Local time and null-recurrent averaging

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Joint work with Zsolt Pajor-Gyulai (Courant)

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- Topologically, the set $\{t \geq 0 : W(t) = 0\}$ is a zero-measure Cantor set.

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  - Random density. For any $f \in L^1(\mathbb{R})$,

$$\int_0^t f(W(s)) ds = \int_{-\infty}^{\infty} f(x) L^W(t,x) dx.$$

Symmetric local time

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- It is a non-decreasing random process which is constant on the set $\{t > 0 : X(t) \neq x\}$.
- Meyer-Tanaka formula says that for any measurable f,

$$\int_0^t f(X(s)) d\langle X \rangle_s = \int_{-\infty}^{\infty} f(x) L^X(t,x) dx.$$

Classical averaging principle

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- Since X is periodic, $X(t/\varepsilon)$ spins very, very fast before Y^ε moves very much. The limiting behavior of Y^ε is essentially averaged over the behavior of X.

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$$\lim_{\varepsilon \to 0} (Y^\varepsilon(t) - Y^\varepsilon(0)) = t\bar{b}$$

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- Let $\dot{y}(t) = \bar{b}(y(t))$.

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- This suggests that

$$\lim_{\varepsilon \to 0} \sup_{t \leq T} |Y^\varepsilon(t) - y(t)| = 0.$$

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- When ε is small $X^\varepsilon(t)$ is very likely near positive or negative infinity.
- If $b_\pm(y) = \lim_{x \to \pm\infty} b(x, y)$ exists, then $\lim_{\varepsilon \to 0} Y^\varepsilon =: Y^0$ solving

$$dY^0(t) = \left(\mathbb{1}_{\{W_1(t) > 0\}} b_+(Y^0(t)) + \mathbb{1}_{\{W_1(t) < 0\}} b_-(Y^0(t)) \right) dt$$

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- Let $a(x,y) = \begin{pmatrix} \varphi(x,y) \\ \sigma(x,y) \end{pmatrix} \begin{pmatrix} \varphi(x,y) \\ \sigma(x,y) \end{pmatrix}^T = \begin{pmatrix} |\varphi(x,y)|^2 & \varphi\sigma^T(x,y) \\ \sigma\varphi^T(x,y) & \sigma\sigma^T(x,y). \end{pmatrix}$

Cesàro limits

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- Then $(\varepsilon\tilde{X}^\varepsilon(t), Y^\varepsilon(t)) \to \bar{Z} = (\bar{X}(t), \bar{Y}(t))$ in law.

$$d\bar{Z}(t) = \begin{pmatrix} 0 \\ \bar{b}(\bar{Z}) \end{pmatrix} + \sqrt{\bar{a}(Z(t))}dW(t)$$

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- Then by Khasminskii, Krylov, $Y^\varepsilon(t) \to y_0$ uniformly on finite time intervals.
- We study nontrivial limiting behavior of the form

$$\frac{Y^\varepsilon(t) - y_0}{\varepsilon^\alpha} \quad \text{or} \quad Y^\varepsilon(t/\varepsilon^\alpha)$$

Zero Cesàro limits example

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- Meyer-Tanaka formula

$$= \int_{-\infty}^{\infty} b(\varepsilon^{-1}x) L^W(t,x)dx = \varepsilon \int_{-\infty}^{\infty} b(x) L^W(t, \varepsilon x)dx.$$

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$$= \int_{-\infty}^{\infty} b(\varepsilon^{-1}x) L^W(t,x)dx = \varepsilon \int_{-\infty}^{\infty} b(x) L^W(t, \varepsilon x)dx.$$

- This implies that

$$\lim_{\varepsilon \to 0} \frac{Y^\varepsilon(t) - Y^\varepsilon(0)}{\varepsilon} = \left(\int_{-\infty}^{\infty} b(x)dx \right) L^W(t,0).$$

Martingale example

- Fast motion is Brownian. Slow motion has no stochastic term

$$\begin{cases} dX^\varepsilon(t) = \varepsilon^{-1} dW_1(t) \\ dY^\varepsilon(t) = \sigma(X^\varepsilon(t)) dW_2. \end{cases}$$

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- We showed on the last slide that this expression is of order ε. Then $Y^\varepsilon(t) - Y^\varepsilon(0)$ is of order $\sqrt{\varepsilon}$.

Martingale example

$$\lim_{\varepsilon \to 0} \frac{Y^\varepsilon(t) - Y^\varepsilon(0)}{\sqrt{\varepsilon}} = V^{W_1}(t)$$

- where V^{W_1} is a martingale with quadratic variation

$$\left\langle V^{W_1} \right\rangle_t = \left(\int_{-\infty}^{\infty} \sigma^2(x) dx \right) L^{W_1}(t).$$

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- Convergence in distribution.

A more general problem

- Consider the following fast-slow system

$$\begin{cases} dX^\varepsilon(t) = \varphi(\varepsilon^{-1}X^\varepsilon(t), Y^\varepsilon(t))dW(t) \\ dY^\varepsilon(t) = (b_1(Y^\varepsilon(t)) + b_2(\varepsilon^{-1}X^\varepsilon(t), Y^\varepsilon(t)))dt \\ \qquad\qquad +\sigma(\varepsilon^{-1}X^\varepsilon(t), Y^\varepsilon(t))dW(t). \end{cases}$$

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$$\begin{cases} d\bar{X}^\varepsilon(t) = \varphi(\varepsilon^{-2}\bar{X}^\varepsilon(t), \bar{Y}^\varepsilon(t))dW(t), \\ d\bar{Y}^\varepsilon(t) = \varepsilon^{-2}b_2(\varepsilon^{-2}\bar{X}^\varepsilon(t), \bar{Y}^\varepsilon(t))dt + \varepsilon^{-1}\sigma(\varepsilon^{-2}\bar{X}^\varepsilon(t), \bar{Y}^\varepsilon(t))dW(t). \end{cases}$$

Long-time limit

Theorem (Zs. Pajor-Gyulai, M.S. 2015)

The long-time limit converges to $(\bar{X}, \bar{Y})$ *solving*

$$\begin{cases} d\bar{X}(t) = \sqrt{\varphi_{\pm}^2(\bar{X}(t), \bar{Y}(t))} dW(t), \\ d\bar{Y}(t) = \left(\int_{-\infty}^{\infty} \frac{b_2}{|\varphi|^2}(x, \bar{Y}(t)) dx \right) L^{\bar{X}}(dt, 0) \\ \quad + \left(\sqrt{\int_{-\infty}^{\infty} \left(\frac{\sigma\sigma^T}{|\varphi|^2} \right) (x, \bar{Y}(t)) dx} \right) dV^{\bar{X}}(t). \end{cases}$$

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$$\left\langle V_i^{\bar{X}}, V_j^{\bar{X}} \right\rangle_t = \delta_{ij} L^{\bar{X}}(t, 0), \quad \left\langle V_i^{\bar{X}}, W \right\rangle_t = 0.$$

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  $L^{\bar{X}}$ is the symmetric local time of $\bar{X}$ at $x = 0$.

Thank you