature condition (P.Li '84)
If $\mathcal{M}$ is a complete Riemannian manifold satisfying $Ric \geq -Cr(x)^2 - C$ for some $C > 0$ and u is L^1, then u is constant.

3. Weighted L^p**-Liouville theorem. (Nadirashvili '85)**
If $\displaystyle\int_{\mathcal{M}} \frac{f(u(x))}{r(x)^2+1} dv(x) < \infty$ for a nonnegative function f on $[0, \infty)$ satisfying $\displaystyle\int_0^\infty 1/f(t)dt < \infty$, then u is constant.

$$\exists p > 1 \text{ s.t. } \int_{\mathcal{M}} \frac{u(x)^p}{r(x)^2+1} dv(x) < \infty \Rightarrow u \text{ const.}$$

[L^1-Liouville theorem and submartingale property]

Prop. If u is a positive, integrable L-subharmonic function and $u(X_t)$ is a submartingale under P_x for a.e. x, then u is constant. Namely vanishing of default function of u implies L^1-Liouville theorem.

Proof.

$$u(x) \le E_x[u(X_t)]$$

for all $0 < t$ and a.e. x.

$$tu(x) \le \int_0^t E_x[u(X_s)]ds.$$

If X is recurrent, ratio ergodic theorem for recurrent Markov processes implies

$$\frac{1}{t}E_x[\int_0^t u(X_s)ds] \to \begin{cases} \dfrac{\int_{\mathcal{M}} u(x)dx}{m(\mathcal{M})} \text{ (if } m(\mathcal{M}) < \infty), \\ 0 \text{ (if } m(\mathcal{M}) = \infty) \end{cases}$$

as $t \to \infty$. In both cases u should be bounded. Then u is a constant. If X is transient, $\frac{1}{t}E_x[\int_0^t u(X_s)ds] \to 0$ as $t \to \infty$ since $E_x[\int_0^\zeta u(X_s)ds] < \infty$ for an integrable function u where ζ is the life time of X. ∎

[Example]

The following example is originally due to Li-Schoen. We give a little modification. Let $\overline{M}$ be a compact 2-dim Riemannian manifold with a metric ds_0^2, $\Delta_{\overline{M}}$ is the Laplacian defined from ds_0^2 and $\overline{X}$ Brownian motion on $\overline{M}$ with its generator $\frac{1}{2}\Delta_{\overline{M}}$. Fix $o \in \overline{M}$. Set

$$g(o,x) = 2\pi \int_0^\infty (p(t,o,x) - \frac{1}{vol(\overline{M})})dt + C,$$

where $p(t,x,y)$ is the transition density of $\overline{X}$ and C is a positive constant such that $g(o,x) > 0$ for all $x \in \overline{M} \setminus \{o\}$. Remark that $g(x,y) \sim \log \frac{1}{d(x,y)^2} \quad (d(x,y) \to 0)$. Note
$\frac{1}{2}\Delta_{\overline{M}} g(o,x) = -2\pi\delta_o(x) + \frac{1}{Vol(\overline{M})}$.

Let M be $\overline{M} \setminus \{o\}$. Take σ be a smooth function on M s.t.

$$\sigma(x) \sim t^{-1}(\log \frac{1}{t})^{-1}(\log \log \frac{1}{t})^{-\alpha} \text{ with } 1/2 < \alpha < 1$$

when $t = d_{\overline{M}}(o, x) \to 0$.

Define a metric $ds^2 = \sigma^2 ds_0^2$ on M. Note that Laplacian Δ_M defined from ds^2 has a form

$$\Delta_M = \sigma^{-2} \Delta_{\overline{M}},$$

where $\Delta_{\overline{M}}$ is defined from ds_0^2. Let X_t be Brownian motion on M with its generator $\dfrac{1}{2}\Delta_M$. Then X_t is a time changed process of $\overline{X}_t$ which is recurrent. Hence X_t is recurrent, in particular, conservative.

(M, ds^2) satisfies

- complete and stochastically complete.
- M is of finite volume w.r.t ds^2.
- $u(x) := g(o, x)$ is a nonnegative smooth subharmonic function on M and integrable w.r.t. ds^2.
- the curvature $\sim -const.r^{\frac{2\alpha}{1-\alpha}} = -cr^{2+\epsilon}$ as $r \to \infty$ $(\epsilon = (4\alpha - 2)/(1 - \alpha) > 0)$.

From these facts we see $u(X_t)$ is a strictly local submartingale and L^1-Liouville property of M fails.

[Our results]

Theorem 1. Suppose X_t is transient and u is a nonnegative L-subharmonic function.

i) Assume there exists $\alpha > 2$ and $0 \leq p < 1$ such that

$$\liminf_{r\to\infty} \frac{1}{r^{2(1-p)}} \log\{m(B(r)) \int_{B(r)} u(x)^{\alpha} dm(x)\} < \infty.$$

If

$$\int_{\mathcal{M}} \frac{u(x)}{(1+r(x))^{2p}} dm(x) < \infty,$$

then $u = 0$.

ii) Assume there exists $\alpha > 2$ such that

$$\liminf_{r\to\infty} \frac{1}{(\log r)^2} \log\{m(B(r)) \int_{B(r)} u(x)^\alpha dm(x)\} < \infty.$$

If

$$\int_{\mathcal{M}} \frac{u(x)}{1 + r(x)^2} dm(x) < \infty,$$

then $u = 0$.

[Brownian motion case]

When $\mathcal{M}$ is a complete Riemannian manifold and u is a nonnegative Δ-subharmonic function, using Ricci curvature condition enables us to simplify the results as follows.

Theorem 2. Suppose $Ric \geq -Cr(x)^2 - C$.

i) Assume

$$\liminf_{r\to\infty} \frac{1}{r^{2(1-p)}} \log vol(B(r)) < \infty \quad (0 \leq \exists p < 1).$$

If

$$\int_M \frac{u(x)}{(1+r(x))^{2p}} dv(x) < \infty,$$

then u is constant.

ii) Assume

$$\liminf_{r\to\infty} \frac{1}{(\log r)^2} \log vol(B(r)) < \infty.$$

If

$$\int_M \frac{u(x)}{1+r(x)^2} dv(x) < \infty,$$

then u is constant.

Rem. When $p = 0$ in i), it implies P.Li's Liouville theorem.

Proofs of Theorem 1 & 2. As for the case of $p = 0$ directly from the submartingale property for $u(X_t)$. For the other case use time-change argument as follows. Let $\rho(t)$ is a non-increasing, positive function on $(0, \infty)$ such that $\int_0^\infty \rho(t)^{1/2} dt = \infty$. Y_t defined by

$$Y_t = X_{\zeta_t^{-1}} \text{ with } \zeta_t = \int_0^t \rho(r(X_s))ds.$$

Note that Y_t has a generator $\frac{1}{2}\rho(r(x))^{-1}L$ which becomes a self-adjoint operator on $L^2(\rho(r(x))dm)$. Define an exhaustion function $\theta(x)$ on $\mathcal{M}$ by

$$\theta(x) = \int_0^{r(x)} \sqrt{\rho(s)}ds.$$

Then $\Gamma(\theta, \theta)$ is bounded. Thus our argument as before is available. Take $\rho(t) = (1+t)^{-2p}$ with $0 \leq p < 1$ in case of i) and with $p = 1$ in case of ii).

§4. Liouville theorems for holomorphic maps.

Let $\mathcal{M}$ be a complete Kähler manifold, $\mathcal{N}$ a Hermitian manifold, and $f : \mathcal{M} \to \mathcal{N}$ a holomorphic map. $R(x) := \inf_{\xi \in T_x\mathcal{M}, ||\xi||=1} Ric(\xi, \xi)$, $B(r) := \{x \in M | r(x) < r\}$, $K(y)$: holomorphic bisectional curvature of $\mathcal{N}$.

Let $e(x) := tr_{g_{\mathcal{M}}} f^* g_{\mathcal{N}}$ (energy density of f). Chern-Lu formula implies

$$\frac{1}{2}\Delta \log e(x) \geq -K(f(x))e(x) + R_-(x) \text{ if } e(x) \neq 0.$$

From this with modifying the method in the previous sections, we have the following.

Theorem 3. Assume Brownian motion on $\mathcal{M}$ is transient. If $K(f(x)) \leq -c_0$ for some $c_0 > 0$, $\int_M R_-(x)dv(x) < \infty$ and

$$\liminf_{r\to\infty} \frac{1}{r^2} \log\{vol(B(r)) \int_{B(r)} R_-(x)^2 dv(x)\} < \infty,$$

then f is constant.

Cor. If $\int_{\mathcal{M}} R_-(x)dv(x) < \infty$ and

$$\liminf_{r\to\infty} \frac{1}{r^2} \log\{vol(B(r)) \int_{B(r)} R_-(x)^2 dv(x)\} < \infty,$$

then every bounded holomorphic function on $\mathcal{M}$ is constant.

In recurrent case we have the following by assuming a Ricci curvature condition.

Theorem 4. Assume $R_{-}(x) \geq -Cr(x)^2 - C$ for some $C > 0$ and Brownian motion on $\mathcal{M}$ is recurrent. If $K(f(x)) \leq -c_0$ for $c_0 > 0$,

$$\int_M |R(x)| dv(x) < \infty, \text{ and } \int_M R(x) dv(x) \geq 0,$$

then f is constant.

Rem. These results a generalization of a Liouville theorem due to Li-Yau('90).

[Problems]

1. Non-symmetric case. Complex Laplacian L on non-Kähler, Hermitian manifolds. $L = \Delta + V$. Girsanov formula does not seem to work well.

2. Difference between the space of L^1-harmonic functions and L^1-subharmonic functions.

3. L^1-Liouville theorem on manifolds with topological constraint.

Murata and Tsuchida conjectures every L^1-harmonic function on comlete Riemannian manifolds with one end should be constant. Cf. Grigoryan showed that every positive L^1-superharmonic function on $\mathcal{M}$ is constant if $\mathcal{M}$ is stochastically complete (i.e. Brownian motion on $\mathcal{M}$ is consevative).