Efficient Computation of Risk Contributions by using MCMC

Taka-aki Koike

Graduate School of Science and Technology,
Keio University, Japan
<taka-aki@math.keio.jp>

BU/KEIO WORKSHOP 2016

Joint work with Prof. Mihoko Minami

Background in Risk Management

Suppose we have a portfolio, consists of d-types of assets:

Let

$X_1, \dots, X_d$: *Randomly occurred Losses* attributed to the asset $j = 1, 2, \dots, d$, respectively

$$S = X_1 + \cdots + X_d$$

: *Total loss over the portfolio,* with *c.d.f.* F

The *Risk* of the portfolio can be measured by $\rho(F)$, where $\rho : F \mapsto \mathbf{R}$ is called a *Risk Measure*

To prepare for the *Risk*, portfolio manager is obliged to hold the amount of capital $\rho(F)$ (so called the *Economic Capital*).

The most popular risk measure ρ is the *Value-at-Risk*, defined by

$$\mathrm{VaR}_p : F \to \mathbf{R} \qquad F \mapsto \inf\{x \in \mathbf{R} : F(x) \geq p\}$$

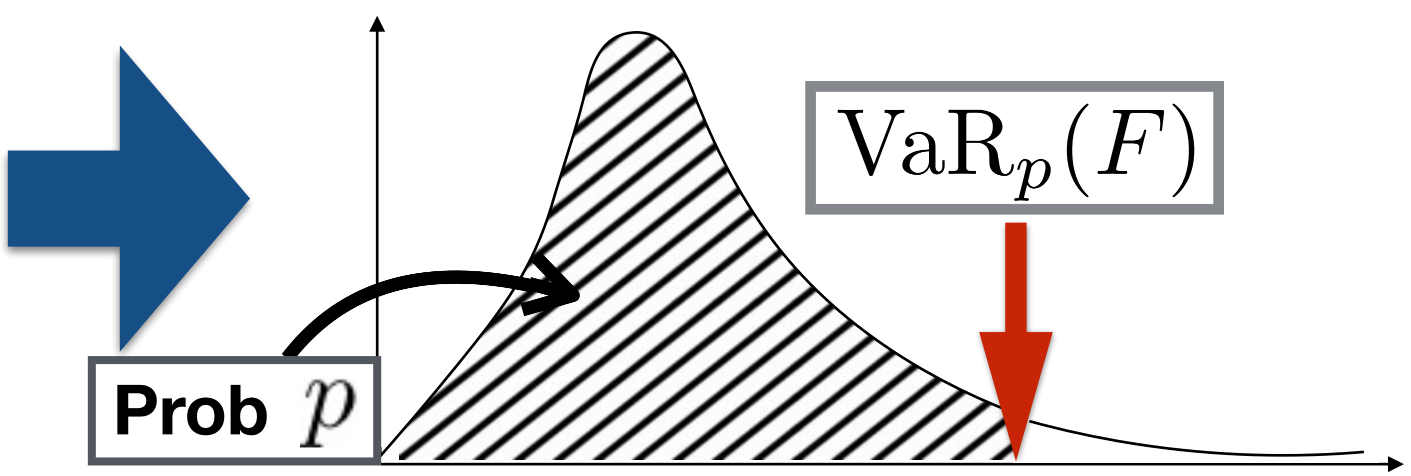
where p is called the confidence level, often is set high such as 0.999:

The Procedure of Portfolio Risk Management

obtain d.f. F of total loss S

Modeling joint d.f.s of loss vector

$$(X_1, \ldots, X_d)$$



compute **Economic Capital**

For more detailed risk analysis, it is required to decompose the economic capital into d *Risk Contributions* (or *Allocated Capital*) $(AC_1, \ldots, AC_d)$, that satisfies

$$\rho(S) = AC_1 + \cdots + AC_d$$

Euler Principle is the most prevalent rule to determine the allocated capitals because of its good economical properties.

When we use VaR as the risk measure ρ , the risk contribution of the asset $j \in \{1, 2, \ldots, d\}$ can be derived by

$$\mathrm{AC}_j^{\mathrm{VaR}_p} = \mathbf{E}[X_j | X_1 + \cdots + X_d = \mathrm{VaR}_p(S)]$$

according to the Euler principle (Tasche, 1999).

The problem throughout this presentation is the following:

Problem

Given

- joint distribution of loss random vector $(X_1, \ldots, X_d)$
- extremely high probability (such as 0.999) p,

how can we compute the VaR contributions

$$\mathrm{AC}_j^{\mathrm{VaR}_p} = \mathbf{E}[X_j | X_1 + \cdots + X_d = \mathrm{VaR}_p(S)]$$

for $j = 1, 2, \ldots, d$?

Outline

Outline

1. Difficulty of the Computation
2. Solution
3. Numerical
4. Conclusion

Difficulty of the Computation

How can we compute the VaR contributions

$$\mathrm{AC}_j^{\mathrm{VaR}_p} = \mathbf{E}[X_j | X_1 + \cdots + X_d = \mathrm{VaR}_p(S)]$$

for $j = 1, 2, \ldots, d$?

- Analytical calculation is quite hard because the joint distribution of (X_j, S) is hardly accessible.
- One can estimate the pseudo VaR contributions:

$$\mathrm{E}[X_j |\ S \in [\mathrm{VaR}_p(S) - \delta, \mathrm{VaR}_p(S) + \delta]\,]$$

for sufficiently small $\delta > 0$ based on Monte Carlo sample of $(X_1, \ldots, X_d)$

⇒ **next page**

Standard Monte Carlo Method

- The d.f. of $\mathbf{X}$ is available, but $\mathbf{X}|S = \mathrm{VaR}_p(S)$ is NOT !

MC for computing VaR and (pseudo) VaR contribution

1) Generate $\mathbf{X_1}, \ldots, \mathbf{X_N}$ from the d.f. of $\mathbf{X}$
2) Compute the component-wise sums $S_1, \ldots, S_N$
3) Estimate $\mathrm{VaR}_p(S)$ by $S_{[Np]}$
4) Take out sample $S_{i_1}, \ldots, S_{i_M} \in [\mathrm{VaR}_p(S) - \delta, \mathrm{VaR}_p(S) + \delta]$
5) Estimate $\mathrm{E}[X_j |\ S \in [\mathrm{VaR}_p(S) - \delta, \mathrm{VaR}_p(S) + \delta]]$
   by sample mean $\frac{1}{M}\sum_{j=1}^{M} \mathbf{X_{i_j}}$

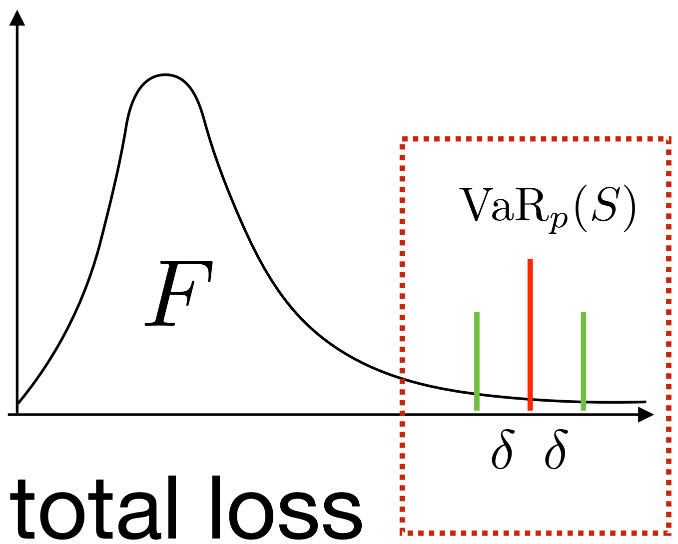
Standard MC method is problematic mainly because

- Only a few observations fall in the interval

$$[\mathrm{VaR}_p(S) - \delta, \mathrm{VaR}_p(S) + \delta]$$

as VaR is the quantile of extremely high probability.

⇒Estimator is quite sensitive to δ



No stable (less volatile) method exists

(except for some special cases; Glasserman, 2005 and Tasche, 2009)

Outline

Solution using MCMC

- Markov chain Monte Carlo (MCMC) is a method to generate sample from a (often intractable) distribution π (called target distribution) by constructing a Markov chain whose stationary distribution is the desired one π.

- Metropolis Hastings (MH) algorithm is a powerful method to generate a Markov chain whose stationary distribution is the desired π.

⇒ **next page**

Metropolis Hastings (MH) Algorithm

input : Initial X_1, size T, target π and proposal function q.

output : Sample $\{\mathbf{X}_t\}_{t=1}^{T} \sim \pi$

<u>**MH for Sampling from π**</u>

1) **For** $t = 1, 2, \ldots, T$

1-1) Generate $X_t^* \sim q(X_t, \cdot)$

1-2) Set $X_{t+1} = X_t^*$ with probability

$$\alpha(X_t, X_t^*) := \frac{\pi(X_t^*) \cdot q(X_t^*, X_t)}{\pi(X_t) \cdot q(X_t, X_t^*)} \wedge 1$$

and $X_{t+1} = X_t$ otherwise.

2) **End For.**

Only the ratio is required

- If we can generate sample $\{\mathbf{X}_t\}_{t=1}^{T}$ from a density $f_{\mathbf{X}|S=\mathrm{VaR}_p(S)}$, then VaR contributions are estimated by

$$\hat{AC}^{\mathrm{VaR}_p} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{X}_t \qquad (T : \text{sample size})$$

$$\left(\begin{aligned} &\approx \int \mathbf{x} \cdot f_{\mathbf{X}|S=\mathrm{VaR}_p(S)} \mathrm{d}\mathbf{X} \\ &= \mathrm{E}[\mathbf{X}|S = \mathrm{VaR}_p(S)] \end{aligned} \right)$$

where $f_{\mathbf{X}|S=\mathrm{VaR}_p(S)}$ is a density of $\mathbf{X} = (X_1 \dots, X_d)$ given $S = \mathrm{VaR}_p(S)$ (assume its existence).

- Unfortunately, the density $f_{\mathbf{X}|S=\mathrm{VaR}_p(S)}$ is hardly tractable as the density of the total loss f_S usually can not be written explicitly.

- However, MH enables to generate sample from $f_{\mathbf{X}|S=\mathrm{VaR}_p(S)}$ because it only requires the ratio of the target density:

$$\frac{\pi(\mathbf{y})}{\pi(\mathbf{x})} = \frac{f_{\mathbf{X}|S=v}(\mathbf{y})}{f_{\mathbf{X}|S=v}(\mathbf{x})} = \frac{f_{\mathbf{X}}(\mathbf{y}) \cdot 1_{[\sum_{j=1}^{d} y_j = v]}}{f_{\mathbf{X}}(\mathbf{x}) \cdot 1_{[\sum_{j=1}^{d} x_j = v]}}$$

write $v = \mathrm{VaR}_p(S)$ and regard it as a given constant

The most cumbersome term f_S disappears

- Define the *v-fiber* by

$$\mathcal{F}_v := \{\mathbf{x} \in \mathbf{R}^d : \sum_{j=1}^{d} x_j = v\}$$

- Set $X_1 \in \mathcal{F}_v$ and define the proposal q so that it holds

$$\mathbf{x} \in \mathcal{F}_v \Rightarrow \{\mathbf{y} \in \mathbf{R}^d : q(\mathbf{x}, \mathbf{y}) > 0\} \subset \mathcal{F}_v$$

- Then one can implement MH only by computing

$$\alpha(X_t, X_t^*) := \frac{f_{\mathbf{X}}(X_t^*) \cdot q(X_t^*, X_t)}{f_{\mathbf{X}}(X_t)\, q(X_t, X_t^*)}$$

Easy to Evaluate!

Outline

Numerical Example

Simple Independent Pareto Model

$d = 3$, independent Pareto distributions with

shape=(4, 4.5, 5) and scale=(1, 1, 1)

MC simulation :

[1] Generate T=1e+07 sample from the risk model above

[2] Estimate $v = \mathrm{VaR}_p(S)$ and pseudo VaR contributions for $\delta = 0.1$

(Part of) MC results :

Estimated VaR	Size of sample on the interval $[v-\delta, v+\delta]$
5.976	1340

MCMC simulation :

[3] Perform MCMC with T=1e+07, $X_1 = (v/3, v/3, v/3)$ and

$$q(\mathbf{x}, \cdot) \sim \mathrm{Unif}(\mathcal{F}_v) \text{ for all } \mathbf{x} \in \mathcal{F}_v$$

[4] Estimate VaR contributions based on MCMC sample.

Estimated VaR contributions :

	MC	MCMC
$X_1 \mid S = v$	3.293	3.264
$X_2 \mid S = v$	1.676	1.733
$X_3 \mid S = v$	1.005	0.979
Sample Size	1340	10^7

※ a.c.f. will be lower than 0.1 if we take every 50 subsamples.

Strong dependence & MultiModality

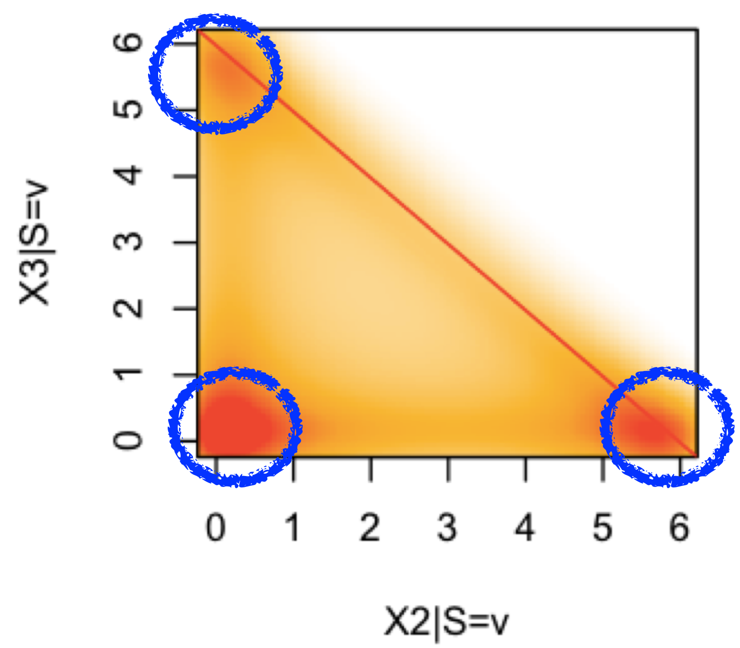


Fig 1. Smoothed scatter plot of MCMC sample

We repeated MC and MCMC simulation 10 times with T=1e+06 to see the stability of our results.

[i] We computed 10 MC and MCMC estimators

[ii] We then compute their sample mean and standard error:

Table 1. Estimated VaR contributions, mean and standard error (in parentheses) over 10 runs.

	MC		MCMC
$X_1 \| S = v$	3.254	$\simeq$	3.259
	(0.234)	$\gneq$	(0.003)
$X_2 \| S = v$	1.737	$\simeq$	1.735
	(0.200)	$\gneq$	(0.019)
$X_3 \| S = v$	0.979	$\simeq$	0.977
	(0.148)	$\gneq$	(0.010)

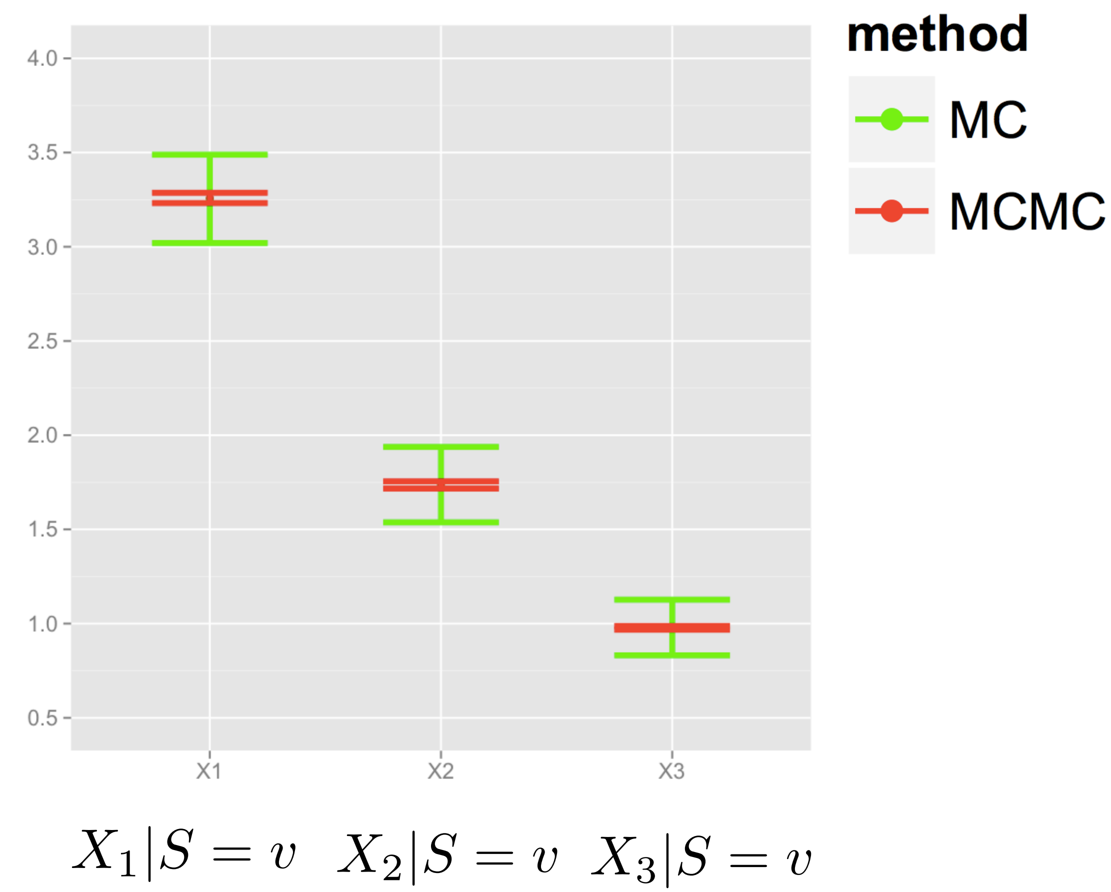


Fig 2. Error-bar plots of (mean+sd, mean-sd)

Outline

1. Difficulty
2. Solution
3. Numerical
4. Conclusion

Conclusion

- In continuous setting, MCMC provides more stable estimator of VaR contributions than standard MC method.
- One can observe some interesting behaviors of conditional densities given extremely high sum, such as strong dependence and multi-modality.
- Many other interesting cases and future works.

Reference

[1] Glasserman, P. (2005). Measuring marginal risk contributions in credit portfolios.
FDIC Center for Financial Research Working Paper, (2005-01).

[2] Tasche, D. (1999). Risk contributions and performance measurement.
Report of the Lehrstuhl für mathematische Statistik, TU München.

[3] Tasche, D. (2009). Capital allocation for credit portfolios with kernel estimators.
Quantitative Finance, 9(5), 581-595.

Supplementary Note 1

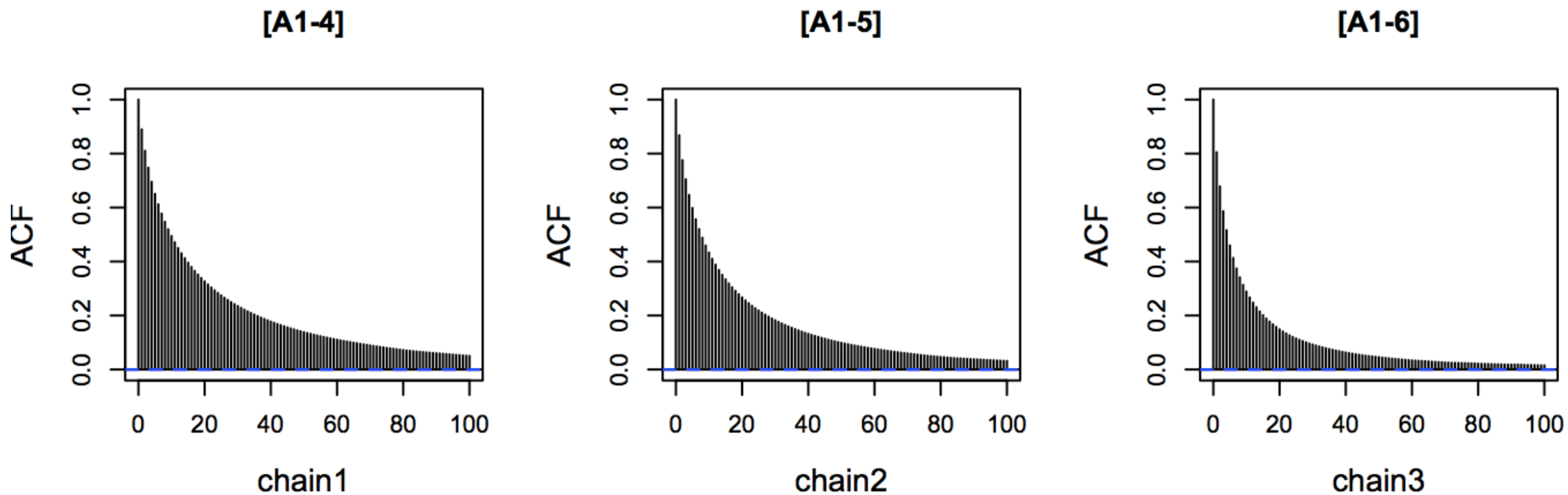


Fig 4. Autocorrelation plots of sample of $(X_1|S = v)$, $(X_2|S = v)$ and $(X_3|S = v)$.

Supplementary Note 2

- Since $X_j \geq 0$ for all j, the v-Fiber $\mathcal{F}_v$ reduces to the following bounded set, called *v-Simplex*.

$$\mathcal{S}_v = \{\mathbf{x} \in \mathbf{R}^d : x_1, \ldots, x_d \geq 0, \sum_{j=1}^{d} x_j = v\}$$
$$\subset [0, v]^d$$