Analysis of Groundwater Levels at a River Without Water

Masayuki Sakai, Mihoko Minami
Keio Univ.

Abstract

We have meteorological data observed at a farm in Tochigi pref in Japan. A strange river flows near this farm. Usually, that river does not have any water, but it floods a few times in a year due to heavy rain. One of the reason why people began recording the data is observing this river. The temperature, moisture, groundwater level, and flood etc are begin observed. The main purpose is to analyze relationship between the groundwater levels and precipitation. As is well known, the precipitation affects changes in the groundwater levels. Our main goal is to estimate the effects that precipitation and other variables have on changes of groundwater levels.

To estimate these changes, we consider a linear gaussian state space model (LGSSM). State space models are widely used for time series analysis. In state space framework, we consider latent states that we cannot observe and data are observed with noise. The main purpose of state space models are estimating state values or distributions of states. In linear gaussian state space models, from property of multivariate normal distribution, we can estimate distributions of states as calculating conditional mean and conditional variance of multivariate normal distribution; these methods are called 'one-ahead-prediction' and 'smoothing'.

At first we fitted LGSSM with predictor and seasonal term to the data. However, we found that this model could not estimate effects of precipitation and seasonal term well. After showing fitting results, we consider whether this model could estimate these effects, and show the results of simulation for confirming the performance of this model.

Objectives

The goal of our research is to estimate seasonal effects and precipitation effects to groundwater levels at a river which normally is without water in Tochigi pref in Japan.

Groundwater levels = seasonal effect + precipitation effect

We have daily meteorological data observed at the farm located in Tochigi pref in Japan. Fig.2 shows that there seems to be some relation between groundwater levels and precipitation. Changes of groundwater levels are said to be affected by precipitation, melting ice and snow from mountain during spring, draining water off from a dam and the other sources. So, it can not be explained only by precipitation.

From our data, precipitation and groundwater levels have clear seasonal trend. Therefore we considered that changes of groundwater levels are explained precipitation effects and seasonal trend. To estimate two effects, we consider the linear gaussian state space model.

Methods

Linear Gaussian State Space Model (LGSSM)

We consider the model, called Linear Gaussian State Space Model (LGSSM). It consists of two equations, called observation equation (1), and the equation (2) is called state equation.

$$y_t = Z_t\theta_t + \epsilon_t\ , \qquad \epsilon_t \sim \mathrm{N}(0, H_t) \quad (1)$$

$$\theta_{t+1} = T_t\theta_t + R_t\eta_t, \quad \eta_t \sim \mathrm{N}(0, Q_t) \quad (2)$$

where

y_t : observation vector $(p \times 1)$
θ_t : state vector $(m \times 1)$
ϵ_t : observation noise $(p \times 1)$
η_t : state noise $(r \times 1)$

Z_t, T_t, R_t, H_t, Q_t : system matrix
$(p \times m)$ $(m \times m)$ $(m \times r)$
$(p \times p)$ $(r \times r)$

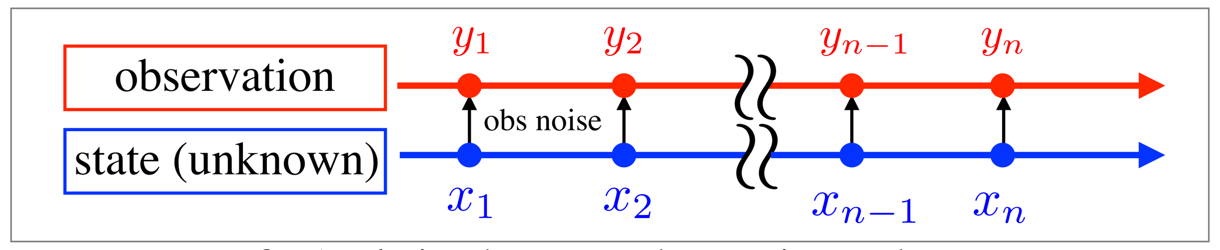

fig.1 relation between observation and state

In State Space framework, we consider a latent states and data consists of the latent states and observation noises. Our concern is estimating distributions that followed by states. To estimate them, we employ a two stage procedure to estimate states called 'One-ahead-prediction' and 'Smoothing' respectively.

Here distributions of observation and state are Gaussian at each time. Then using property of multivariate normal distribution, we can calculate conditional expectation and variance of state distributions as follows.

Notation

$$Y_{[1:t]} = (Y_1, \ldots, Y_t)^T, \quad \text{for} \quad t = 1, \ldots, n$$

$$v_t = y_t - Z_t a_t\,, \qquad F_t = \mathrm{Var}(v_t|Y_{1:t-1}) = Z_t P_t Z_t^T + H_t$$

$$K_t = T_t P_t Z_t^T F_t^{-1},$$

$$\theta_t|Y_{[1:t-1]} \sim \mathrm{N}(\tilde{\theta}_t, P_t), \quad \theta_t|Y_{[1:n]} \sim \mathrm{N}(\hat{\theta}_t, V_t)$$

One-ahead-prediction

→ Estimating state distributions θ_t given $Y_{[1:t]}$

$$\mathrm{E}(\theta_{t+1}|Y_{[1:t]}) = T_t\tilde{\theta}_t + K_t v_t$$

$$\mathrm{Var}(\theta_{t+1}|Y_{[1:t]}) = T_t P_t (T_t - K_t Z_t)^T + R_t Q_t R_t^T$$

Smoothing

→ Estimating state distributions θ_t given $Y_{[1:n]}$

$$\mathrm{E}(\theta_t|Y_{[1:n]}) = \tilde{\theta}_t + \sum_{j=t}^{n} \mathrm{Cov}(x_t, v_j) F_j^{-1} v_j$$

$$\mathrm{Var}(\theta_t|Y_{[1:n]}) = P_t - \sum_{j=t}^{n} \mathrm{Cov}(x_t, v_j) F_j^{-1} \mathrm{Cov}(x_t, v_j)^T$$

Results

Groundwater levels and Precipitation data

From fig2, we see that there seems to be some relation between groundwater levels and precipitation.

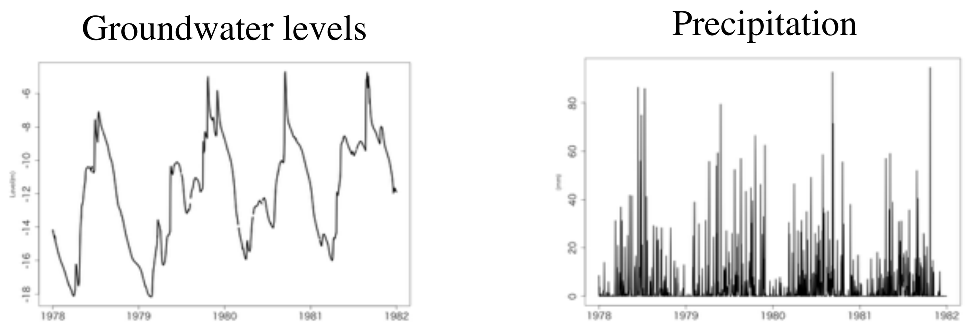

fig.2 left : Groundwater levels, right : Precipitation

Groundwater levels and precipitation clearly have annual cyclic trend. Precipitation has also similar trend. We consider including precipitation as predictor and seasonal term in linear gaussian state space model for changes of groundwater levels. We define system matrix, state vector and noise vector in observation and state equation as follows.

Model

$$Z_t = \begin{bmatrix} 1 & X_t & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \theta_t = \begin{bmatrix} \alpha_t & \beta_t & \gamma_t & C_1(t+1) & C_1^*(t+1) & C_2(t+1) & C_2^*(t+1) \end{bmatrix}^T, \quad H_t = \sigma_\epsilon^2$$

$$T_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cos(t\omega_1) & \sin(t\omega_1 & 0 & 0 \\ 0 & 0 & 0 & -\sin(t\omega_1) & \cos(t\omega_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos(t\omega_2) & \sin(t\omega_2) \\ 0 & 0 & 0 & 0 & 0 & -\sin(t\omega_2) & \cos(t\omega_2) \end{bmatrix}, \quad \eta_t = \begin{bmatrix} \eta_{t,\alpha} \\ \eta_{t,\beta} \\ \eta_{t,\gamma} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Q_t = \left[\begin{array}{ccc|c} \sigma_\alpha & 0 & 0 & \\ \beta & \sigma_\beta & 0 & 0 \\ 0 & 0 & \sigma_\gamma & \\ \hline & 0 & & 0 \end{array}\right]$$

where X_t is predictor (precipitation), α_t and β_t are coefficients, and γ_t is seasonal term. We set j = 1,2, and s = 365(annual). Now, response variable y_t is groundwater levels and is scholar. Therefore H_t is scholar. R_t is identify matrix.

$$\gamma_t = \sum_{j=1}^{2} C_j(t) \ , \quad \begin{aligned} C_j(t) &= a_j\cos(t\omega_j) + b_j\sin(t\omega_j), \\ C_j^*(t) &= -a_j\sin(t\omega_j) + b_k\cos(t\omega_j), \end{aligned} \quad \omega_j = \frac{2\pi j}{s}$$

and then

$$\begin{bmatrix} C_j(t+1) \\ C_j^*(t+1) \end{bmatrix} = \begin{bmatrix} \cos(\omega_j) & \sin(\omega_j) \\ -\sin(\omega_j) & \cos(\omega_j) \end{bmatrix} \begin{bmatrix} C_j(t) \\ C_j^*(t) \end{bmatrix}$$

Results

We show results after smoothing procedure in fig.3. symbol "^" stand for estimated value. Black solid lines are smoothing conditional mean, and blue dashed lines are 95% confidence interval.

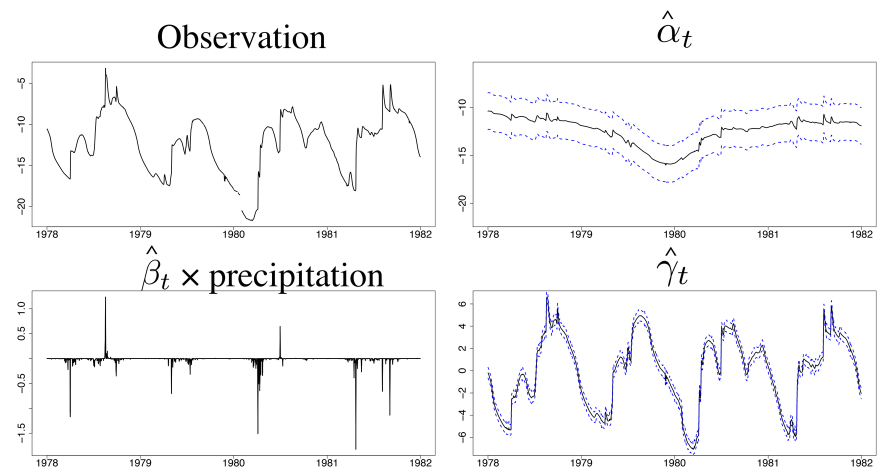

fig.3 Smoothed mean (black solid line) and 95% confidence interval (blue dashed line) of elements of state vector θ_t .

$\hat{\gamma}_t$ are annual cycle trend. contrary to our expectations, coefficient of precipitation nearly zero.

To investigate the validity of this model, we conducted a simulation with and sample data y_t^+ generated using precipitation data and following the model. We show the simulation results after smoothing in fig.4. symbol "+" stand for generated data for simulation. We found that the model is able to estimate properly two effects if the model was right for the data. Discussion for the simulation results is in next part.

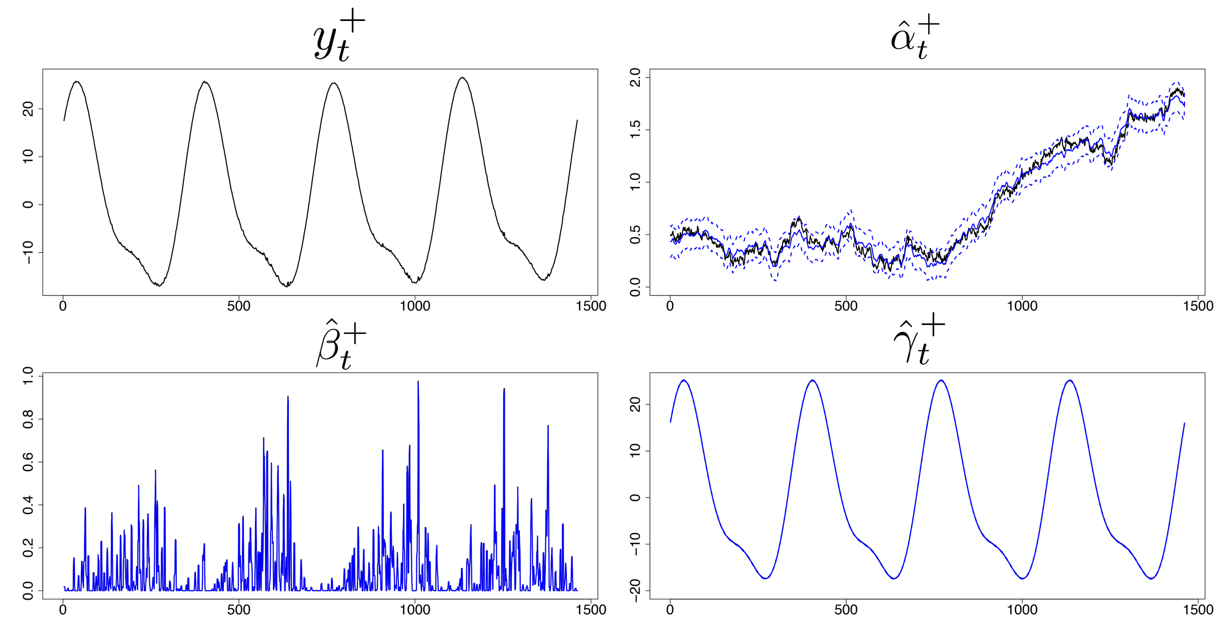

fig.4 Simulation results. Blue solid and dashed lines show smoothing mean and 95% confidence interval. Black lines show true values.

Conclusion and Future works

We estimated effects of seasonal term and precipitation on changes of groundwater levels by smoothing. According to our results, precipitation rarely has any effects on groundwater levels. It is intuitively strange. By simulation, we find this model is able to estimate effects of seasonal term and precipitation. One of the reasons why it is occurred is seasonal trend of precipitation are included cyclic rend in this model like simulation results.

We confirm that this model is able to estimate properly each variance and seasonal trend if model is right by simulation. So, main feature works is that considering annual cyclic trend of precipitation and extending this model to non-gaussian and non-linear state space model.

References

[1] G.Petris, S.Petrone, P.Campagnoli. (2009). Dynamic Linear Model with R. Splinger
[2] J.Durbin, and S.J.koopman (2012), Time Series Analysis By State Space Method, Oxford University Press
[3] J.F.Commandeur, and S.J.Koopman (2007). An Introduction to State Space Time Series Analysis, Oxford University Press
[5] M.Owor, R.G.Taylor, C.Tindimugaya, and D.Mwesigwa (2009). Rainfall intensity and groundwater recharge: empirical evidence from the Uper Nile Basin. Environmental Reraserch Letters. 4(2009)
[6] N.MATSUMOTO, Interpolation Method for Missing Data in Time Seriesof Groundwater Level and Barometric Pressure, Geological Survey of Japan,