Differentiability of flows and sensitivity analysis for reflected Brownian motions in polyhedral cones

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Differentiability of flows and sensitivity analysis

Differentiability of flows and sensitivity analysis for diffusions is a classical topic in stochastic analysis. Consider the parameterized SDE:

$$X^\alpha(t) = x_0(\alpha) + \int_0^t b(\alpha, X^\alpha(s))ds + \int_0^t \sigma(\alpha, X^\alpha(s))dW(s).$$

A natural question is: what are the pathwise effects of perturbations to the parameter α? Also of interest in applications, including in mathematical finance.

This question was studied by Elworthy, Bismut, Ikeda and Watanabe, Kunita and others in the late 1970s and early 1980s.

One can ask these same questions for stochastic processes that are constrained to lie in some region. In this talk I focus on reflected Brownian motions (RBMs) constrained to lie in a polyhedral cone.

Reflected Brownian motions (RBMs)

RBMs in convex polyhedral cones (e.g., nonnegative orthant) arise in a variety of applications, including

- As "heavy-traffic" limits in queueing networks
- Math finance (e.g., Atlas model)

Focus of this talk

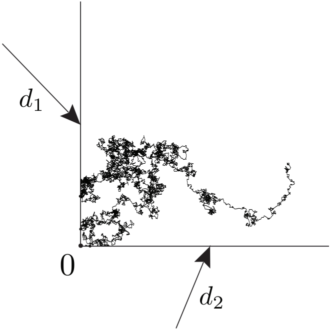
Understand the pathwise effects of perturbations to parameters (e.g., initial condition, drift, covariance, directions of reflection) that describe an RBM and study implications for computations of sensitivities of functions of RBMs

RBM: Description

Roughly speaking: An RBM Z in a polyhedral cone G:

- Behaves like a Brownian motion (BM) in the interior G°.
- Is constrained to remain in G by a **regulator** process that acts when Z is at the boundary ∂G in specified directions that are constant along each face of G.

Example: Nonnegative quadrant



RBM: Definition

An RBM in a polyhedral cone G with faces $F_1, \ldots, F_J$,

- initial condition z_0 in G,
- drift vector b,
- positive definite covariance matrix $\sigma\sigma^T$, and
- invertible reflection matrix $R \doteq \begin{pmatrix} d_1 & \cdots & d_J \end{pmatrix}$,

is a process $Z = \{Z(t), t \geq 0\}$ that satisfies

$$Z(t) = z_0 + bt + \sigma W(t) + RY(t) \in G,$$

where $W = \{W(t), t \geq 0\}$ is a BM and $Y = \{Y(t), t \geq 0\}$ is a **regulator** process that satisfies, for each i,

- $Y_i(0) = 0$ and Y_i is nondecreasing,
- Y_i can only increase when Z lies in face F_i.

Dupuis and Ishii '91 provide broad geometric conditions on $\{d_i\}$ under which there is a pathwise unique RBM with driving BM W.

Pathwise derivatives of RBMs

Fix a BM W. For each

- initial condition z_0 in G,
- drift vector b,
- dispersion matrix σ (s.t. $\sigma\sigma^T$ is positive definite),
- invertible reflection matrix R (satisfying geometric conditions),

let $Z^{z_0,b,\sigma,R}$ denote the associated RBM in G with driving BM W.

For $t \geq 0$, we seek to compute and characterize pathwise derivatives of $Z^{z_0,b,\sigma,R}(t)$ with respect to z_0, b, σ and R.

Implications for derivatives of stochastic flows and sensitivities of expectations of certain functions f of RBMs, e.g.:

$$\frac{d}{db}\mathbb{E}\left[f(Z^{z_0,b,\sigma,R}(t))\right] = \mathbb{E}\left[Df(Z^{z_0,b,\sigma,R}(t))\frac{d}{db}Z^{z_0,b,\sigma,R}(t)\right].$$

Prior results

- Deuschel and Zambotti ('03) characterized derivatives of flows of RBMs with state-dependent drift in the orthant with normal reflection.
- Andres ('09) characterized derivatives of flows of RBMs with state-dependent drift in a broad class of polyhedral domains with oblique reflection, but only up to the first hitting time of the nonsmooth part of the boundary.
- Dieker and Gao ('14) characterized sensitivities of reflected diffusions in the orthant with certain reflection matrices ($\mathcal{M}$-matrices) to perturbations of the drift in the direction $-\mathbf{1}$.

Our goal is to unify and extend these results.

RBMs defined via the Skorokhod map

Under our geometric conditions, given x in $\mathbb{C} = C([0,\infty),\mathbb{R}^J)$ with $x(0) \in G$, there is a unique pair (z, y) in $\mathbb{C} \times \mathbb{C}$ such that

$$z(t) = x(t) + Ry(t) \in G,$$

and for each i,

- $y_i(0) = 0$ and y_i is nondecreasing,
- y_i can only increase when z lies in face F_i.

We refer to the mapping $\Gamma : x \mapsto z$ as the **Skorokhod map (SM)**. In addition, the SM satisfies a certain Lipschitz continuity property.

Define the process $X = \{X(t), t \geq 0\}$ by

$$X(t) = z_0 + bt + \sigma W(t).$$

Then $Z = \Gamma(X)$.

Example: 1-d RBM via 1-d SM

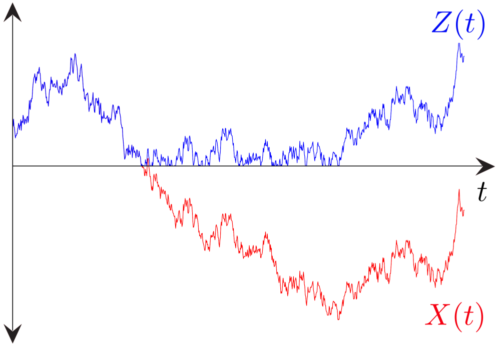
The 1-d SM has the following well-known explicit form:

$$\Gamma_1(x)(t) = x(t) + \sup_{0 \le s \le t} (-x(s)) \vee 0$$

Given an initial condition z_0, drift b and variance σ^2, define

$$X(t) = z_0 + bt + \sigma W(t)$$

Then a 1-d RBM Z can be defined pathwise by $Z = \Gamma_1(X)$.



Perturbed RBMs via the SM

For $\varepsilon > 0$ small, let Z^ε denote the RBM with **perturbed**

- initial condition $z_0 + \varepsilon y_0$,
- drift vector $b + \varepsilon a$,
- dispersion matrix $\sigma + \varepsilon \rho$,
- reflection matrix $R + \varepsilon S$.

Then

$$\begin{aligned} Z^\varepsilon(t) &= z_0 + \varepsilon y_0 + (b + \varepsilon a)t + (\sigma + \varepsilon \rho)W(t) + (R + \varepsilon S)Y^\varepsilon(t) \\ &= X(t) + \varepsilon \psi^\varepsilon(t) + RY^\varepsilon(t) \\ &= \Gamma(X + \varepsilon \psi^\varepsilon)(t) \end{aligned}$$

where X is defined as before and

$$\psi^\varepsilon(t) = y_0 + at + \rho W(t) + SY^\varepsilon(t).$$

Due to Lipschitz continuity of the SM, $\psi^\varepsilon \to \psi$ uoc where

$$\psi(t) = y_0 + at + \rho W(t) + SY(t).$$

Pathwise derivatives of RBMs

For $\varepsilon > 0$ and $t \geq 0$, we have (assuming the limit exists)

$$\lim_{\varepsilon \downarrow 0} \frac{Z^\varepsilon(t) - Z(t)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\Gamma(X + \varepsilon\psi^\varepsilon)(t) - \Gamma(X)(t)}{\varepsilon}.$$

Since $\psi^\varepsilon \to \psi$ uoc and Γ is Lipschitz continuous, we have

$$\lim_{\varepsilon \downarrow 0} \frac{Z^\varepsilon(t) - Z(t)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\Gamma(X + \varepsilon\psi)(t) - \Gamma(X)(t)}{\varepsilon}.$$

Given $x, \psi \in \mathbb{C}$ with $x(0) \in G$, define $\nabla_\psi \Gamma(x) : [0, \infty) \mapsto \mathbb{R}^J$ by

$$\nabla_\psi \Gamma(x)(t) = \lim_{\varepsilon \downarrow 0} \frac{\Gamma(x + \varepsilon\psi)(t) - \Gamma(x)(t)}{\varepsilon}, \qquad t \geq 0,$$

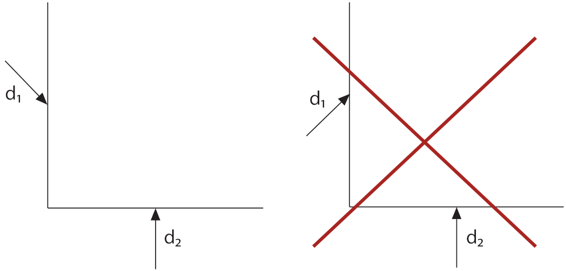
provided the limits exist. We refer to $\nabla_\psi \Gamma(x)$ as the **directional derivative of Γ at x in the direction ψ.**

Directional derivatives of SMs

Directional derivatives of SMs have been studied in the:

- One-dimensional setting: Mandelbaum and Massey ('95) and Whitt ('02)
- Multidimensional setting: Mandelbaum and Ramanan ('10) for a large class of SMs

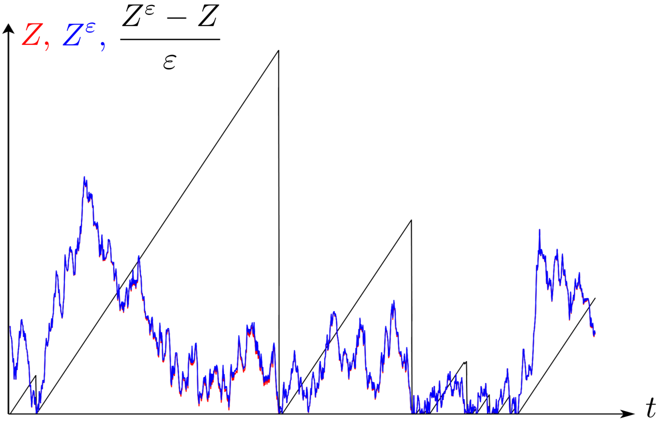
The class considered by Mandelbaum and Ramanan excludes some important cases (and their techniques cannot be extended):



How to go beyond?
A new approach is needed...

1-d directional derivatives

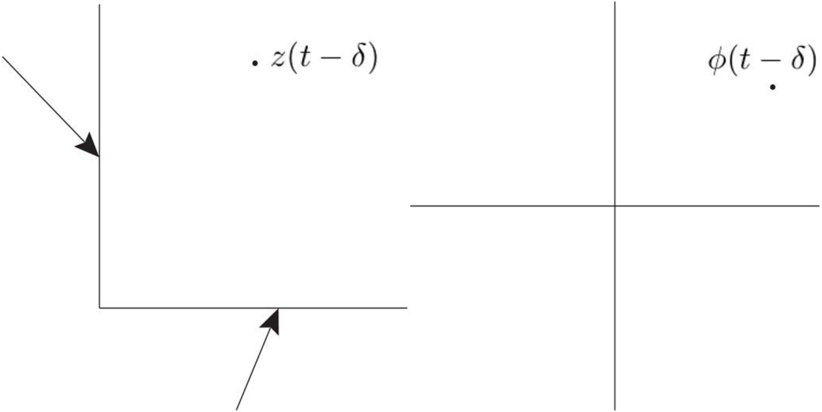
Suppose x is such that whenever $z(t) = \Gamma_1(x)(t) = 0$, the regulator term y is non-constant at t (holds when X is a BM). Then whenever $z(t) = 0$, $\nabla_\psi \Gamma_1(x)$ is projected to zero. This uses results on the directional derivatives of the 1-d SM.



Here Z is an RBM with drift b, Z^ε is an RBM with drift $b + \varepsilon$.

Projections in multidimensional setting

In the multidimensional setting, whenever z "hits" the relative interior of face F_i, $\nabla_\psi \Gamma(x)$ is projected to the hyperplane $H_i = \text{span}(F_i)$ along the direction $\text{span}(d_i)$.



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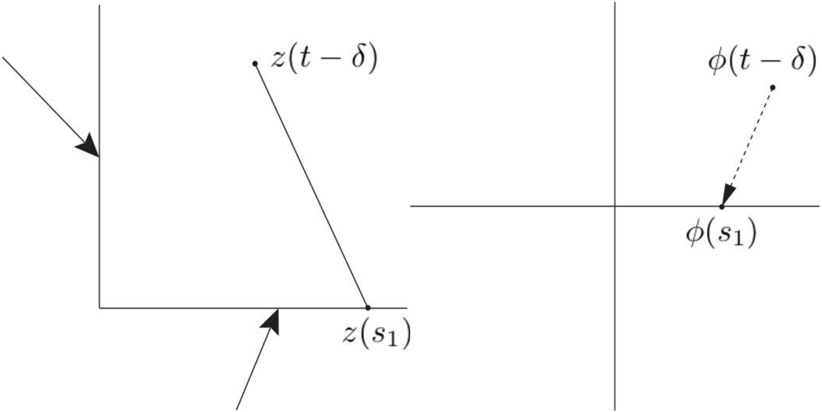

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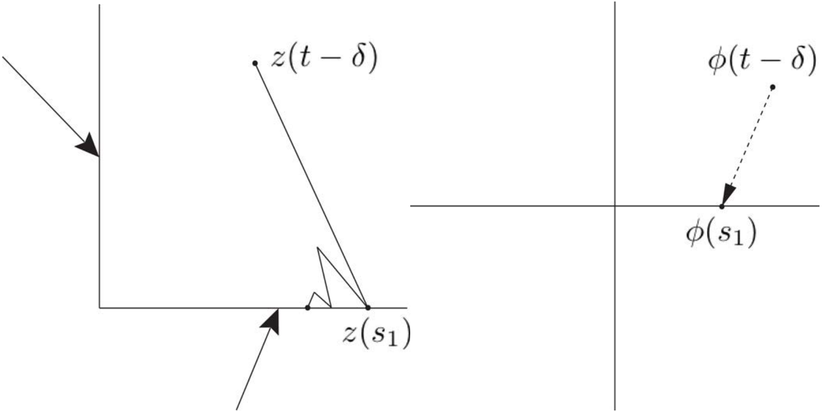

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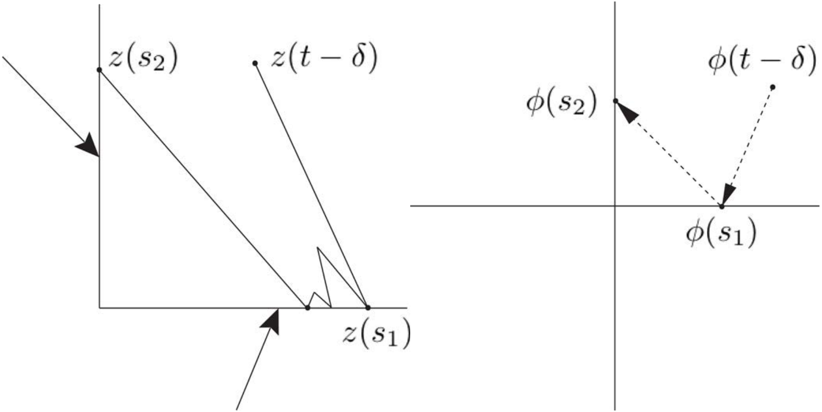
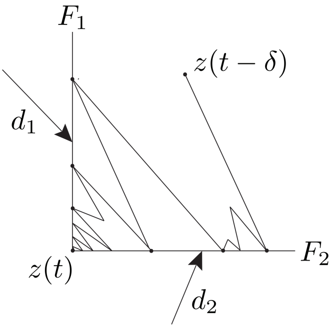


What happens at edges and corners?

Boundary jitter property

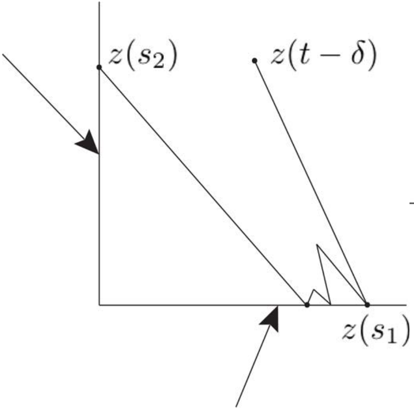
We assume the paths z satisfy the following boundary jitter property: If the $z(t) \in F_i \cap F_j$ for some $i \neq j$, then for each $\delta \in (0, t)$, there exists $s \in (t - \delta, t)$ such that $z(s)$ lies in the relative interior of face F_i.

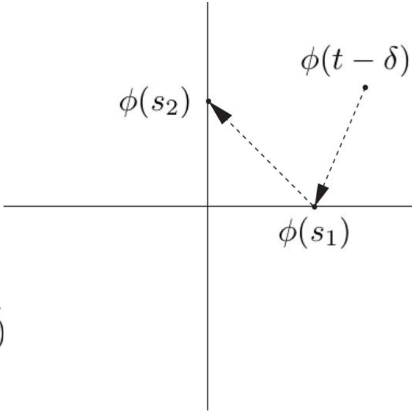


Lemma (L. and Ramanan, 2016): An RBM Z a.s. satisfies the boundary jitter property.

Directional derivatives at nonsmooth parts of ∂G

The projection property along with geometric properties on the directions of reflection $\{d_i\}$ ensures that if $z(t) \in F_i \cap F_j$, then $\phi(t) \doteq \nabla_\psi \Gamma(x)(t+) \in H_i \cap H_j$.





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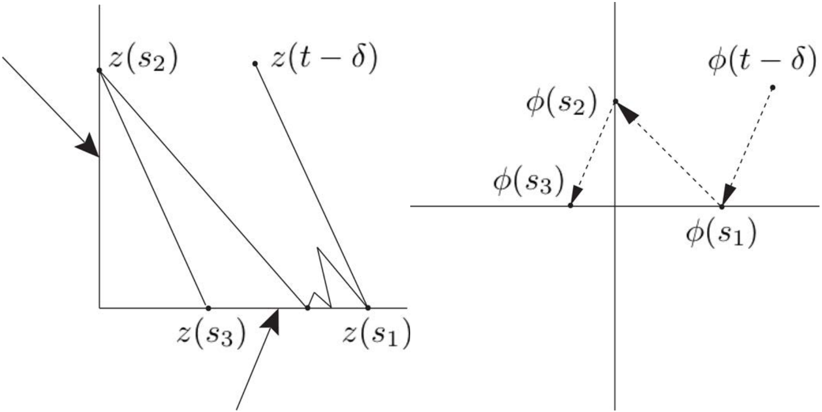

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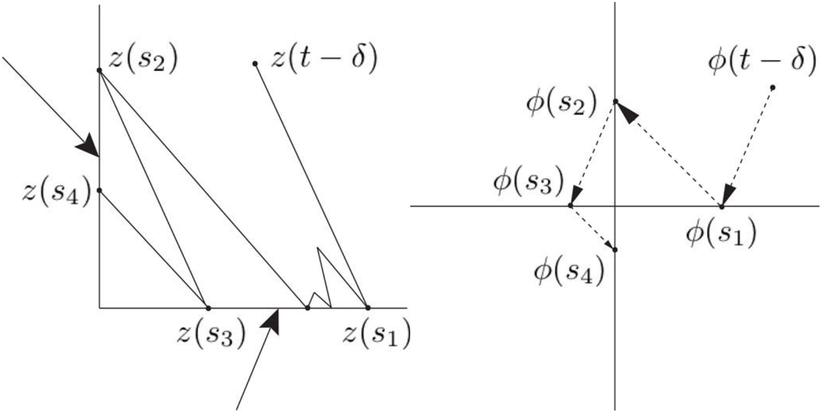

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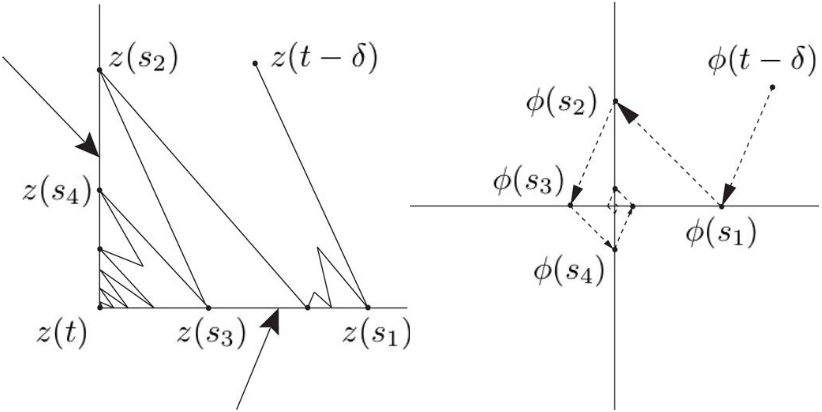

Directional derivatives of multidimensional SMs

For $x_0 \in G$, let $I(x_0) = \{i : x_0 \in F_i\}$ and $H_{x_0} \doteq \bigcap_{i \in I(x_0)} H_i$.

Theorem (L. and Ramanan, '16) Let $x \in \mathbb{C}$ satisfy $x(0) \in G$ and $z \doteq \Gamma(x)$ satisfy the boundary jitter property. Then for all $\psi \in \mathbb{C}$,

$$\nabla_\psi \Gamma(x)(t) = \lim_{\varepsilon \downarrow 0} \frac{\Gamma(x + \varepsilon\psi)(t) - \Gamma(x)(t)}{\varepsilon}$$

exists for all $t \geq 0$ and is left and/or right continuous at each $t > 0$. If $\phi(t) = \nabla_\psi \Gamma(x)(t+)$ for all $t \geq 0$, then ϕ is the unique function that satisfies:

$$\phi(t) = \psi(t) + R\eta(t) \in H_{z(t)},$$

where η is a right continuous function such that η_i is constant on intervals where z does not lie in face F_i, i.e.,

$$\eta_i(t) - \eta_i(s) = 0 \qquad \text{if } z(u) \notin F_i \text{ for all } u \in (s,t].$$

Furthermore, for fixed z, the mapping $\psi \mapsto \phi$ is **linear**.

Pathwise derivatives of RBMs

Corollary (L. and Ramanan, '16) Almost surely

$$\lim_{\varepsilon \downarrow 0} \frac{Z^{\varepsilon}(t) - Z(t)}{\varepsilon} = \nabla_{\psi}\Gamma(X)(t)$$

for all $t \geq 0$. If $\phi(t) = \nabla_{\psi}\Gamma(X)(t+)$ for all $t \geq 0$, then ϕ is the unique right continuous process satisfying

$$\phi(t) = y_0 + at + \rho W(t) + SY(t) + R\eta(t) \in H_{Z(t)},$$

where η is a right continuous process such that η_i is constant on intervals where Z does not lie in face F_i, i.e.,

$$\eta_i(t) - \eta_i(s) = 0 \qquad \text{if } Z(u) \notin F_i \text{ for all } u \in (s, t].$$

Summary and comments

- Proved the boundary jitter property for RBMs in a large class of polyhedral cones.
- Characterized pathwise derivatives of these RBMs.
- Method is readily adapted for state-dependent drift. State-dependent covariance presents unresolved technical challenges.
- Can compute certain pathwise derivatives for other reflected processes that satisfy the boundary jitter property.
- Current work: steady state analysis of the joint Markov process (Z, ϕ) and numerical methods for sensitivities.

Thank you