Analysis of Groundwater Levels at a River Without Water
Masayuki Sakai, Mihoko Minami
Keio Univ.

Abstract

We have meteorological data observed at a farm in Tocchi Pref in Japan. A strange river flows near this farm. Usually, that river does not have any water, but it floods a few times in a year due to heavy rain. One of the reasons why people began recording the data is observing this river. The temperature, moisture, groundwater level, and flood etc are begun observed. The main purpose is to analyze relationship between the groundwater levels and precipitation. As is well known, the precipitation affects changes in the groundwater levels. Our main goal is to estimate the effects that precipitation and other variables have on changes of groundwater levels.

To estimate these changes, we consider a linear gaussian state space model (LGSSM). State space models are widely used for time series analysis. In state space framework, we consider latent states that we cannot observe and data are observed with noise. The main purpose of state space models are estimating state values or distributions of states. In linear gaussian state space models, from property of multivariate normal distribution, we can estimate distributions of states as calculating conditional mean and conditional variance of multivariate normal distribution; these methods are called ‘one-ahead-prediction’ and ‘smoothing’.

At first we fitted LGSSM with predictor and seasonal term to the data. However, we found that this model could not estimate effects of precipitation and seasonal term well. After showing fitting results, we consider whether this model could estimate these effects, and show the results of simulation for confirming the performance of this model.

Methods

Linear Gaussian State Space Model (LGSSM)

We consider the model, called Linear Gaussian State Space Model (LGSSM). It consists of two equations, called observation equation (1), and the equation (2) is called state equation.

\[
y_t = Z_t \theta_t + \epsilon_t, \quad \epsilon_t \sim N(0, H_t)
\]

\[
\theta_{t+1} = T_t \theta_t + R_t \eta_t, \quad \eta_t \sim N(0, Q_t)
\]

where

- \( y_t \): observation vector (\( p \times 1 \))
- \( \theta_t \): state vector (\( m \times 1 \))
- \( \epsilon_t \): observation noise (\( p \times 1 \))
- \( \eta_t \): state noise (\( p \times 1 \))
- \( Z_t \): measurement matrix
- \( T_t \): transition matrix
- \( H_t \): observation noise matrix
- \( R_t \): state noise matrix

In State Space framework, we consider a latent states and data consists of the latent states and observation noises. Our concern is estimating distributions that followed by states. To estimate them, we employ a two stage procedure to estimate states called ‘one-ahead-prediction’ and ‘smoothing’ respectively. Here distributions of observation and state are Gaussian at each time. Then using property of multivariate normal distribution, we can calculate conditional expectation and variance of state distributions as follows.

\[
Y_{1|1} = (Y_1, \ldots, Y_{t+1})^T \\
\varepsilon_t = y_t - Z_t \theta_t \\
F_t = Var(\varepsilon_t|Y_{1:t}) = Z_t P_t Z_t^T + H_t
\]

\[
K_t = T_t P_t Z_t^T \\
\theta_t|Y_{1:t+1} \sim N(\hat{\theta}_t, P_t) \\
\theta_t|Y_{1:t} \sim N(\hat{\theta}_t, P_t)
\]

One-ahead-prediction

Estimating state distributions \( \theta_t \) given \( Y_{1:t} \)

\[
E(\theta_{t+1}|Y_{1:t}) = T_t \hat{\theta}_t + K_t \varepsilon_t
\]

\[
Var(\theta_{t+1}|Y_{1:t}) = T_t P_t (T_t - K_t Z_t)^T + R_t Q_t R_t^T
\]

Smoothing

Estimating state distributions \( \theta_t \) given \( Y_{1:n} \)

\[
E(\theta_t|Y_{1:n}) = \hat{\theta}_t + \sum_{j=t}^n Cov(x_t, \varepsilon_j) F_j^{-1} \varepsilon_j
\]

\[
Var(\theta_t|Y_{1:n}) = P_t - \sum_{j=t}^n Cov(x_t, \varepsilon_j) F_j^{-1} Cov(x_t, \varepsilon_j)^T
\]

Groundwater levels and precipitation data

From fig.2, we see that there seems to be some relation between groundwater levels and precipitation.

Groundwater levels

Precipitation

Model

\[
Y_t = \alpha_1 + \beta_1 P_t + \gamma_1 \sin(\omega_1 t) + \epsilon_t
\]

\[
\beta_t = \alpha_2 + \beta_2 P_t + \gamma_2 \sin(\omega_2 t) + \eta_t
\]

Results

We show results after smoothing procedure in fig.3. symbol ‘*’ stand for estimated value. Black solid lines are smoothing conditional mean, and blue dashed lines are 95% confidence interval.

Conclusion and Future works

We estimated effects of seasonal term and precipitation on changes of groundwater levels by smoothing. According to our results, precipitation rarely has any effects on groundwater levels. It is intuitively strange. By simulation, we find this model is able to estimate effects of seasonal term and precipitation. One of the reasons why it is occurred is seasonal trend of precipitation are included cyclic trend in this model like simulation results.

We confirm that this model is able to estimate properly each variance and seasonal trend if model is right by simulation. So, main feature works is that considering annual cyclic trend of precipitation and extending this model to non-gaussian and non-linear state space model.

References